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A Second Step to Mathematical Olympiad Problems

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A SECOND STEP TO MATHEMATICAL OLYMPIAD PROBLEMS

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that leads on to the solution of certain Diophantine equations of the form $as + bt = r$.

You should remember (if you don't, it's probably a good idea to go and check it out), that to find the highest factor of 78 and 30 we adopt the following approach

$$78 = 2 \times 30 + 18,$$
$$30 = 1 \times 18 + 12,$$
$$18 = 1 \times 12 + 6,$$
$$12 = 2 \times 6.$$

Since 6 is the last non-zero remainder in this process, then $6 = (78, 30)$. That is, 6 is the highest common factor of 78 and 30. But working back from there we see that

$$\begin{aligned} 6 &= 18 - 1 \times 12 \\ &= 18 - (30 - 1 \times 18) \\ &= 2 \times 18 - 30 \\ &= 2 \times (78 - 2 \times 30) - 30 \\ &= 2 \times 78 - 5 \times 30. \end{aligned}$$

In other words, if we want to solve $78a + 30b = 6$, one solution is $a = 2$, $b = -5$. But Chapter 4 of *First Step* showed that there are in fact an infinite number of solutions of $78a + 30b = 6$. It also showed how to find all these solutions.

Exercises

5. Using the Euclidean Algorithm, find the highest common factor of the following pairs of numbers.
   (i) 78, 45; (ii) 121, 33; (iii) 151, 72.
6. Use the last exercise to find all solutions of the following equations.
   (i) $78a + 45b = 3$; (ii) $121a + 33b = 11$; (iii) $151a + 72b = 1$.

The other thing in Chapter 4 of *First Step* that I wanted to mention here is the technique that enabled us to find things like $\sum_{r=1}^{n} r^2$.

13. Find the partial fraction decomposition for
$$\frac{1}{x(x-1)(x+1)}.$$

14. Are the partial fraction decompositions of Exercises 12 and 13 unique? In other words, are there other families of solutions, as there were for fractions of the type $\frac{1}{12}$ or is there just one solution each time?

Actually, the method I showed you earlier is not the quickest. I lied! Apologies. (This just shows that you can never trust a mathematician. You should check everything that I say. If nothing else, I do make errors from time to time — even in books.) It turns out that ***the cover up method*** is quicker. However, this is a deep dark secret, reserved for the most sophisticated mathematical gurus. It is with some hesitation that I reveal it now to you. I sincerely expect great retribution will be heaped on my head. Undoubtedly I will be struck off all sophisticated mathematical gurus' Christmas cards' lists.

Now take $\frac{2}{1-x^2} = \frac{1}{(1-x)(1+x)}$.

Remember I wanted to find a and b in $\frac{a}{1-x} + \frac{b}{1+x}$.

Previously I concocted a common denominator and then substituted $x = 1$ and $x = -1$. Let's now do this at source. If we put $x = 1$ in $\frac{2}{(1-x)(1+x)}$ the world (or at least your maths teacher) will explode because it looks as if we're going to have to divide by zero! As you all know, this is the worst sin in mathematics land and condemns the perpetrator to instant mathematical Hades.

So let's hide our sin by covering up the $1-x$!

Putting $x = 1$ in $\frac{2}{(\;\times\;)(1+x)}$ gives $\frac{2}{(\;\times\;)(1+1)} = 1$.

This turns out to be the coefficient of $\frac{1}{1-x}$ in the partial fraction decomposition of $\frac{2}{1-x^2}$.

Do the same covering up with $x = -1$. We get $\frac{2}{(1-(-1))(\;\times\;)} = 1$. Again 1 is the coefficient of $\frac{1}{1+x}$ in the decomposition of $\frac{2}{1-x^2}$.

So $\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$.

Let's repeat that on $\frac{1}{x(x-1)}$. Now clearly $\frac{1}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1}$ and we can do this the legitimate way that I showed you earlier. But let's live dangerously. Try the cover up method. To find a we let $x = 0$. So we see that $\frac{1}{(\;\times\;)(0-1)} = -1 = a$.

To find b we let $x = 1$. So we get $\frac{1}{1(\;\times\;)} = 1$. So $b = 1$.

Is it really true that $\frac{1}{x(x-1)} = -\frac{1}{x} + \frac{1}{x-1}$?

Exercises

25. Show that the following recurring decimals are fractions.
    (i) $0.\dot{1}$; (ii) $0.\dot{7}$; (iii) $0.\dot{2}\dot{1}$; (iv) $0.\dot{2}0\dot{1}$.
26. Show that ***every*** repeating decimal is a fraction, or find a repeating decimal which isn't a fraction.

If we look a little more closely at repeating decimals we see something interesting. Look at $0.\dot{3}$, for instance. Here we have $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$. To get the next term what do we have to do? Multiply by $\frac{1}{10}$. This goes on and on for ever.

The same sort of thing happens with $0.\dot{7}\dot{1}$, though here we multiply by $\frac{1}{100}$ each time. Let's generalise that. Clearly we can look at numbers that change by a factor of 10^{-3} or 10^{-4} and so on. However, we can do things more generally. Why don't we multiply by some number r? So we get, say

$$3 + 3r + 3r^2 + 3r^3 + 3r^4 + \cdots + 3r^{n-1} + \cdots$$

But more generally still, start with a as the first term, not 3. Then we get

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

How can we sum this infinite expression? Is it a fraction? Will a sum always exist?

Incidentally, an expression like $\sum_{n=1}^{\infty} ar^{n-1}$, which goes on for ever, is said to be the sum of an ***infinite geometric progression***. The a is called the ***first term*** and the r is the ***common ratio*** (between consecutive terms).

The sum $\sum_{n=1}^{\infty} ar^{n-1}$ is the same as $a + ar + ar^2 + ar^3 + \cdots$. The sum is considered to be going on for ever. Hence you never actually get to put $n = \infty$ in the summation. The ∞ at the top of the $\sum$ just warns you that there's an awful lot of adding to do.

Exercises

27. Let $S = \sum_{n=1}^{\infty} ar^{n-1}$ where $a = 2$ and $r = \frac{1}{3}$. Using the technique of Example 1, express S as a fraction.
    (The "technique of Example 1" was multiply by 100. What corresponds to 100 in this exercise?)
28. Express the following as fractions.
    (i) $S = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}$; (ii) $S = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}$.

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42. Find recurrence relations for the following sequences.
    (i) $5, 10, 20, 40, 80, \ldots$; (ii) $5, 15, 45, 135, 405, \ldots$;
    (iii) $5, 7, 9, 11, 13, \ldots$; (iv) $5, 7, 12, 19, 31, \ldots$;
    (v) $5, 7, 17, 31, 65, \ldots$; (vi) $5, 7, 24, 62, 172, \ldots$.

What would be nice would be a method of solving recurrence relations so that we could find the value of the n-th term. This would make it easy then to calculate, say, the 594th term of the Fibonacci sequence. It would save us adding term to term via equation (2) until we got as far as 594. It would also mean that next week, when we wanted the 638th term, we wouldn't have to use the recurrence relation and start all over again.

Finding the n-th term of the Fibonacci sequence is not too easy. So let's work up to it. Let's try to find an expression for the general term of the sequence which has $a_1 = 1$ and recurrence relation $a_n = 2a_{n-1}$.

First let us see what the first few terms look like. If $a_1 = 1$, then $a_2 = 2 \times a_1 = 2$. So $a_3 = 2 \times a_2 = 4$. The sequence must be $1, 2, 4, 8, 16, \ldots$. So what is a_n (apart from being $2a_{n-1}$)? It's clearly some power of 2. Is it 2^n or 2^{n+1} or something close to one of these?

If $a_n = 2^n$, then $a_1 = 2$. Clearly $a_n = 2^{n+1}$ won't work either. It's always too big. What if we try something smaller like $a_n = 2^{n-1}$? Then $a_1 = 2^0 = 1$, $a_2 = 2^1 = 2$, $a_3 = 2^2 = 4$ and so on. That looks pretty good.

Exercise

43. Guess the n-th term of the sequences defined by the recurrence relations below.
    (i) $a_n = 2a_{n-1}, a_1 = 2$; (ii) $a_n = 3a_{n-1}, a_1 = 1$;
    (iii) $a_n = 4a_{n-1}, a_1 = 3$; (iv) $a_n = 5a_{n-1}, a_1 = -1$.

But in the sequence defined by $a_n = 2a_{n-1}$ with $a_1 = 1$, how can we be absolutely sure that $a_n = 2^{n-1}$? It certainly fits the pattern alright. But maybe, after starting nicely, it runs off the rails. Remember the number of regions into which lines divide a circle? That looked easy in Chapter 6 of *First Step* till we tried $n = 6$, and then the 2^{n-1} pattern blew up in our faces.

So, if it's true here that $a_n = 2^{n-1}$, how can we *prove* it? Do we have the tools? Well, something like this suggests we try mathematical induction (again see Chapter 6).

Claim. *If $a_n = 2a_{n-1}$ with $a_1 = 1$, then $a_n = 2^{n-1}$.*

Proof. Step 1. Check the case $n = 1$. Now $a_1 = 1$ — we were told this. Further $2^{1-1} = 2^0 = 1$. So $a_n = 2^{n-1}$ for $n = 1$.

Example 3. Find the general term of the sequence defined by the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \quad \text{where } a_1 = 1 \text{ and } a_2 = 2.$$

Again, try to find a few terms. This won't always help us with the general term but it will give us some idea of what sequence it is we're dealing with. And it will provide a useful check.

