Le Hai Chau
Le Hai Khoi

Vol. 5 | Mathematical Olympiad Series

Selected Problems of the Vietnamese Mathematical Olympiad (1962–2009)

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Foreword

The International Mathematical Olympiad (IMO) - an annual international mathematical competition primarily for high school students - has a history of more than half a century and is the oldest of all international science Olympiads. Having attracted the participation of more than 100 countries and territories, not only has the IMO been instrumental in promoting interest in mathematics among high school students, it has also been successful in the identification of mathematical talent. For example, since 1990, at least one of the Fields Medalists in every batch had participated in an IMO earlier and won a medal.

Vietnam began participating in the IMO in 1974 and has consistently done very well. Up to 2009, the Vietnamese team had already won 44 gold, 82 silver and 57 bronze medals at the IMO - an impressive performance that places it among the top ten countries in the cumulative medal tally. This is probably related to the fact that there is a well-established tradition in mathematical competitions in Vietnam - the Vietnamese Mathematical Olympiad (VMO) started in 1962. The VMO and the Vietnamese IMO teams have also helped to identify many outstanding mathematical talents from Vietnam, including Ngo Bao Chau, whose proof of the Fundamental Lemma in Langland's program made it to the list of Top Ten Scientific Discoveries of 2009 of Time magazine.

It is therefore good news that selected problems from the VMO are now made more readily available through this book.

One of the authors - Le Hai Chau - is a highly respected mathematics educator in Vietnam with extensive experience in the development of mathematical talent. He started working in the Ministry of Education of Vietnam in 1955, and has been involved in the VMO and IMO as a setter of problems and the leader of the Vietnamese team to several IMO. He has published many mathematics books, including textbooks for secondary and high school students, and has played an important role in the development of mathematical education in Vietnam. For his contributions, he was bestowed the nation's highest honour of "People's Teacher" by the government of Vietnam in 2008. Personally, I have witnessed first-hand the kind of great respect expressed by teachers and mathematicians in Vietnam whenever the name "Le Hai Chau" is mentioned.

Le Hai Chau's passion for mathematics is no doubt one of the main reasons that his son Le Hai Khoi - the other author of this book - also fell in love with mathematics. He has been a member of a Vietnamese IMO team, and chose to be a mathematician for his career. With a PhD in mathematics from Russia, Le Hai Khoi has worked in both Vietnam and Singapore, where he is based currently. Like his father, Le Hai Khoi also has a keen interest in discovering and nurturing mathematical talent.

I congratulate the authors for the successful completion of this book. I trust that many young minds will find it interesting, stimulating and enriching.

San Ling

Singapore, Feb 2010

Preface

In 1962, the first Vietnamese Mathematical Olympiad (VMO) was held in Hanoi. Since then the Vietnam Ministry of Education has, jointly with the Vietnamese Mathematical Society (VMS), organized annually (except in 1973) this competition. The best winners of VMO then participated in the Selection Test to form a team to represent Vietnam at the International Mathematical Olympiad (IMO), in which Vietnam took part for the first time in 1974. After 33 participations (except in 1977 and 1981) Vietnamese students have won almost 200 medals, among them over 40 gold.

This books contains about 230 selected problems from more than 45 competitions. These problems are divided into five sections following the classification of the IMO: Algebra, Analysis, Number Theory, Combinatorics, and Geometry.

It should be noted that the problems presented in this book are of average level of difficulty. In the future we hope to prepare another book containing more difficult problems of the VMO, as well as some problems of the Selection Tests for forming the Vietnamese teams for the IMO.

We also note that from 1990 the VMO has been divided into two echelons. The first echelon is for students of the big cities and provinces, while the second echelon is for students of the smaller cities and highland regions. Problems for the second echelon are denoted with the letter B.

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hard work to prepare this book for publication. We also thank Mr. Wong Fook Sung, Albert of Temasek Polytechnic, for his copyediting of the book.

Last but not least, we are responsible for any typos, errors,... in the book, and hope to receive the reader's feedback.

The Authors
Hanoi and Singapore, Dec 2009

Contents

Chapter 1

The Gifted Students

On the first school opening day of the Democratic Republic of Vietnam in September 1946, President Ho Chi Minh sent a letter to all students, stating: "Whether or not Vietnam becomes glorious and the Vietnamese nation becomes gloriously paired with other wealthy nations over five continents, will depend mainly on students' effort to study". Later Uncle Ho reminds all the people: "It takes 10 years for trees to grow. It takes 100 years to "cultivate" a person's career".

Uncle Ho's teachings encouraged millions of Vietnamese teachers and students to spend all efforts in "Teaching Well and Studying Well", even through all the years of war against aggressors.