So $a_3 = 2 \times 2 - 1 = 3, a_4 = 2 \times 3 - 2 = 4, a_5 = 2 \times 4 - 3 = 5$. It looks like $a_n = n$. But can that be right?

What does the theorem say? In the present case $A = 2$ and $B = -1$. So we have to solve the quadratic $x^2 = 2x - 1$. Rearranging we get $x^2 - 2x + 1 = 0$ or $(x-1)^2 = 0$. The only root of this equation is 1. Thus we use the second part of the theorem with $\alpha = 1$ to find that

$$a_n = (K + nL)1^n.$$

As before we find K and L by using a_1 and a_2.

$$1 = a_1 = K + L,$$
$$2 = a_2 = K + 2L.$$

Solving gives $K = 0$ and $L = 1$. So $a_n = n$. How about that?

Exercises

50. Use the theorem to find the general term of the sequences described below.
    (i) $a_n = 4a_{n-1} - 3a_{n-2}$, $a_1 = 1$, $a_2 = 1$;
    (ii) $a_n = 4a_{n-1} - 3a_{n-2}$, $a_1 = 3$, $a_2 = 9$;
    (iii) $a_n = 4a_{n-1} - 3a_{n-2}$, $a_1 = 1$, $a_2 = 9$;
    (iv) $a_n = 5a_{n-1} - 6a_{n-2}$, $a_1 = 0$, $a_2 = 1$;
    (v) $a_n = 4a_{n-1} - 4a_{n-2}$, $a_1 = 0$, $a_2 = 1$.
51. Use the theorem to find the general term of the Fibonacci sequence.
52. Let a_n be the number of n-digit numbers that can be made using only 1 and 2, if no consecutive 2's are allowed. Show that $a_1 = 2, a_2 = 3$ and $a_3 = 5$. Find a recurrence relation for a_n.
53. Let a_n be the number of ways of hanging red and white shirts on a line so that no two red shirts are next to each other. Find a recurrence relation for a_n.

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Example 6. Find the general term of the power series

$$f(x) = \frac{x}{1 - 5x - 6x^2}.$$

The trick is: partial fractions! You see, $1 - 5x + 6x^2 = (1 - 3x)(1 - 2x)$. So

$$f(x) = \frac{1}{(1 - 3x)(1 - 2x)} = \frac{1}{1 - 3x} - \frac{1}{1 - 2x}.$$

(I used the, shsh!, cover up method.)

By the extended Binomial Theorem we learn that

$$\frac{1}{1 - 3x} = \sum_{n=0}^{\infty} (3x)^n \quad \text{and} \quad \frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n.$$

Hence $f(x) = \sum_{n=0}^{\infty}(3^n x^n - 2^n x^n) = \sum_{n=0}^{\infty}(3^n - 2^n)x^n$.

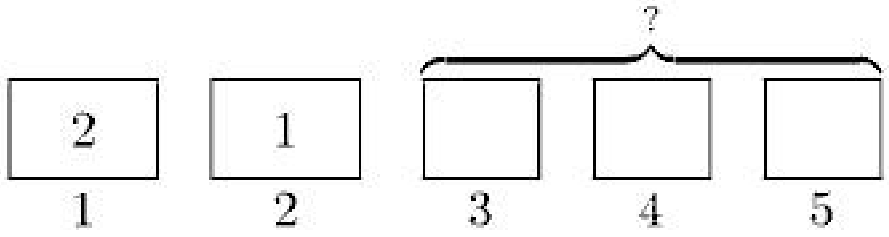
So that means that the coefficient of x^n is, in fact, $3^n - 2^n$.

In general then, we try to use the partial fraction decomposition of $f(x)$. When we've done that, we can expand using the extended Binomial Theorem. If we then gather the x^n terms together, we've won ourselves a_n.

Exercises

62. Use the above method to find an expression for a_n for each of the recurrence relations in Exercise 60.
63. Use generating functions to find the general terms of the following sequences.
    (i) $a_n = 5a_{n-1} - 6a_{n-2}$, $a_1 = 0$, $a_2 = 1$;
    (ii) $a_n = 2a_{n-1} + 3a_{n-2}$, $a_1 = 1$, $a_2 = 2$;
    (iii) $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1 = 1$, $a_2 = 4$;
    (iv) $a_n = 4a_{n-1} + 5a_{n-2}$, $a_1 = 1$, $a_2 = 3$;
    (v) $a_n = 2a_{n-1} + 3$, $a_1 = 1$;
    (vi) $a_n = 3a_{n-1} + 2$, $a_1 = 2$.
64. Use the generating function $F(x) = \frac{x}{1-x-x^2}$ to find the general term of the Fibonacci sequence.
65. Colour the squares of a $1 \times n$ chessboard either red or white or blue. How many ways are there of doing this if no two white squares are adjacent?
66. I have given you two methods to solve recurrence relations. Are they really the same method? Is one quicker than the other? Is one more useful than the other?

interchanged. I've shown this in the diagram.



In how many ways can the postman get the letters to houses 3, 4, 5 the wrong way? It's just 2 isn't it? But more importantly, what does that 2 represent? Isn't it just the number of ways of delivering three letters wrongly to three houses. So we should perhaps really think of 2 as p_3. After all we're trying to get a recurrence relation for p_n. By starting with five houses, we hope to get p_5 in terms of smaller p_n. That way we might see how to get a general recurrence relation.

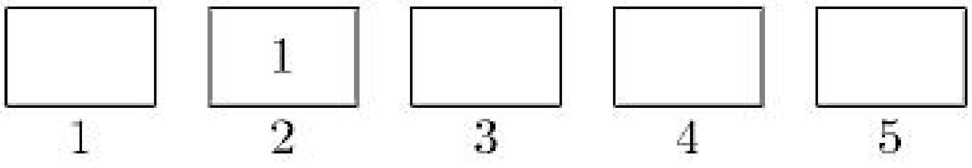
So we've found p_3 wrong deliveries when 1 and 2 are swapped. How many possible swaps are there with 1? Well, there's 1 and 2, 1 and 3, 1 and 4, 1 and 5. That's four altogether.

If the postman goes wrong by swapping 1 with some other letter, then he can go wrong in $4p_3$ ways.

Exercise

73. The postie is faced with a street with six houses.
    (a) How many completely wrong deliveries can he make if he swaps letter 1 with letter 5?
    Repeat this with n houses.
    (b) How many completely wrong deliveries can he make, if he inadvertently swaps letter 1 with another letter in the six house street?
    Generalise.

What then, if letter 1 isn't swapped with any other letter? Suppose letter 1 gets delivered to house 2 but letter 2 doesn't get delivered to house 1. In how many ways can the postman go wrong now?



Certainly 3 can't go to 3, 4 can't go to 4 and 5 can't go to 5. What's more 2 can't go to 1. Hmm. What if we readdressed letter 2 as letter 1 for a minute. (That's safe because letter 1 has actually been delivered to house 2.) We now see that the new 1 can't go to 1, 3 can't go to 3, 4 can't got to 4 and 5

19. In the spirit of this section we want to rearrange things so that there is a cancelling. Show that $a_k = 2ka_k - 2(k+1)a_{k+1}$. I'll give you the rest of the solution later.
    [This problem was submitted by Iceland in the 29th IMO in Canberra. It was not used for that Olympiad.]
20. (i) $0.1\dot{6}$; (ii) $0.\dot{1}$; (iii) $0.\dot{0}\dot{9}$; (iv) 0.2; (v) $0.\dot{1}4285\dot{7}$.
21. (i) Yes; (ii) Yes; (iii) Yes; (iv) Yes.
22. $\frac{27}{83} = 0.2 + \text{rem}.21 = 0.32 + \text{rem}.44 = 0.325 + \text{rem}.25$
    $= 0.3253 + \text{rem}.1 = 0.32530 + \text{rem}.10 = 0.32531 + \text{rem}.17$
    $= 0.325312048 + \text{rem}.16 = 0.3253120481 + \text{rem}.77$
    $= 0.32531204819 + \text{rem}.23 = 0.325312048192 + \text{rem}.64$
    $= 0.3253120481927 + \text{rem}.59 = \cdots$
    Wait a minute. This is all boring and tedious and totally unnecessary. Look at those remainders. Every one of them is naturally under 83. Eventually a remainder will be zero OR a remainder must come up that we've seen before. That'll have to happen in less than or equal to 83 divisions. When it happens the whole sequence will go round again till the same remainder occurs. Then off we go again and again. So $\frac{27}{83}$ must be a repeating decimal.
23. The argument is exactly the same as that of Exercise 22.
24. Yes. Yes. Yes. (But why?)
25. (i) If $S = 0.\dot{1}$, then $10S = 1 + S$. Hence $S = \dfrac{1}{9}$; (ii) If $S = 0.\dot{7}$, then $10S = 7 + S$. Hence $S = \dfrac{7}{9}$; (iii) $S = \dfrac{7}{33}$; (iv) $S = \dfrac{67}{333}$.
26. Surely every repeating decimal is a fraction. But how to prove it? It's fairly obvious how to do each ***particular*** example. But how to do it in general?
27. $S = 2 + \frac{2}{3} + \frac{2}{9} + \cdots$, then $3S = 6 + 2 + \frac{2}{3} + \frac{2}{9} + \cdots = 6 + S$. Hence $S = 3$.
28. (i) $\dfrac{4}{3}$; (ii) 3.
29. (i) $\dfrac{5}{2}$; (ii) $\dfrac{7}{2}$.
30. If $S = a + ar + ar^2 + \cdots$, then $rS = ar + ar^2 + \cdots$. Hence $S - rS = a$. This give $S = \frac{a}{1-r}$. Obviously $r \neq 1$ because dividing by zero is the most heinous of mathematical crimes. Anyway, if $r = 1$, then $rS = S$ and so subtracting rS from S is fairly useless. Clearly if $r = 1$, then S is infinite.
    There's something screwy too if $r > 1$. What's going on here? So $S = \frac{a}{1-r}$ provided $r < 1$.

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54. There are a_{n-1} such numbers which start with a 1. There are $2a_{n-2}$ such numbers that start with a 2 (they actually start with 21, 23). There are $2a_{n-2}$ such numbers which start with a 3. Hence $a_n = a_{n-1} + 4a_{n-2}$.
55. There are a_{n-1} which start with B and another a_{n-1} which start with C. There are $2a_{n-2}$ which start with A (they start AB or AC). Hence $a_n = 2a_{n-1} + 2a_{n-2}$.

    Clearly $a_1 = 3$ and $a_2 = 8$.

    Using the theorem, $x^2 - 2x - 2 = 0$ gives $\alpha = 1 + \sqrt{3}$, $\beta = 1 - \sqrt{3}$. Then $K = \frac{1}{6}(3 + 2\sqrt{3})$, $L = \frac{1}{6}(3 - 2\sqrt{3})$. Hence $a_n = \frac{1}{6}[(3 + 2\sqrt{3})(1 + \sqrt{3})^n + (3 - 2\sqrt{3})(1 - \sqrt{3})^n]$.
56. Let's try Step 3 in the induction, where $\alpha \neq \beta$. So we may assume that $a_k = K\alpha^k + L\beta^k$. Then
    $a_{k+1} = Aa_k + Ba_{k-1} = A(K\alpha^k + L\beta^k) + B(K\alpha^{k-1} + L\beta^{k-1})$
    $= (A\alpha + \beta)K\alpha^{k-1} + (A\beta + B)\beta^{k-1}$.

    So all we've got to do is to show that $A\alpha + B = \alpha^2$ and $A\beta + B = \beta^2$ and we are done. How to do that? (See after Exercise 19 (revisited).) So, in the meantime, let's try Step 3, when $\alpha = \beta$.

    We can assume that $a_k = (K + kL)\alpha^k$. Hence
    $a_{k+1} = Aa_k + Ba_{k-1} = A[K + kL]\alpha^k + B[K + (k-1)L]\alpha^{k-1}$
    $= \{[A\alpha + K]K + [Ak\alpha + B(k-1)]L\}\alpha^{k-1}$.

    We now have to show that $A\alpha + K = \alpha^2$ and $Ak\alpha + B(k-1) = (k+1)\alpha^2$. At least the former requirement is the same as in the $\alpha \neq \beta$ case!
57. Using the Theorem we get $a_n = \frac{1}{2\sqrt{2}}[(1 + \sqrt{2})^n - (1 - \sqrt{2})^n]$.

    The Binomial Theorem expansion for $(1 + \sqrt{2})^n$ is $\sum_{r=0}^{n} {}^nC_r(\sqrt{2})^r$.
    Hence $a_n = \frac{1}{2\sqrt{2}}\left[\sum_{r=0}^{n} {}^nC_r(\sqrt{2}^r) - \sum_{r=0}^{n} {}^nC_r(-1)^r(\sqrt{2}^r)\right]$.

    If r is even, the terms ${}^nC_r(\sqrt{2})^r$ and ${}^nC_r(-1)^r(\sqrt{2})^r$ cancel. If r is odd, they give $2{}^nC_r(\sqrt{2})^r$. So

$$\begin{aligned} a_n &= \frac{2}{2\sqrt{2}}\left[{}^nC_1\sqrt{2} + {}^nC_3(\sqrt{2})^3 + {}^nC_5(\sqrt{2})^5 + \cdots\right] \\ &= {}^nC_1 + 2{}^nC_3 + 4{}^nC_5 + 8{}^nC_7 + \cdots \\ &= n + \textstyle\sum_{p\geq 1} 2^p\,{}^nC_{2p+1} \end{aligned}$$

    where the summation keeps going up to the last odd integer less than or equal to n.

    Let $n = 2^k m$, where m is odd. Then $a_n = 2^k m + \sum_{p\geq 1} 2^p\,{}^nC_{2p+1}$. We therefore have to show that the summation is of the form $2^{k+1}m$ for some integer m. (Why?)

Now $2^p\,{}^nC_{2p+1}$ is an integer and $2p+1$ isn't a factor of 2^p. Hence $\frac{m}{2p+1}\ {}^{n-1}C_{2p}$ is an integer. This means that $2^{p\,n}C_{2p+1}$ is divisible by 2^{k+p}. So $a_n = n + \sum_{p\geq 1} 2^p\ {}^nC_{2p+1} = 2^k m + 2^{k+1}M$, for some integer M. The result now follows.

60. (revisited) (i) This is done in Example 6;
(ii) Since
$(1-2x-3x^2)^{-1} = (1-3x)^{-1}(1+x)^{-1} = \frac{3}{4}(1-3x)^{-1} + \frac{1}{4}(1+x)^{-1}$,
we have
$(4x-3x^2)(1-2x-3x^2)^{-1} = \sum_{n=1}^{\infty} \frac{1}{4}[(-1)^{n-1}7 + 3^{n+1}]x^n$;
(iii) $(2x-6x^2)\left[\frac{5/6}{1-5x} + \frac{1/6}{1+x}\right] = \sum_{n=1}^{\infty} \frac{1}{6}[4\times 5^{n-1} + 8(-1)^{n-1}]x^n$;
(iv) $x^2(1-x-x^2)^{-1} = xF(x) = \frac{x}{(\alpha-\beta)}\left[\sum_{n=1}^{\infty}(\alpha^n-\beta^n)\,x^n\right]$. (See Exercise 64 for full details.)

64. (revisited) But $1-x-x^2 = (1-\alpha x)(1-\beta x) = 1-(\alpha+\beta)x+\alpha\beta x$. Hence $\alpha\beta = -1$ and $\alpha+\beta = 1$. So $\alpha - \frac{1}{\alpha} = 1$. This gives $\alpha^2-\alpha-1 = 0$. Therefore $\alpha = \frac{1\pm\sqrt{5}}{2}$ and $\beta = \frac{1\pm\sqrt{5}}{2}$.
Without loss of generality, choose $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.
Hence $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, which is what we got in Exercise 51. (Would we get the same result if we had chosen $\alpha = \frac{1-\sqrt{5}}{2}$ and $\beta = \frac{1+\sqrt{5}}{2}$? Why?)

66. (revisited) Try $x^3 = Ax^2 + Bx + C$.

67. (revisited) (a) The crucial step is
$a_{i+j+1} = a_{i+j} + a_{i+j-1} = (a_i a_{j-1} + a_{i+1}a_j) + (a_i a_{j-2} + a_{i+1}a_j)$.
Alternatively: (a) Use the fact that $a_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$.
(b) $x^n - y^n = (x-y)(x^{n-1}y + \cdots + y^{n-1})$, so let $x = \alpha^k$ and $y = \beta^k$.
(c) Can this be done this way?

56. (revisited again) We want to show that $-B = \alpha^2$. Well, remember that $x^2 - Ax - B = 0$ and that $(x-\alpha)^2 = 0$. Hence $x^2 - 2\alpha x + \alpha^2 = 0$. Since this last quadratic is precisely the same as $x^2 - Ax - B = 0$, $A = 2\alpha$ but, more importantly, $-B = \alpha^2$.

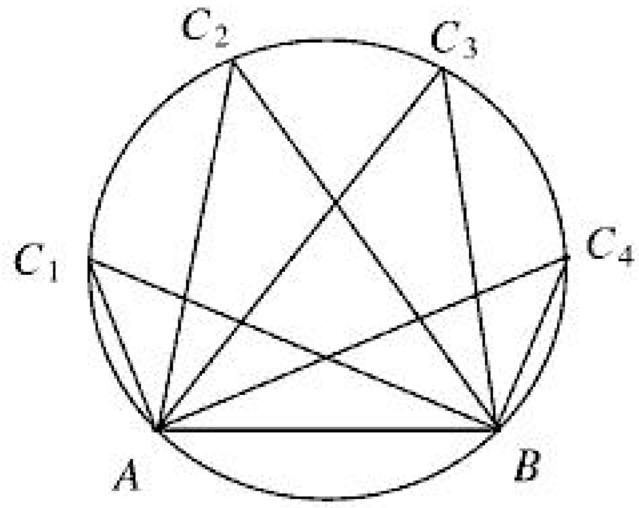

Figure 2.1.

Try drawing some circles, electronically or otherwise. Take any chord AB and see if all the angles ACB are the same, where C is any point on a segment of the circle.

If this is true, of course, there's got to be a reason for it. How could we possibly show that all those angles are the same? What is constant about the circle? Certainly AB is. What if we looked at the isosceles triangle with base AB and vertex C on the circle? What about looking at the triangle AOB, where O is the circumcentre of the triangle? (See Figure 2.2.)

How does that help?

Well, ΔAOC is an isosceles triangle as $AO = OC$ (since they are radii), so $\angle OAC = \angle OCA$. Hmm. But $\angle DOA = \angle OAC + \angle OCA$ (because $\angle DOA$ is an external angle opposite the two internal angles $\angle OAC$ and $\angle AOC$). So $\angle DOA = 2\angle OCA$. But that means that $\angle DOB = 2\angle OCB$ too. So $\angle AOB = 2\angle ACB$.