1.1 The Vietnamese Mathematical Olympiad

1. The Vietnamese Mathematical Olympiad was organized by the Ministry of Education for the first time in the academic year 1961-1962. It was for gifted students of the final year of Secondary School (grade 7) and of High School (grade 10), at that period, with the objectives to:

1. discover and train gifted students in mathematics,

2. encourage the "Teaching Well and Studying Well" campaign for mathematics in schools.

Nowadays, the Vietnamese education system includes three levels totaling 12 years:

- Primary School: from grade 1 to grade 5.
- Secondary School: from grade 6 to grade 9.
- High School: from grade 10 to grade 12.

At the end of grade 12, students must take the final graduate exams, and only those who passed the exams are allowed to take the entrance exams to universities and colleges.

The competition is organized annually, via the following stages.

Stage 1. At the beginning of each academic year, all schools classify students, discover and train gifted students in mathematics.

Stage 2. Districts select gifted students in mathematics from the final year of Primary School, and the first year of Secondary and High Schools to form their teams for training in facultative hours (not during the official learning hours), following the program and materials provided by local (provincial) Departments of Education.

Stage 3. Gifted students are selected from city/province level to participate in a mathematical competition (for year-end students of each level). This competition is organized completely by the local city/province (set-up questions, script marking and rewards).

Stage 4. The National Mathematical Olympiad for students of the final grades of Secondary and High Schools is organized by the Ministry of Education. The national jury is formed for this to be in charge of posing questions, marking papers and suggesting prizes. The olympiad is held over two days. Each day students solve three problems in three hours. There are 2 types of awards: Individual prize and Team prize, each consists of First, Second, Third and Honorable prizes.

2. During the first few years, the Ministry of Education assigned the first-named author, Ministry's Inspector for Mathematics, to take charge in organizing the Olympiad, from setting the questions to marking the papers. When the Vietnamese Mathematical Society was established (Jan 1964), the Ministry invited the VMS to join in. Professor Le Van Thiem, the first Director of Vietnam Institute of Mathematics, was nominated as a chair of the jury. Since then, the VMO is organized annually by the Ministry of Education, even during years of fierce war.

For the reader to imagine the content of the national competition, the full questions of the first 1962 and the latest 2009 Olympiad are presented here.

The first Mathematical Olympiad, 1962

Problem 1. Prove that

$$\frac{1}{\frac{1}{a}+\frac{1}{b}}+\frac{1}{\frac{1}{c}+\frac{1}{d}}\le\frac{1}{\frac{1}{a+c}+\frac{1}{b+d}},$$

for all positive real numbers a, b, c, d.

Problem 2. Find the first derivative at $x = -1$ of the function

$$f(x) = (1+x)\sqrt{2+x^2}\,\sqrt[3]{3+x^3}.$$

Problem 3. Let $ABCD$ be a tetrahedron, A', B' the orthogonal projections of A, B on the opposite faces, respectively. Prove that AA' and BB' intersect each other if and only if $AB \perp CD$.

Do AA' and BB' intersect each other if $AC = AD = BC = BD$?

Problem 4. Given a pyramid $SABCD$ such that the base $ABCD$ is a square with the center O, and $SO \perp ABCD$. The height SO is h and the angle between SAB and $ABCD$ is α. The plane passing through the edge AB is perpendicular to the opposite face SCD. Find the volume of the prescribed pyramid. Investigate the obtained formula.

Problem 5. Solve the equation

$$\sin^6 x + \cos^6 x = \frac{1}{4}.$$

The Mathematical Olympiad, 2009

Problem 1. Solve the system

$$\begin{cases} \frac{1}{\sqrt{1+2x^2}}+\frac{1}{\sqrt{1+2y^2}}=\frac{1}{\sqrt{1+2xy}}, \\ \sqrt{x(1-2x)}+\sqrt{y(1-2y)}=\frac{2}{9}. \end{cases}$$

Problem 2. Let a sequence (x_n) be defined by

$$x_1 = \frac{1}{2},\ x_n = \frac{\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}}{2},\ n \ge 2.$$

Prove that a sequence (y_n) defined by $y_n = \sum_{i=1}^{n}\frac{1}{x_i^2}$ converges and find its limit.

Problem 3. In the plane given two fixed points $A \neq B$ and a variable point C satisfying condition $\widehat{ACB} = \alpha$ ($\alpha \in (0°, 180°)$ is constant). The in-circle of the triangle ABC centered at I is tangent to AB, BC and CA at D, E and F respectively. The lines AI, BI intersect the line EF at M, N respectively.

1) Prove that a line segment MN has a constant length.

2) Prove that the circum-circle of a triangle DMN always passes through some fixed point.

Problem 4. Three real numbers a, b, c satisfy the following conditions: for each positive integer n, the sum $a^n + b^n + c^n$ is an integer. Prove that there exist three integers p, q, r such that a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$.