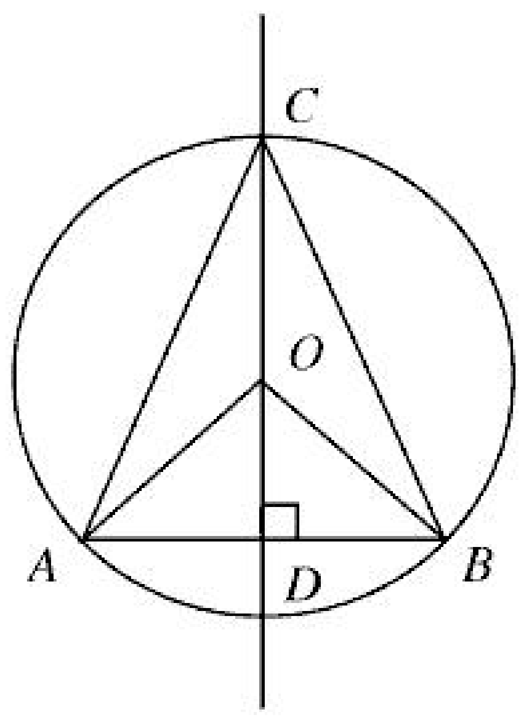

Figure 2.2.

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Exercises

21. Draw the two tangents from the point D external to a circle, centre O. Suppose the tangents touch the circle at E and F. What is the relation between the lengths of the line segments DE and DF? What is the relation between $\angle ODE$ and $\angle ODF$?
22. Show how to find the incentre of ΔABC.
Construct the incircle of ΔABC using ruler and compass.
23. In Exercise 17 we found an equation linking the area of the ΔABC and R, the circumradius. Can you find an equation which links the area of ΔABC and r, the inradius?
24. What is the size of the inradius for the triangle with sides $3, 4, 5$?
25. Is R ever equal to $\frac{5}{2}r$? Is R ever equal to $2r$?

In Exercise 22 we needed to look at the bisectors of the angles of a triangle. Why not look at the bisectors of the ***sides*** of a triangle? A line from a vertex of a triangle to the midpoint of the opposite side, is called a ***median***. In Figure 2.5, AL, BM, CN are all medians, so $AN = NB, BL = LC, CM = MA$. Figure 2.5 rather seems to suggest that the three medians of a triangle are concurrent.

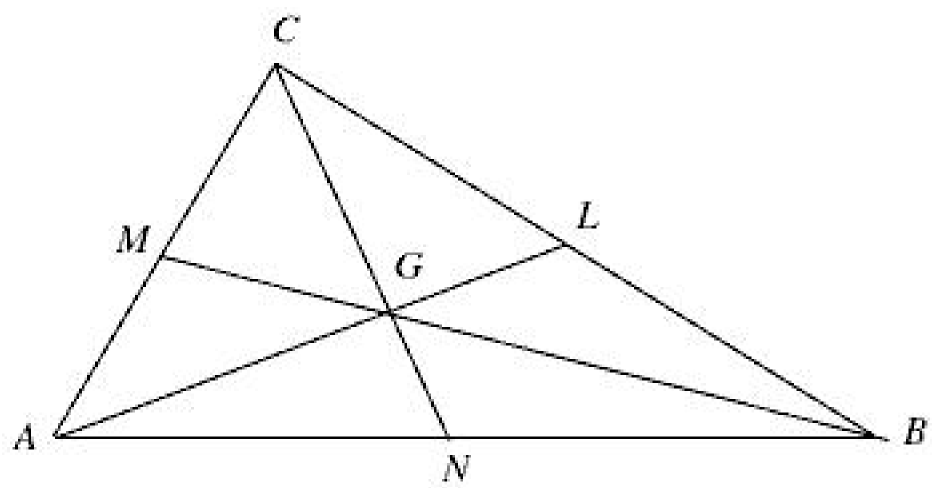

Figure 2.5.

To see whether this is true or not look at Figure 2.6.

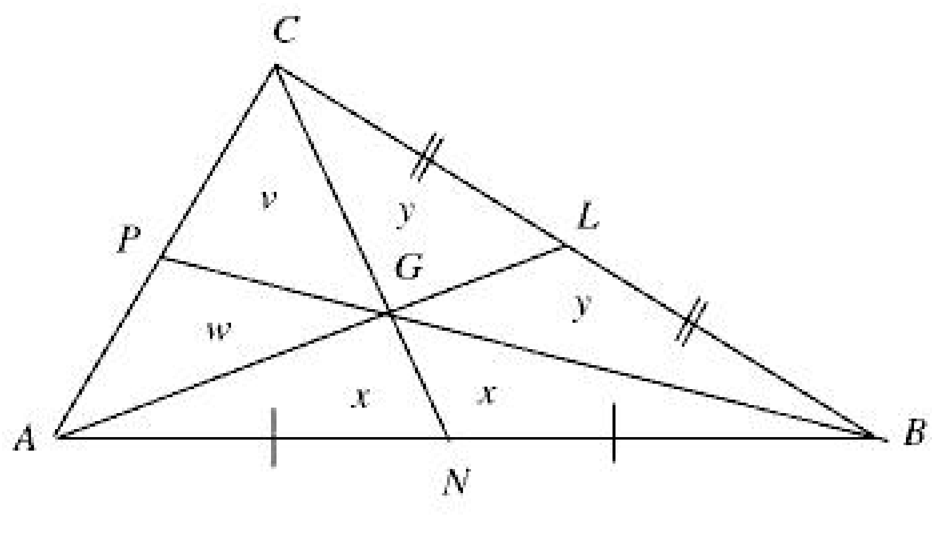

Figure 2.6.

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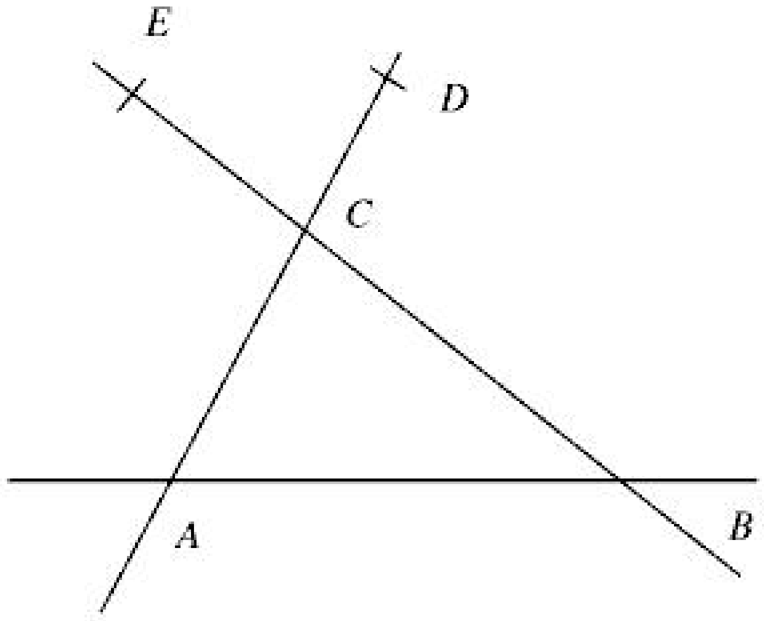


Figure 2.9.

I might as well think of $\angle ACE$ and $\angle BCD$ as the only worthwhile ***external*** angles of the ΔABC at C.

Continuing in this vein, I can see that I_bI_c bisects the external angle at A and I_cI_a bisects the external angle at B.

Exercises

39. Let AB, AC meet the excircle with centre I_a at X, Y, respectively. Show that $AX + AY = a + b + c$ (where a, b, c represent the usual sidelengths in ΔABC).
40. Show that $\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}$, where r is the radius of the incircle of ΔABC and the other "rs" are the radii of the three excircles.
41. Draw the triangle with vertices I_a, I_b, I_c.
    (a) Show that A, B, C are the feet of the altitudes drawn from I_a, I_b, I_c, respectively to the opposite sides of $\Delta I_aI_bI_c$.
    (b) Show that the orthocentre (see Exercise 32) of $\Delta I_aI_bI_c$ is the incentre of ΔABC.
42. If ΔABC is an obtuse angled triangle, do all the results of the last exercise hold?

2.4. The 6-Point Circle?

In this section I want to explore another circle related to a triangle. I'll start this by looking at what I can get from an equilateral triangle. There is a diagram in Figure 2.10 that will help me focus my attention. I'll investigate the circle with centre O, that passes through the feet of the medians, L, M and N, on BC, CA, and AB, respectively.

As for side lengths, $CO = 5$ because it is the radius of the circle. Now Δ's ABC and ACT are similar because of their angles. So $\frac{CT}{BC} = \frac{AC}{AB}$. Hence $CT = (16 \times 12)/20 = 48/5$.

Applying the Cosine Rule in ΔCTO we find that $OT = 5$ and T, the foot of the altitude from C to AB is on the circle. So $S = T$!

Exercises

49. Check the details of the Cosine Rule to make sure that $OT = 5$.
50. Does what we have done above apply to ***every*** right angled triangle ABC? Does the circle through the midpoints of the sides of ***any*** right angled triangle pass through the vertex, C, with the right angle and through the foot of the perpendicular from C to AB?
51. Is there anything interesting about U and V, the internal points where AL and BM, respectively, cut the circle?

Finally in this section, let's look at the isosceles triangle that has side lengths 2, 2 and $2\sqrt{3}$, see Figure 2.14. When this has been done, I'll look over what has been achieved by investigating these three examples.

Because of the symmetry of isosceles triangles, AL is both the median and the altitude from A to BC. In this triangle, the altitudes CT and BS meet AL outside the triangle at the orthocentre G. Where is the centre of the circle through L, M and N, and do S and T lie on this circle?

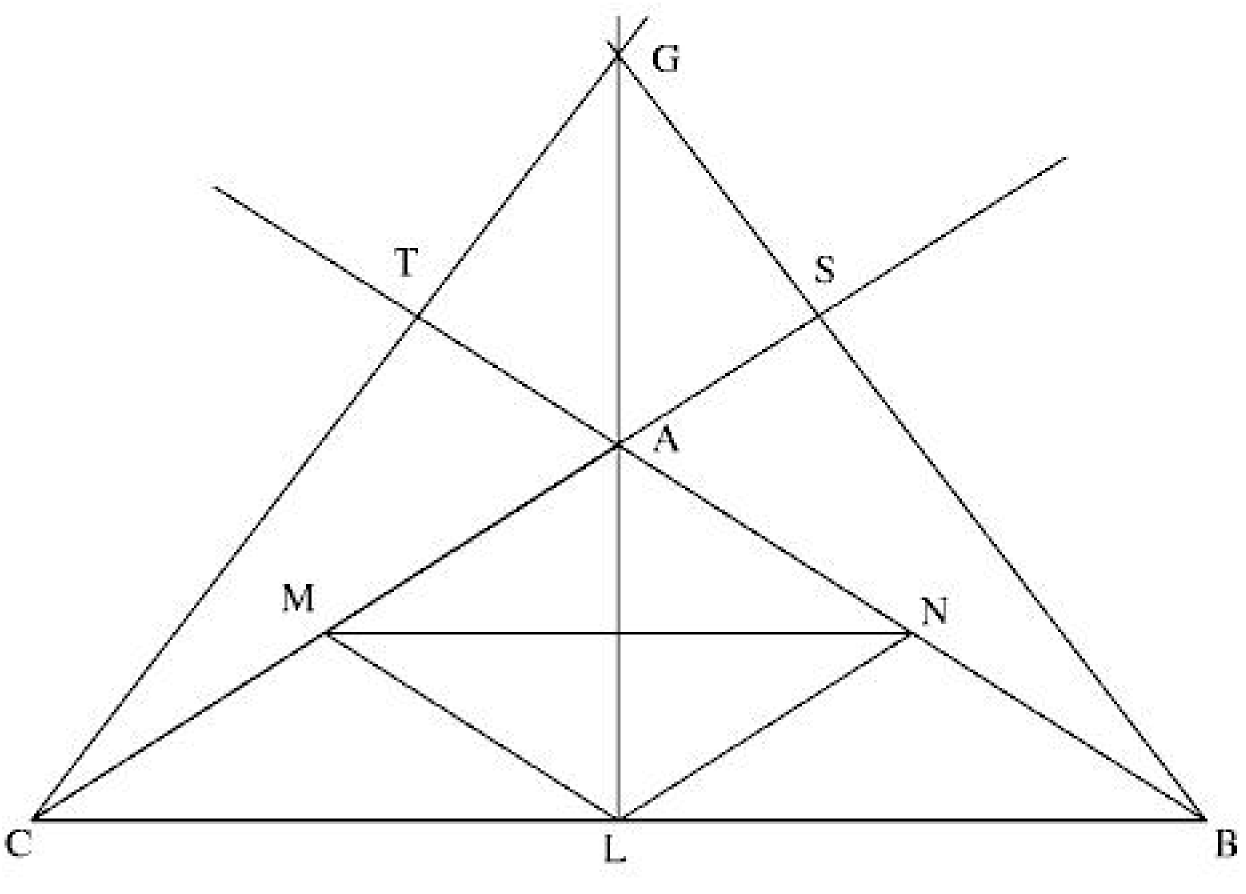


Figure 2.14.

Exercises

66. ABC is a triangle, the bisector of angle A meets the circumcircle of ΔABC in A_1. Points B_1 and C_1 are defined similarly. Let AA_1 meet the line that bisects the two external angles at B and C, at A_0. Points B_0 and C_0 are defined similarly.

$$\text{Prove that area of } \Delta A_0B_0C_0 = 2 \times \text{ area of hexagon } AC_1BA_1CB_1 \geq 4 \times \text{ area } \Delta ABC.$$

(IMO, 1989; submitted by Australia)

67. Triangle ABC is right angled at A and D is the foot of the altitude from A. The straight line joining the incentres of the triangles ABD, ACD intersects the sides AB, AC at the points K, L, respectively. S and T denote the area of the triangles ABC and AKL, respectively. Show that $S \geq 2T$.

(IMO, 1988; submitted by Greece)

68. The triangle ABC is inscribed in a circle. The interior bisectors of the angles A, B and C meet the circle again at A', B' and C', respectively. Prove that the area of triangle $A'B'C'$ is greater than or equal to the area of triangle ABC.

(Submitted by Canada in 1988)

69. Let ABC be an acute angled triangle. Three lines L_A, L_B and L_C are constructed through the vertices, A, B and C, respectively, according to the following prescription. Let H be the foot of the altitude drawn from the vertex A to the side BC; let S_A be the circle with diameter AH; let S_A meet the sides AB and AC at M and N, respectively, where M and N are distinct from A; then L_A is the line through A perpendicular to MN. The lines L_B and L_C are constructed similarly. Prove that L_A, L_B and L_C are concurrent.

(Submitted by Iceland in 1988)

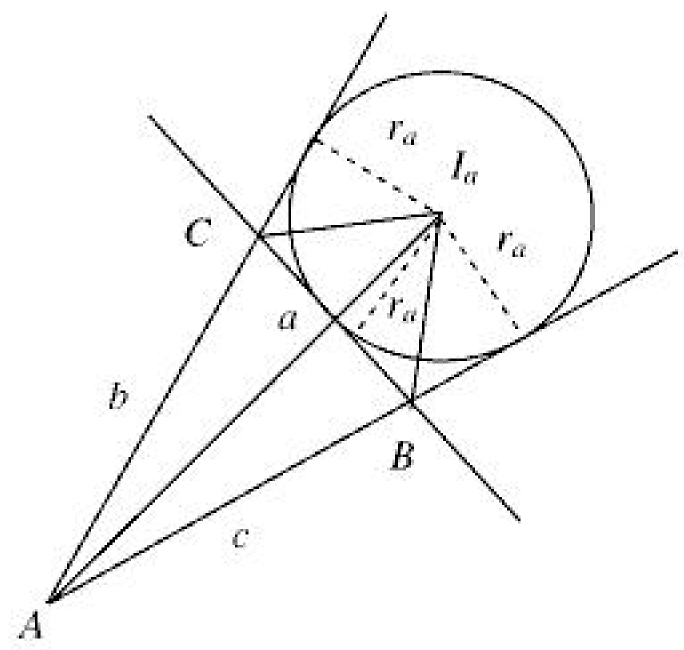
70. The triangle ABC has a right angle at C. The point P is located on segment AC such that triangles PBA and PBC have congruent inscribed circles. Express the length $x = PC$ in terms of $a = BC, b = CA$ and $c = AB$.

(Submitted by USA in 1988)

71. Vertex A of the acute triangle ABC is equidistant from the circumcentre O and the orthocentre H. Determine all possible values for the measure of angle A.

(Submitted by USA in 1989)

34. Compare angles.
35. (a) Not much of a triangle.
    (b) Yes.
36. The tangents will meet at X and Y. So it's angle bisecting time again.
37. How many would you like?
38. (a) Take a triangle and extend the sides to infinity.
    (b) How many unique centres can you find?
    (c) This must be the rest. But where could the centres be?
39. Let Z be the point where the excircle with centre I_a touches the tangent BC.
40. First get a relation between Δ and r_a similar to that for Δ and r, viz., $\Delta = \frac{1}{2}(a+b+c)r$. Then repeat for Δ and r_b, and Δ and r_c.



41. It's all about bisecting angles again.
42. Why not? Draw a diagram and see.
43. Congruent triangles?
44. What do you know about the centroid or maybe just side lengths.
45. Use known side lengths.
46. What can you say about the quadrilateral $CLNM$?
47. What things are special about a triangle?
48. Repeat the arguments of the 12, 16, 20 triangle and see what happens.
49. State the Cosine Rule and substitute what you know here.
50. Go back to Exercise 45.
51. Do they represent anything that might be generalized?
52. Perpendicular bisectors. But of what?
53. Sure are!
54. Ah now, what might we expect? They are on a circle? Put the points in on a diagram and see what it looks like.
55. Just recall what all the definitions are.

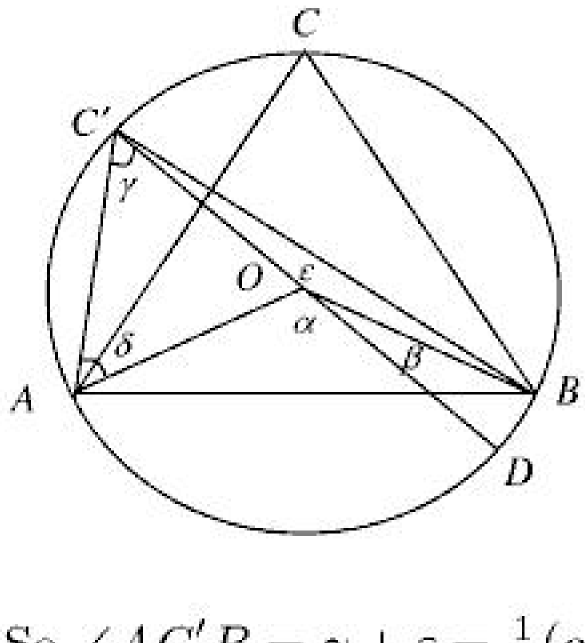
Does this mean that in an acute angled triangle (where all angles are smaller than 90°), the circumcentre is inside the triangle? So for obtuse angled triangles, is the circumcircle ***outside*** the triangle?