Problem 5. Let n be a positive integer. Denote by T the set of the first $2n$ positive integers. How many subsets S are there such that $S \subset T$ and there are no $a, b \in S$ with $|a - b| \in \{1, n\}$? (Remark: the empty set $\emptyset$ is considered as a subset that has such a property).

3. The Ministry of Education regularly provided documents guiding the teaching and training of gifted students, as well as organizing seminars and workshops on discovering and training students. Below are some experiences from those events.

How to study mathematics wisely?

Intelligence is a synthesis of man's intellectual abilities such as observation, memory, imagination, and particularly the thinking ability, whose most fundamental characteristic is the ability of independent and creative thinking.

A student who studies mathematics intelligently manifests himself in the following ways:

- Grasping fundamental knowledge accurately, systematically, understanding, remembering and wisely applying the mathematical knowledge in his real life activities,

- Capable of analyzing and synthesizing, i.e., discovering and solving a problem or an issue by himself, as well as having critical thinking skills,

- Capable of creative thinking, i.e., not limiting to old methods.

However, one should not exaggerate the importance of intelligence. “An average aptitude is sufficient for a man to grasp mathematics in secondary school if he has good guidance and good books" (A. Kolmogorov, Russian Academician).

3.1. **In Arithmetic**.

The following puzzles can help sharpen intellectual abilities:

a) *A fruit basket contains 5 oranges. Distribute these 5 oranges to 5 children so that each of them has 1 orange, yet there still remains 1 orange in the basket.*

The solution is to give 4 oranges to 4 children, and the fruit basket with 1 orange to be given to the fifth child.

b) *Some people come together for a dinner. There are family ties among them: 2 of them are fathers, 2 are sons, 2 are uncles, 2 are nephews, 1 is grandfather, 1 is elder brother, 1 is young brother. So there are 12 people! True or false? How are they related?*

In fact, A is the father of B's, and C is the father of D's and the nephew of A's; A is C's uncle.

3.2. **In Algebra**.

a) After learning the identity $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, the student can easily solve the following problem.

Prove the relation $x^3 + y^3 + z^3 = 3xyz$ *if* $x + y + z = 0$.

Clearly, from $x + y + z = 0$ it follows that $z = -(x+y)$. Substituting this into the left-hand side of the relation to be proved, we get

$$\begin{aligned} x^3 + y^3 + z^3 &= x^3 + y^3 - (x+y)^3 \\ &= x^3 + y^3 - x^3 - 3x^2y - 3xy^2 - y^3 \\ &= -3xy(x+y) \\ &= 3xyz. \end{aligned}$$

Consequently, when facing the problem of *factorizing the expression* $(x-y)^3 + (y-z)^3 + (z-x)^3$, we can easily arrive at the result $3(x-y)(y-z)(z-x)$.

Similarly, we can prove that $(x-y)^5 + (y-z)^5 + (z-x)^5$ is divisible by $5(x-y)(y-z)(z-x)$ for all x, y, z that are distinct integers.

Another example. *It is seen that* $3^2 + 4^2 = 5^2, 5^2 + 12^2 = 13^2, 7^2 + 24^2 = 25^2, 9^2 + 40^2 = 41^2$. *State a general rule suggested by these examples and prove it.*

A possible relation is

$$(2n+1)^2 + [2n(n+1)]^2 = [2n(n+1)+1]^2, \ n \geq 1.$$

Similarly, with the following problem: *It is seen that* $1^2 = \frac{1\cdot2\cdot3}{6}, 1^2+3^2 = \frac{3\cdot4\cdot5}{6}, 1^2+3^2+5^2 = \frac{5\cdot6\cdot7}{6}$. *State a general law suggested by these examples and prove it.*

We can find the general law

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},\ n \geq 1.$$

From this, we find again another formula:

$$2^2 + 4^2 + \cdots + (2n)^2 = 2^2(1^2 + 2^2 + \cdots + n^2) = \frac{2n(n+1)(2n+1)}{3}.$$

b) Let's now consider the following problem (VMO, 1996).

Solve the system.

$$\begin{cases} \sqrt{3x}\left(1 + \frac{1}{x+y}\right) = 2, \\ \sqrt{7y}\left(1 - \frac{1}{x+y}\right) = 4\sqrt{2}. \end{cases}$$

A brief solution is as follows.