I settle this question later on. Can you settle it now? If necessary, revisit this problem after completing Exercise 8.

4. (a) Half the length of the hypotenuse.
   (e) We know that the circumradius is $\frac{\sqrt{3}}{3}\ell$, where ℓ is the length of the side of the equilateral triangle.
5. What is the circumcentre of $\Delta AC'B$? So it's likely that $\angle AC'B = 90°$.
6. If you're measuring the angle, then $\angle AC'B$ probably won't always be 60°. However it ought to be close enough to conjecture that $\angle AC'B = \angle ACB = 60°$. So how can this be proved?

   Complete the diagram below, where $\alpha = \angle AOB, \beta = \angle DOB$, $\gamma = \angle OC'A, \delta = \angle OAC'$ and $\varepsilon = \angle OC'B$. Now $\angle OAB = \angle OBA = 30°$, since we know that the circumradii OA and OB, bisect $\angle$'s CAB, CBA, respectively. But $\gamma = \delta$ since $\Delta AOC'$ is isosceles. Further $\alpha = \gamma + \delta$ (external angle equals the sum of the two interior and opposite angles).

   Hence $\gamma = \delta = \frac{1}{2}\alpha$.



   Similarly $\varepsilon = \frac{1}{2}\beta$. So $\angle AC'B = \gamma + \varepsilon = \frac{1}{2}(\alpha + \beta)$.

   Finally in $\Delta AOB, \angle AOB = 180° - \angle OAB - \angle OBA = 120°$. Hence $\angle AC'B = \frac{1}{2}(\alpha + \beta) = \frac{1}{2}120° = 60°$.

   (For another solution to this exercise, see Exercise 6 revisited on page 45.)
7. You probably know $\angle AC'B$ always equals $\angle ACB$. But can you prove it? If you can, you can check out your answer in the next piece of text.
8. This is proved in part of Exercise 6, where $\alpha + \beta$ happens to be 120°. However, the fact that $\gamma + \varepsilon = \frac{1}{2}(\alpha + \beta)$ doesn't rely on the fact that $\alpha + \beta = 120°$ but only on $\alpha + \beta < 180°$. Now show the same proof holds for $\alpha + \beta \geq 180°$.

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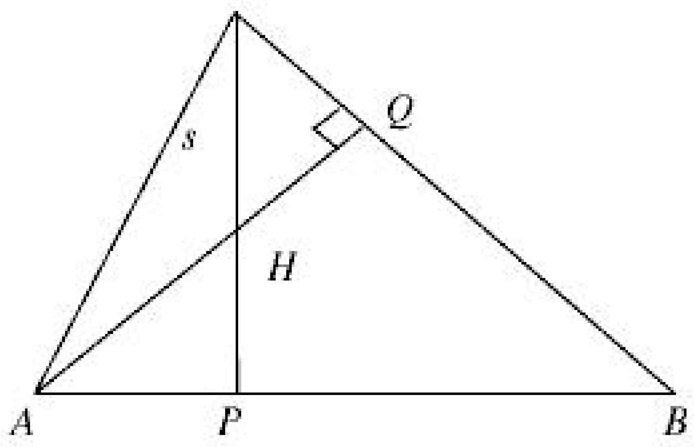
of *First Step*). Where these bisectors meet is the incentre, I. Drop the perpendicular from I to any side of the triangle to obtain the inradius.

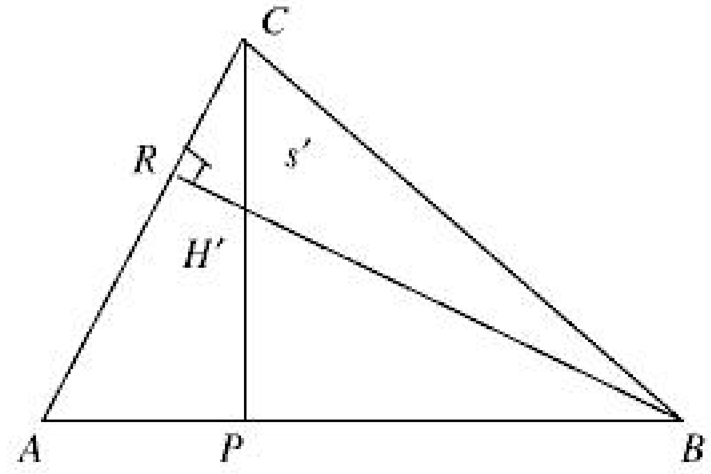
23. The area of $\Delta AIB = \frac{1}{2}cr$. Similarly, area $\Delta BIC = \frac{1}{2}ar$ and area $\Delta CIA = \frac{1}{2}br$. Hence $\Delta = \frac{1}{2}r(a+b+c)$, where Δ is the area of ΔABC.
24. $\Delta = 6, a = 3, b = 4, c = 5$, so $r = 1$. (Is there a way of doing this directly?)
25. $R = \frac{5}{2}r$ for the 3, 4, 5 triangle. After all, we showed that $r = 1$ and R is just half the length of the hypotenuse.

    For equilateral triangles, $R = 2r$. (This can be checked out directly from the formulae for R and r which involve Δ.)

    For what right angled triangles is $R = 2r$?
26. Δ's CML, CAB are similar because they both have the angle C and corresponding sides are in the same ratio $(1:2)$. Hence $\angle CML = \angle CAB$. So $ML \| AB$.
27. Using the arguments of the last Exercise, Δ's MNA, LNB are congruent to ΔCML and similar to ΔCAB. ΔNLM is congruent to ΔCML by the *SSS* argument. Hence we have four congruent triangles in ΔABC.
28. Since the diagonals of a parallelogram bisect each other, AL must bisect MN ($AMLN$ is a parallelogram since $ML \| AN$ and $ML = AN$).
29. Δ's CXL, CYB are similar because they both have the angle C and $\angle CLX = \angle CBY$. But $\mathrm{CL}:\mathrm{CB} = 1:2$, so $\mathrm{CX}:\mathrm{CY} = 1:2$.
30. $v + w = 2x$ and $v = w$. Hence $v = x$. So $v = w = x = y$ and all the triangles have the same area.
31. Δ's AGB, LGB have the same altitude. Any difference in their areas will be caused by the difference in their bases, AG and LG. Now $\frac{\text{area } \Delta AGB}{\text{area } \Delta LGB} = \frac{2x}{y} = 2$, since $x = y$. Hence $\frac{AG}{LG} = 2$.

    Similarly, it can be proved that $\frac{BG}{GM} = \frac{CG}{GN} = 2$. Hence the centroid of a triangle is two thirds of the way along a median (from a vertex).
32. Let $\alpha = 90^\circ - A$, $\beta = 90^\circ - B$, $\gamma = 90^\circ - C$, $s = CH$, and $s' = CH'$. In $\Delta AQC, \angle CAQ = 90^\circ - C = \gamma$ and $CQ = AC \sin\gamma$. From ΔCPB, $\angle BCP = 90^\circ - B = \beta$.





47. Clearly C lies on the circle because it is a vertex of the rectangle $MNLC$. But does C have any relevance apart from the fact that it is at the right angle of ΔABC?

   And what can we say about (i) the unnamed point where the circle and ΔABC intersect, call this point S; (ii) the point (not L) where the circle and AL intersect, call this point U; and (iii) the point (not M) where the circle and BM intersect, call this point V?

48. See what you can do with this. We'll return to it later.
49. Now $OT^2 = OC^2 + CT^2 - 2 \times OC \times CT \cos(90^\circ - 2\beta) = 5^2 + (48/5)^2 - 2 \times 5 \times 48/5 \times \cos(90^\circ - 2\beta)$. So we had better worry about that cosine term. Now $\cos(90^\circ - 2\beta) = \sin 2\beta = 2 \sin\beta \cos\beta$. And we can get these values form the original triangle: $\sin\beta = 12/20$ and $\cos\beta = 16/20$. This then gives us

$$\begin{aligned} OT^2 &= 5^2 + (48/5)^2 - 2 \times 5 \times 48/5 \times 2 \times 12/20 \times 16/20 \\ &= 5^2 + (48/5)^2 - 2 \times 5 \times 48/5 \times 2 \times 3/5 \times 4/5 \\ &= 5^2 + (48/5)^2 - 2 \times 48/5 \times 2 \times 3 \times 4/5 \\ &= 5^2 + (48/5)^2 - (48/5)^2 = 5^2. \end{aligned}$$

   (Note the value of not simplifying the expression until the last moment.)

   Miraculously $OT = 5$ and so S is the foot of the altitude from C to AB. How about that?

   (But there is a better way to do this problem.)

50. This follows on from the last Solution. It is just a matter of changing the lengths of the sides to variables a, b, c, say.
51. I can't find anything but if you do, please let me know.
52. O has to be on the intersection of the perpendicular bisectors of LM, MN and NL. AL is the perpendicular bisector of MN, so O has to be somewhere on AL.

   Let F be the midpoint of LN. Then $LF = FN = 1/2$. Since $\angle ALN = 60^\circ$, ΔOFL has sides $1/2$, 1, and $\sqrt{3}/2$. So O is on AL and 1 unit up from BC so the radius of the circle is 1. But A is 1 from BC on AL, so $O = A$.

   Since ΔBCT is a right angled triangle with right angle at T, and $\angle TBC = \angle ABC = 30^\circ$, $\angle BCT = 60^\circ$. Hence $\angle ACT = 30^\circ$ and $\angle CAT = 60^\circ$. But $AC = 2$, so $AT = 1$. Hence T does lie on the circle centre O. By symmetry, so does S.

53. It's not too difficult now to show that $AG = 2$. So there is another point on the circle, R, where $TR = 1 (= RG)$.

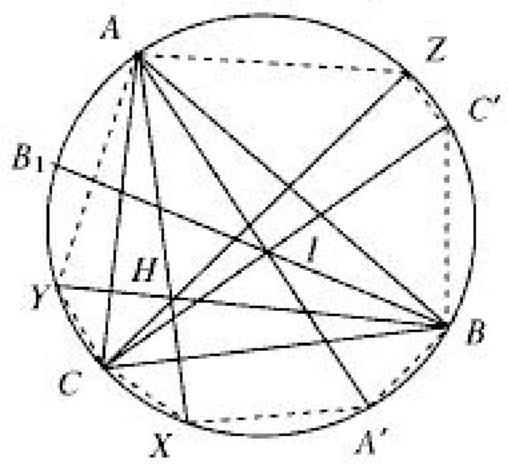
   In this case it's worth noting that $CT = TG = \sqrt{3}, T$ and S, as well as R are half way between a vertex of the triangle and the orthocentre.

Considering (1) we get

$$\text{area } \Delta IA_1B = \text{area } \Delta A_0A_1B.$$

Repeating this argument for the six triangles that have a vertex at I and adding them together gives the required equality.

To prove the inequality, draw the three altitudes in triangle ABC, intersecting in H. Let X be the mirror image of H on the side BC, Y its mirror image on AC and Z on AB. Then X, Y, Z are points on the circumcircle of ΔABC. (This is because $\angle CXB = \angle CHB = 180° - A$ from the cyclic quadrilateral formed by the altitudes.)



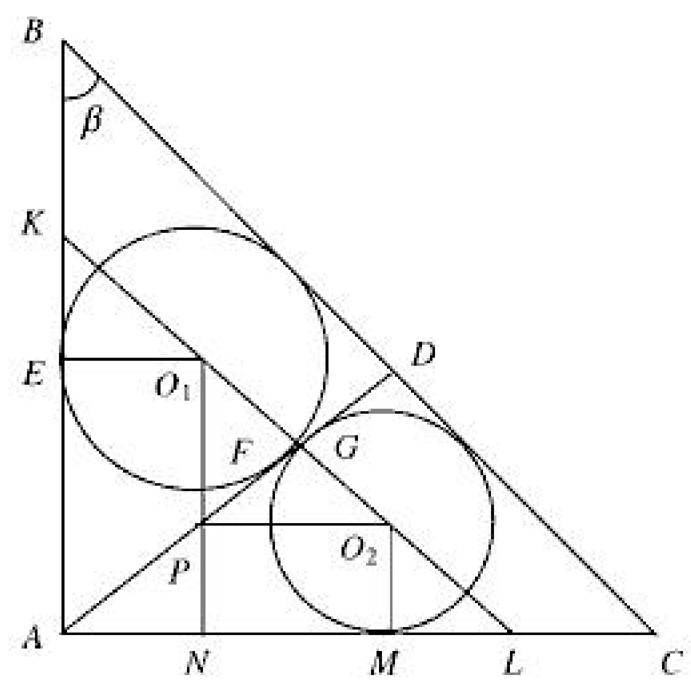
Because A_1 is the midpoint of arc BC, area $\Delta BA_1C \geq$ area ΔBXC.

So area hexagon $AC_1BA_1CB_1 \geq$ area hexagon $AZBXCY$

$$= 2 \times \text{area } (\Delta BHC + \Delta CHA + \Delta AHB)$$

$$= 2 \times \text{area } \Delta ABC.$$

67. Let $AB = c, AC = b, BC = a$ and $AD = h$, the circle inscribed in ΔABD by C_1 and that inscribed in ΔADC by C_2. Let O_1, O_2 be the centres of C_1 and C_2, respectively, E and F the points of contact of C_1 with AB and AD, respectively, and G and M the points of contact of C_2 with AD and AC, respectively. Let O_1N be perpendicular to AC and O_2P perpendicular to NO_1.



A similar argument shows that $\angle CAB > 90°$ and so ΔABC is obtuse.

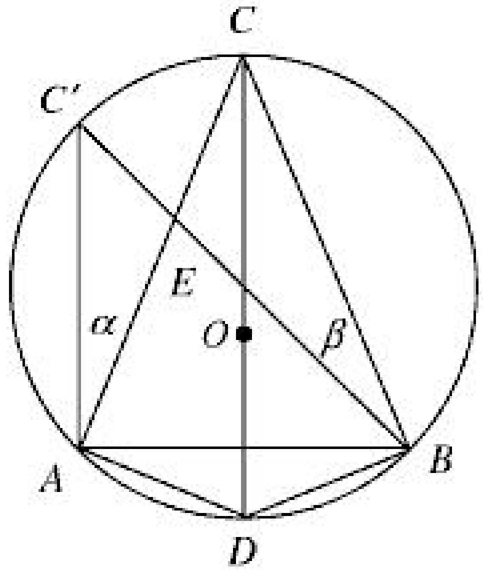
We can now easily prove that an acute angle triangle contains its circumcircle. Suppose the triangle is acute but the circumcircle lies **outside** the triangle. Then by the argument of the last paragraph, the triangle is obtuse. This contradicts the original assumption about the triangle.

So we have a triangle is acute if and only if its circumcentre lies inside the triangle.

In a similar way we can prove that a necessary and sufficient condition for a triangle to be obtuse is that its circumcircle is outside the triangle. What are necessary and sufficient conditions for a triangle to be right angled?

6. (revisited) Let CD be a diameter of the circumcircle of the equilateral triangle ABC.

Since $ADBC'$ is a cyclic quadrilateral, $\angle DAC' + \angle DBC' = 180°$. But $\angle DAC' = \angle DAC + \alpha = 90° + \alpha$. Similarly $\angle DBC' = 90° - \beta$. Hence $90° + \alpha + 90° - \beta = 180°$. So $\alpha = \beta$.



In Δ's $AC'E$, BCE we have $\alpha = \beta$ and $\angle AEC' = \angle BEC$ (vertically opposite angles). Hence $\angle AC'B = \angle ACB$ $(= 60°)$.

(Does this proof generalise to triangles other than equilateral triangles?)

18. (revisited) In the diagram of the solution of Exercise 18, draw CO extended to C' on the circumcircle. The proof that $\frac{b}{\sin B} = 2R$ follows as in Exercise 15 using $\Delta CC'A$. For $\frac{a}{\sin A} = 2R$, use $\Delta CC'B$.

35. (b) (revisited) Clearly there are problems with a $45°, 45°, 90°$ triangle. But this is covered by part (a).

36. If and only if the line XY is perpendicular to the other two lines. (Suppose XY is drawn as in Figure 2.7. Then the circle in region II touches XY **above** the midpoint of XY while the circle in region VI touches XY **below** the midpoint.)

Exercises

8. Invent three more generalisations of the six circles problem.
9. Invent three more extensions of the six circles problem.
10. Let a set of six numbers be ***nice*** if the numbers can be put into the six circles so that the sums of the three numbers on any of the three sides of the equilateral triangle are the same. Describe nice sets. Prove that what you have conjectured is true.
11. In what other problems in this book did you need to (i) experiment; or (ii) conjecture?
    Did you ever find a counter-example?
    Did you ever give up?
    Now that you know about generalisations and extensions, are there any problems that you have solved and you can now generalise or extend?

3.5. More on Research Methods

Perhaps not surprisingly there is more to mathematical research than the skeleton of Figure 3.2. In fact the same is true for solving problems. In this section I want to say something about heuristics. Now ***heuristics*** are ideas that are worth trying. They won't always get you to a solution of your problem but they may well help. They are worth a try anyway. So let me look at a few of the more commonly used ones. Here is a list *of heuristics*, in no particular order, to start off with.

(i) think of a similar problem
(ii) guess and check
(iii) draw a diagram
(iv) try some algebra
(v) act it out
(vi) make a systematic list
(vii) look for patterns
(viii) consider a simpler problem
(ix) consider an extreme case.

(i) *Think of a similar problem*: This, of course, is the first thing that you think about when you are doing an exam. "What sort of question is this?" "Is it something like the one I was revising?" And in trying to solve a problem, you should always start by asking "Have I seen anything like this before?" It is just possible that some problem that you have done recently is similar in some way to the one you are trying to solve. There may be

Now the green frogs want to get to where the brown frogs are and vice versa. Frogs can move by going to an adjacent lily pad that is empty or by jumping over a frog onto an empty lily pad on the other side of the jumped frog. One final rule, frogs can't go back. If they start moving to the left, they have to keep moving in that direction. And the problem is, can the green and brown frogs swap places?

A very good strategy here is to put seven chairs in a row to represent the lily pads, and get two groups of three students to sit on the chairs and ***act the problem out.*** What are the advantages of doing this?

First of all it has all the advantages of a (three-dimensional) diagram. It helps you see what is going on and it helps you remember what you have done. It's also easy to see when you have made a wrong move and to diagnose it was wrong. Second, it helps you to see ideas. Third, it helps you to see patterns and enables you to make obvious generalisations and extensions. What happens if we have four frogs on each side or five or n? What happens if we have three frogs on one side and five on the other? How about m and n frogs on different sides? And fourth, it seems to be more fun this way. And don't underestimate the importance of enjoying what you are doing in maths.