With the condition $x, y > 0$ we have the equivalent system

$$\begin{cases} \frac{1}{x+y} = \frac{1}{\sqrt{3x}} - \frac{2\sqrt{2}}{\sqrt{7y}}, \\ 1 = \frac{1}{\sqrt{3x}} + \frac{2\sqrt{2}}{\sqrt{7y}}. \end{cases}$$

Multiplying these two equations, we get $7y^2 - 38xy - 24x^2 = 0$, or $(y - 6x)(7y+4) = 0$, which gives $y = 6x$ (as $7y + 4 > 0$). Hence,

$$x = \frac{11+4\sqrt{7}}{21}, y = \frac{22+8\sqrt{7}}{7}.$$

Another solution is as follows.

Put $u = \sqrt{x}, v = \sqrt{y}$, then the system becomes

$$\begin{cases} u\left(1 + \frac{1}{u^2+v^2}\right) = \frac{2}{\sqrt{3}}, \\ v\left(1 - \frac{1}{u^2+v^2}\right) = \frac{4\sqrt{2}}{\sqrt{7}}. \end{cases}$$

But u^2+v^2 is the square of absolute value of the complex number $z = u+iv$. Thus

$$u + iv + \frac{u - iv}{u^2+v^2} = \frac{2}{\sqrt{3}} + i\frac{4\sqrt{2}}{\sqrt{7}}. \tag{1}$$

Note that

$$\frac{u - iv}{u^2 + v^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z},$$

so the equation (1) becomes

$$z + \frac{1}{z} = \frac{2}{\sqrt{3}} + i\frac{4\sqrt{2}}{\sqrt{7}},$$

or

$$z^2 - \left(\frac{2}{\sqrt{3}} + i\frac{4\sqrt{2}}{\sqrt{7}}\right) z + 1 = 0.$$

The solutions are

$$z = \left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right) + i\left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right),$$

with corresponding (+) and (−) signs. This shows that the initial system has the following solutions

$$x = \left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right)^2, \; y = \left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right)^2.$$

The new studying style for mathematics for students is: to overcome difficulties, to think independently, to learn and practise, as well as to have a study plan.

3.3. **In Geometry**.

When students have finished the chapter about *quadrilaterals*, they can ask and answer by themselves the following questions.

- *If we join the midpoints of the adjacent sides of a quadrilateral, a parallelogram, a rectangle, a lozenge, a square and an isosceles trapezoid, what figures will we obtain?*

- *Which quadrilateral has the sum of interior angles equal the sum of exterior angles?*

Finally, between memory and intelligence, *it is necessary to memorize in a clever manner.* Specifically:

a) In order to memorize well, one must understand. For example, *to have* $(x + y)^4$ *we must know that it is deduced from* $(x + y)^3(x + y)$.

b) Have a thorough grasp of the relationship between notions of the same kind. For instance, with the relation $\sin^2 \alpha + \cos^2 \alpha = 1$ we can prove that it is wrong to write $\sin^2 \frac{\alpha}{2} + \cos^2 \frac{\alpha}{2} = \frac{1}{2}$ (!).

c) Remember by figures. For example, if we have grasped trigonometric circle concepts, we can easily remember all the formulas to find the roots of basic trigonometric equations.

4. The international experience shows there is no need for a scientist to be old in order to be a wise mathematician. Therefore, we need to pay attention to discover young gifted students and to develop their talents.

Below are examples of 2 gifted students in mathematics in Vietnam:

Case 1.

Twenty years ago, the Ministry of Education was informed by an Education Department of one province that a grade 2 pupil in a village has passed maths level of the final grade of high school. The Ministry of Education assigned the first-named author of this book, the Ministry's Inspector for Mathematics, to visit Quat Dong village, which is 20 km away from Hanoi, to assess this pupil's capability in maths. A lot of curious people from the village gathered at the pupil's house. The following is an extract of that interview using the house yard instead of a blackboard.

- Inspector: *Is* $x^2 - 6x + 8$ *a quadratic polynomial? Can you factorize it?*
- Pupil: This is a quadratic polynomial. I can add 1 and subtract 1 from this expression.

- Inspector: *Why did you do it this way?*
- Pupil: For the given expression, if I add 1, I would have $x^2 - 6x + 9 = (x-3)^2$, and subtracted 1, the expression is unchanged. Then I could have $(x-3)^2 - 1$, and using the rule "a difference of two squares is a product of its sum and difference", it becomes

$$(x-3)^2 - 1 = (x-3+1)(x-3-1) = (x-2)(x-4).$$

(The pupil explained very clearly, which showed that he understood well about what could be done).

- Inspector: *Do you think there is a better way to solve this problem?*
- Pupil: I can solve the quadratic equation $x^2 - 6x + 8 = 0$ following the general one $ax^2 + bx + c = 0$. Here $b = 6 = 2b'$, so I use the discriminant $\Delta' = b'^2 - ac$ and a formula $x_{1,2} = \frac{-b' \pm \sqrt{\Delta'}}{a}$ to get the answer.
(He has found out correctly two roots 2 and 4).