There are several "crossing the river" problems that can be solved in a similar way (if you don't know about them you can look them up on the web by looking for "crossing the river problems"). Take the man who wants to get his fox, goose and some grain across a river but his boat is too small to take more than one at a time. But he can't leave the fox and the goose because the fox will eat the goose and he can't leave the goose and the hay because the goose will eat the hay. How can he get all of his belongings across the river?

But "act it out" is also a good strategy to understand game problems. In games with two players it's often better to have another person there "competing against" you. They will often see things from a completely different perspective and bring ideas to the game that you may not have thought of.

For instance, think about the Nim-like "21 Game". Here there are 21 pieces on the table and each player takes a turn to remove some. Both players have to take 1, or 2 pieces from the table. The winner is the person who takes away the last piece. Does the first or second player win? I'm assuming that both players play optimally, that is, as well as is possible.

Even if you don't have another person on hand to tackle such a game, it is probably better to play this game against yourself with actual pieces.

In "Understanding the Problem", he emphasised the fundamental need to know what the problem was asking before trying to make any further progress. Of course it is necessary to solve the problem in hand rather than one that you might have thought up that appears to be the one in hand. This is clearly the case in examinations or tests. So before you get too far down the problem solving track you need to satisfy yourself that you have understood the problem and know what it is asking.

This first step involves reading the problem several times, not just at the start but throughout the process so as to not stray from the track. It also involves the experimentation that I suggested in Section 3.1. It might go as far as getting a first conjecture.

Then it is important to have a plan of where you hope to go. You might first plan to get to understand the problem by doing a few simple examples. That might be followed by finding a conjecture. But then, how are you going to prove the conjecture is true? What mathematical ideas might help you? Algebra? Can I set up a Proof by Contradiction or even by Mathematical Induction[a]? When have these types of proof worked before? Will they work here? So how do they fit into my plan?

At first it might be useful to actually say what that plan is but as you get more experienced you will have a better idea of what you are trying to do and probably won't need to write it down.

After having given some thought as to what your plan might look like, then you are set to carry the plan out. This won't necessarily go smoothly. Maybe, say, you had thought to use a Proof by Contradiction, and you can't set it up as you thought you might. Then you just have to go back and plan again. It may be too that you don't know some piece of mathematics that is fundamental to a solution. That is a good time to look up some books or the internet or ask someone who might know.

But if your plan works and you solve your problem, then it is time to look back. First it is worth checking that the problem you have just solved is the problem you were actually asked to solve. If not, it's back to the drawing board. If it is, then you might check that you have solved it correctly. Did you make any errors? If not, perhaps there is a better way to solve it. (It's amazing the number of times that a mathematician has produced a theorem and when writing it up to publish it in a journal, finds a better proof.) After that, you can think about where this problem that you have solved might lead. Are there any other interesting things that might come out of this?

[a]See Chapter 6 of *First Step*.

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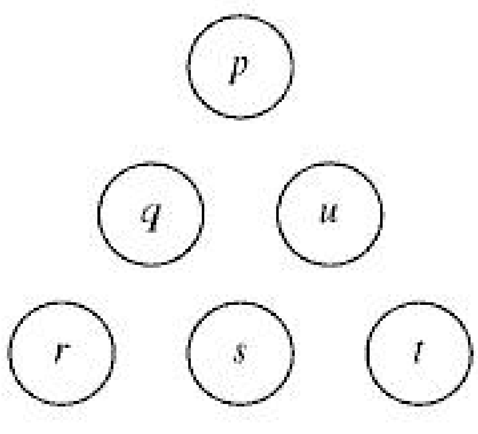
3. I don't know what you did but you might have tried systematically to put the numbers in all possible ways. You probably used lots of diagrams with six circles in them. Was there any way that you found to be systematic? What about using algebra in some way? Did you have any other ideas on how to tackle the problem?
4. Yes there are only four snarks, but I'll hold back on showing you why for the moment.
5. Since the problem places importance on the sums of the numbers on each side, this property might be worth considering. Why are the only side sums 9, 10, 11 and 12?
6. What is the smallest side sum for a side containing 6?
7. Nine can only (see "systematic list" in Section 3.4) be written as the sum of three different numbers from 1 to 6 as follows:

$$6 + 2 + 1; \quad 5 + 3 + 1; \quad 4 + 3 + 2.$$

These sums can then only go on one of the three sides of the triangle. The corner (vertex) numbers are the ones that are repeated (1, 2, 3). These sums lead to a unique answer.

Show how to do this for the other side sums.
8. (i) Take any six numbers (not necessarily whole numbers) and use them instead of $\{1, 2, 3, 4, 5, 6\}$ in the original problem; (ii) put three circles along the edge of any regular polygon; (iii) what else did you get?
9. (i) Put four circles on each side of an equilateral triangle to give another eight circle problem; (ii) try putting circles along the edge of a pyramid; (iii) put nine circles in a 3 by 3 array. Is it possible the three circled numbers in each of the rows and columns add up to the same value?
10. A set of numbers is nice, if and only it is of the form $\{a, b, c, a + d, b + d, c + d\}$.

First of all it is easy to see that if we put the numbers a, b and c in the corners of the triangle and $a + d, b + d, c + d$ in the middle of the sides opposite a, b and c, respectively, we get a snark. So $\{a, b, c, a + d, b + d, c + d\}$ is nice.



Chapter 4

Number Theory 2

There is a chapter dealing with Number Theory (Number Theory 1) in Chapter 4 of *First Step* that will serve as an introduction to this one. It will especially be useful to make sure that you are familiar with the congruence/modulo notation $a \equiv b \pmod{n}$.

Mainly here I deal with Euler's Theorem and Wilson's Theorem. Euler's Theorem is a generalisation of Fermat's Little Theorem (see Chapter 4 of *First Step* and Section 4.1 below). Wilson's Theorem is somehow related. These results are largely concerned with powers of a number modulo a prime. So on with a little more Number Theory.

4.1. A Problem

In the Number Theory chapter referred to above, I talked about Fermat's Little Theorem. Just to refresh your memory, here it is again.

Fermat's Little Theorem (F.L.T.). *If p is a prime and $1 \leq a < p$, then $a^p \equiv a\ (mod\ p)$.*

I just want to juggle that around a bit. If $a^p \equiv a \pmod{p}$ then, what this really means is, that $a^p = a + kp$ where $k \in \mathbb{Z}$, the set of integers. So

$$kp = a^p - a = a(a^{p-1} - 1).$$

But a is smaller than p, so $(a, p) = 1$. (Remember $(a, b) = c$ means that c is the highest common factor of a and b. So $(a, p) = 1$ means that a and p have no factors in common.) Since a is a factor of the right-hand side it must be a factor of kp. Now a and p have no common factors so a must be a factor of k. So $\frac{k}{a} = k'$ is an integer.

We know that $\phi(3) = 2$ and $\phi(13) = 12$. The following 24 numbers are relatively prime to 39: $1, 2, 4, 5, 7, 8, 10, 11, 14, 16, 17, 19, 20, 22, 23, 25, 28, 29, 31, 32, 34, 35, 37, 38$.

But let's look at those 24 numbers to see if we can see ***why*** there are 2×12 of them and to see ***how*** they are connected to $\phi(3)$ and $\phi(13)$.

First of all, any number can be written in the form $3m + r$, where $0 \le r \le 2$ (see Chapter 4 of *First Step*). Now $3m + r$ is relatively prime to 3 if and only if r is relatively prime to 3. This happens when $r = 1$ or 2.

What are the numbers less than 39 which are relatively prime to 3? They are the numbers $\{3m + 1\colon 0 \le m \le 12\} \cup \{3m + 2\colon 0 \le m \le 12\}$.

The numbers of the form $3m + 1$ are $1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37$. Of these only 13 is not relatively prime to 13. This gives 12 numbers less than 39 and relatively prime to 39. Note that $12 = \phi(13)$.

Similarly, in the set $\{3m + 2\colon 0 \le m \le 12\}$, only 26 is not relatively prime to 13 and therefore to 39. Hence we have another $12 = \phi(13)$ numbers which are relatively prime to 39.

Hence $\phi(39) = 24 = 2 \times 12 = \phi(3)\phi(13)$. Note that there are $\phi(3)$ sets $\{3m+1\}$, $\{3m+2\}$ relatively prime to 3 and in each of these sets there are $\phi(13)$ numbers relatively prime to 13.

Exercises

10. Use the arguments of Example 1 to show that
    (i) $\phi(15) = \phi(3)\phi(5)$; (ii) $\phi(20) = \phi(4)\phi(5)$;
    (iii) $\phi(60) = \phi(4)\phi(15)$; (iv) $\phi(210) = \phi(14)\phi(15)$.
11. Can the arguments of Example 1 be used to show that $\phi(24) = \phi(4)\phi(6)$?
12. (a) Prove that if p and q are primes with $q = ps + 1$ for some s, then $\phi(pq) = \phi(p)\phi(q)$.
    (b) Prove that if p and q are primes, then $\phi(pq) = \phi(p)\phi(q)$.
13. (a) For what $m, n \le 25$ is it true that $\phi(mn) = \phi(m)\phi(n)$?
    (b) In general, for what m, n is it true that $\phi(mn) = \phi(m)\phi(n)$?
14. (a) If p, q, r are prime numbers, find an expression for $\phi(pqr)$ in terms of p, q, r.
    (b) Repeat (a) with (i) $\phi(p^2q)$; (ii) $\phi(p^2q^3r)$; (iii) $\phi(p^\alpha)$.

So can we find a formula for $\phi(n)$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$?

Theorem 1. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where the p_i are distinct primes and the α_i are natural numbers then $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_r})$.*

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