- Inspector: *During the computation of the roots, you might be wrong. Is there any way to verify the answer?*

- Pupil: I can use the Viète formula, as the sum of two roots is $-\frac{b}{2a}$ and the product of two roots is $\frac{c}{a}$, etc.

Then the interview changed focus to Geometry.

- Inspector: *Can you solve the following geometrical problem: Consider a triangle ABC with the side BC fixed, and where the vertex A is allowed to vary. Find the locus of the centroid G of a triangle ABC.*

The pupil drew a figure on the house yard, thinking for a while and commented as follows:

- Pupil: Did you intentionally give a wrong problem?
- Inspector: *Yes, how did you know?*
- Pupil: In the problem a vertex A varies, but we have to know how it varies to arrive at the answer.

- Inspector: *How do you think would A vary?*
- Pupil: If A is on the line parallel to the base BC, then the locus is obviously another line parallel to BC and away from BC the one-third of the distance of the line of A to BC, because ...

- Inspector: *Good! But if A varies on a circle centered at the midpoint I of BC and of a given radius, then what will be the locus of G?*
- Pupil: Oh, then the locus of G is a circle concentric to the given one, and of the radius of one-third of this given circle.

That pupil of grade 2 was Pham Ngoc Anh. The Ministry of Education trained him in an independent way, allowed him to "skip" some grades, and sent him to a university overseas. He was the youngest student who entered to university and also was the youngest PhD in mathematics of Vietnam. Dr. Pham Ngoc Anh is now working for the Institute of Mathematics, Hungarian Academy of Sciences.

Case 2.

Some years ago in Hanoi there was a rumor about a five-year-old boy, who could solve high school mathematical problems. One day the Vietnam TV representative sent a correspondent to the first-named author of this book and invited him to be one of the juries for a direct interview of that boy at the TV broadcasting studio.

Below is an extract of that interview. The first-named author of this book, the MOE's inspector, gave him ten short mathematical questions. Note that the boy, at that time did not know how to read.

- Inspector: *Of how many seconds does consist one day of 24 hours?*
- Pupil: (computing in his head) $86,400$ seconds.

(In fact he did a multiplication $3,600 \times 24$).

- Inspector: *In a championship there are 20 football teams, each of which has to play 19 matches with other 19 teams. How many matches are there in total?*

- Pupil: (thinking and computing in his head) 190.

(In fact he did the following computation $19 + 18 + \cdots + 2 + 1 = (19 + 1) + (18 + 2) + \cdots + (11 + 9) + 10 = 190$).

However, for the following question: *"One snail is at the bottom of the water-well in* $10m$ *depth. During the day time snail climbed up* $3m$*, but by night snail climbed down* $2m$*. After how many days did the snail go over the water-well?"* The boy said that the answer was 10 days, as each day the snail could climbed up $3m - 2m = 1m$.

In fact the boy was incorrect, as it required only 7 days (!). Anyway, he had strong capabilities and very good memory. However, as he could not read, his mathematical reasoning was essentially limited.

1.2 High Schools for the Gifted in Maths

1. The Ministry of Education has strong emphasis on the discovery and developmental activities for mathematically gifted students. So besides organizing annually national Olympiad for school students to select talents, the Ministry decided to establish classes for gifted students in mathematics, starting from Hanoi, in two universities (nowadays, VNU-Hanoi and Hanoi University of Education), and after that extending to other cities.

These classes for gifted students in mathematics allow us to identify quickly and develop centrally good students in maths nation-wide. In the provinces, these classes are formed by the local Department of Education and usually assigned to some top local schools to manage it, while classes from universities are enrolled and developed by universities themselves.

2. There are some experiences about these classes for the gifted in mathematics.

a) The number of students selected depends on the qualification of available gifted students, and it emphasizes on the quality of maths teachers who satisfy two conditions:

- having good capability in maths,
- having rich experience in teaching.

b) Always care about students' ethics, because gifted students tend to be too proud; students should be encouraged to be humble always. There is a Vietnamese saying: "Be humble to go further".

c) To follow strictly the policy: "Develop maths talents on the basis of total education". Avoiding the situation that maths gifted students only study maths, ignoring other subjects. Always remember total education "knowledge, ethic, health and beauty".

d) To transfer the spirit to students: "Daring but careful, confident but humble, aggressive but truthful".

e) So as not to miss talents, it is necessary to select every year during the training period by organizing a supplementary contest to add new good students, and at the same time to pass those unqualified to normal classes. Also to give prizes to students with good achievements in study and ethic development is a good way to encourage students.

g) Teaching maths must light up the fire in the students' mind. We must know how to teach wisely and help students to study intelligently.

For example, in teaching surds equations, when we have to deal with the equation

$$\frac{1-ax}{1+ax}\sqrt{\frac{1+bx}{1-bx}}=1$$

that leads to

$$\frac{(1-ax)^2}{(1+ax)^2}=\frac{1-bx}{1+bx},$$

one should pay attention to the fact that if we do a cross-multiplication and expand the obtained expression, then we could have very complicated computations. Instead, it would be better to use the property of ratios

$$\frac{a}{b}=\frac{c}{d} \implies \frac{a+b}{a-b}=\frac{c+d}{c-d}$$

to get much a simpler equation

$$\frac{-4ax}{2(1+a^2-x^2)}=\frac{-2bx}{2}.$$

The last equation is quite easy to solve.

Another example, for a trigonometric problem, is to prove the following relation in a triangle

$$\sin^2 A+\sin^2 B+\sin^2 C=2(\cos A\cos B\cos C+1). \qquad (2)$$

We can pose a question to the students whether there is a similar relation, like

$$\cos^2 A+\cos^2 B+\cos^2 C=2(\sin A\sin B\sin C+1). \qquad (3)$$

From this we can show students that since (2)+(3) $= 2(\cos A\cos B\cos C + \sin A\sin B\sin C) = -1$, it is impossible if a triangle ABC is acute.

After that we can ask students if such a relation does exist in the case of obtuse triangle, etc.

It would be nice if we could give students some so-called "generalized" exercises with several questions to encourage students to think deeper. For example, when teaching tetrahedra, we can pose the following problem.

Given a tetrahedron $ABCD$ whose trihedral angle at the vertex A is a right angle.

1. Prove that if $AH \perp (BCD)$, then H is the orthocenter of triangle BCD.
2. Prove that if H is the orthocenter of triangle BCD, then $AH \perp (BCD)$.
3. Prove that if $AH \perp (BCD)$, then $\frac{1}{AH^2} = \frac{1}{AB^2} + \frac{1}{AC^2} + \frac{1}{AD^2}$.
4. Let α, β, γ be angles between AH and AB, AC, AD, respectively. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. How does this relation vary when H is an arbitrary point in the triangle BCD?
5. Let x, y, z be dihedral angles of sides CD, DB, DC respectively. Prove that $\cos^2 x + \cos^2 y + \cos^2 z = 1$.
6. Prove that $\frac{S_{ABC}}{S_{BCD}} = \frac{S^2_{ABC}}{S^2_{BCD}}$.
7. Prove that $S^2_{BCD} = S^2_{ABC} + S^2_{ACD} + S^2_{ADB}$.
8. Prove that for a triangle BCD there hold $a^2\tan B = b^2\tan C = c^2\tan D$, with $AD = a, AB = b, AC = c$.
9. Prove that BCD is an acute triangle.
10. Take points B', C', D' on AB, AC, AD respectively so that $AB\cdot AB' = AC\cdot AC' = AD\cdot AD'$. Let G, H and G', H' be the centroid and orthocenter of triangles BCD and $B'C'D'$ respectively. Prove that the three points A, G, H' and the three points A, G', H are collinear.
11. Find the maximum value of the expression
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos^2\alpha\cos^2\beta\cos^2\gamma.$$

Another example: *For what p do the two quadratic equations $x^2 - px + 1 = 0$ and $x^2 - x + p = 0$ have the same (real) roots?* At the first glance, $p = 1$ is an answer. But for $p = 1$ the equation $x^2 - x + 1 = 0$ has no (real) roots.

1.3 Participating in IMO

1. In the beginning of 1974, while the Vietnam War was still being fought fiercely in the South, the Democratic Republic of Germany invited Vietnam to participate in the 16th International Mathematical Olympiad (IMO). It was the first time our country sent an IMO team of gifted students in mathematics led by the first-named author of this book, Inspector for Maths of the Ministry of Education.

Two days before departure, on the night of June 20, 1974, the team was granted a meeting with Prime Minister Pham Van Dong at the Presidential Palace. The meeting made a very deep impression. The Prime Minister encouraged the students to be "self-confident" and calm. Students promised to do their best for the first challenging trial. The first Vietnamese team comprised five students selected from a contest for gifted students of provinces from the Northern Vietnam and two university-attached classes.

In the afternoon of July 15, 1974 in the Grand House of Berlin at Alexander Square, the Vietnamese team attained the first "glorious feat of arms": 1 gold, 1 silver and 2 bronze medals, and the last student was only short of one point from obtaining the bronze medal.

"The Weekly Post" of Germany, issued on August 28, 1974 wrote:

"People with the loudest applause welcomed a Vietnamese team of five students, participating in the competition for the first time, already winning four medals: one gold, one silver and two bronze. How do you explain this phenomena that high school students of a country experiencing a devastating war, could have such good mathematical knowledge?".

Many German and foreign journalists in Berlin asked us three questions:

a) *Is it true that the U.S. was said to have bombed Vietnam back to the stone age?*

- Yes, it is. But we are not afraid of this.

b) *Why could your students study under such adverse circumstances?*

- During the bombing, our students went down into the tunnel. After the bombing, they climbed out to continue their class. The paper is the ground and the pen is a bamboo stick. So you can write as much as you like.

c) *Why do Vietnamese students study so well?*

- Mathematics does not need a lab, just a clever mind. Vietnamese students are intelligent and that's why they study well.

It is impossible to imagine, from a country devastated by the American B-52 flying stratofortress airplanes, from the evacuated schools, the bombing and the fierce battles, the flickering light of the oil lamps in the night, lacking in everything, how the first IMO student team can be the first, second and third in the world.

Due to the struggle in the country, Vietnamese students must leave the nice schools in the capital and other urban areas and evacuate to remote rural areas, to study in temporary bamboo classrooms, weathered by wind and rain, surrounded by interlaced communication trenches, and by a series of A-shaped bamboo tunnels. Without tables and chairs, many students have to sit on the brick ground.

Each student from kindergarten through university must wear a hat made from the rice-straw to avoid ordnance.

The tunnels are usually dug through the classrooms, under the bamboo tables. When the alarm sounded, students would evacuate into the tunnels to stay in the dungeons underneath the backyards. They heard the airplane roar and whiz, they saw bombshells falling continuously from the sky. During the war, many children who like mathematics, solved mathematical problems on sedge-mats laid on the ground in the tunnels. Lack of papers, pictures were drawn on the ground. Lack of pens, bamboo sticks were used to write on the ground, because they do not need a laboratory, students can learn anywhere, anytime.

During the years of the war, Vietnamese education continues to enhance discovery and foster gifted students in mathematics, although these students have no contact nor the knowledge of the achievements of the world's mathematics. Every year, the competition in mathematics for Secondary and High Schools were organized regularly. There were competitions and grading sessions held in the midst of aggressive battles.

Vietnamese students have a good model in studying mathematics. They share a common feature: dissatisfied with a quick solution, but keen in finding alternative solutions, or to suggest new problems from a given one.

Material hardship and the threat of American ordnance could not kill the dreams of young Vietnamese students.

2. Since the first participation in IMO 1974, Vietnam has participated in 33 other International Mathematical Olympiads and Vietnamese students have obtained, besides gold, silver and bronze medals, three other prizes: the unique special prize at IMO 1979 in Great Britain, a prize for the youngest student and the unique team prize at IMO 1978 in Romania.

It is worth mentioning that the 48-th IMO, which for the first time, was organized by Vietnam in 2007, where the Vietnamese team obtained 3 gold and 3 silver medals. In that Olympiad, Vietnam "mobilized" over 30 former winners of the IMO and VMO working abroad, together with over

30 mathematicians from the Institute of Mathematics and other universities of Vietnam in a coordinated effort as jury to mark the competition papers. This effort of the Local Vietnamese Organizing Committee was highly appreciated by other countries.

Finally, we would like to recall Uncle Ho's saying in his letter of October 15, 1968 to all educators: "Despite difficulties, we still have to try our best to teach well and study well. On the basis of using education to improve cultural and professional life, aimed at solving practical problems of our country, and in future, to record the significant achievements of science and technology".

Photo 1.1:

The first-named author (left) and Prof. Ling San in Singapore, 2008

Photo 1.2:

The second-named author (right)
coordinating papers at the 48-th IMO in Vietnam, 2007

Chapter 2

Basic Notions and Facts

In this chapter the most basic notions and facts in Algebra, Analysis, Number Theory, Combinatorics, Plane and Solid Geometry (from High School Program in Mathematics) are presented.

2.1 Algebra

2.1.1 Important inequalities

1) Mean quantities

Four types of mean are often used:

- The arithmetic mean of n numbers $a_1, a_2, \ldots, a_n$,

$$A(a) = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

- The geometric mean of n nonnegative real numbers,

$$G(a) = \sqrt[n]{a_1 a_2 \cdots a_n}.$$

- The harmonic mean of n positive real numbers,

$$H(a) = \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

- The square mean of n real numbers,

$$S(a) = \frac{\sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}}{n}.$$

We have the following relationships:

$$\begin{aligned} &S(a) \geq A(a) \ \text{ for real numbers } a_1, a_2, \ldots, a_n, \text{ and} \\ &G(a) \geq H(a) \ \text{ for positive real numbers } a_1, a_2, \ldots, a_n. \end{aligned}$$

For each of these, the equality occurs if and only if $a_1 = \cdots = a_n$.

2) Arithmetic-Geometric Mean (or Cauchy) inequality

For nonnegative real numbers $a_1, a_2, \ldots, a_n$

$$A(a) \geq G(a).$$

The equality occurs if and only if all a_i's are equal.

From this it follows that

(i) Positive real numbers with a constant sum have their product maximum if and only if they all are equal.

(ii) Positive real numbers with a constant product have their sum minimum if and only if they all are equal.

3) Cauchy-Schwarz inequality

For any real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$, there always holds

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

The equality occurs if and only if either $a_1 = kb_1, a_2 = kb_2, \ldots, a_n = kb_n$ or $b_1 = ka_1, b_2 = ka_2, \ldots, b_n = ka_n$ for some real number k.

4) Bernoulli inequality

For any $a > -1$ and positive integer n we have

$$(1 + a)^n \geq 1 + na.$$

The equality occurs if and only if either $a = 0$ or $n = 1$.

5) Hölder inequality

For any real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and any positive real numbers p, q with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ there holds

$$|a_1b_1 + a_2b_2 + \cdots + a_nb_n| \leq$$

$$(|a_1|^p + |a_2|^p + \cdots + |a_n|^p)^{1/p} \cdot (|b_1|^q + |b_2|^q + \cdots + |b_n|^q)^{1/q}.$$

The equality occurs if and only if either $a_1 = kb_1, a_2 = kb_2, \ldots, a_n = kb_n$ or $b_1 = ka_1, b_2 = ka_2, \ldots, b_n = ka_n$ for some real number k.

The Cauchy-Schwarz inequality is a special case of this inequality when $p = q = 2$.

2.1.2 Polynomials

1) Definition

A **polynomial of degree** n is a function of the form

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where n is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, \ldots, a_n$ are called the *coefficients* of the polynomial. The number a_0 is the *constant coefficient* or *constant term*. The number a_n, the coefficient of the highest power, is the *leading coefficient*, and the term a_nx^n is the *leading term*.

2) Properties

- The sum of polynomials is a polynomial.
- The product of polynomials is a polynomial.
- The derivative of a polynomial is a polynomial.
- Any primitive or antiderivative of a polynomial is a polynomial.

A number c is called a root of *multiplicity* k of $P(x)$ if there is a polynomial $Q(x)$ such that $Q(c) \neq 0$ and $P(x) = (x - c)^kQ(x)$. If $k = 1$, then c is called a *simple* root of $P(x)$.

3) Polynomials with integer coefficients

Suppose $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ ($a_n \neq 0$) is a polynomial with integer coefficients, and $x = \frac{p}{q}$ is a rational root of $P(x)$. Then p divides a_0 and q divides a_n.

From this it follows that if $a_n = 1$, then any rational root of $P(x)$ must be an integer (and is a divisor of a_0).

2.2 Analysis

2.2.1 Convex and concave functions

1) Definition

A real function defined on (a,b) is said to be *convex* if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}, \ \forall x,y \in (a,b).$$

If the opposite inequality holds, then f is called *concave*.

2) Properties

If $f(x)$ and $g(x)$ are convex functions on (a,b), then so are $h(x) = f(x)+g(x)$ and $M(x) = \max\{f(x), g(x)\}$.

If $f(x)$ and $g(x)$ are convex functions on (a,b) and if $g(x)$ is nondecreasing on (a,b), then $h(x) = g\big(f(x)\big)$ is convex on (a,b).

3) Jensen inequality

If a function $f(x)$ is convex on (a,b) and $\lambda_1, \ldots, \lambda_n$ are nonnegative real numbers with $\lambda_1 + \cdots + \lambda_n = 1$, then

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \le \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n),$$

for all x_i's in (a,b).

If $f(x)$ is concave, the inequality is reversed.

2.2.2 Weierstrass theorem

1) Monotone sequences

A sequence is said to be *monotonic* if it is one of the following: nonincreasing, nondecreasing.

A sequence (x_n) is said to be *bounded* if there are real numbers m, M such that $m \le x_n \le M$ for all n.

2) Necessary condition for convergence

If a sequence converges, then it is bounded.

3) Weierstrass theorem

A bounded monotonic sequence always converges.

2.2.3 Functional equations

Given two functions $f(x), g(x)$ such that the domain of definition of f contains the range of g. The *composition* of f and g is defined by

$$(f \circ g)(x) := f\big(g(x)\big).$$

If $f = g$ we write f^2 instead of $f \circ f$.

The composition of functions has an associative property.

Also

$$f^n(x) := \underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}}(x) = \underbrace{f(f(\ldots f}_{n \text{ times}}(x))), \ n \geq 1.$$

Solving a functional equation means to find an unknown function in the equation.

2.3 Number Theory

2.3.1 Prime Numbers

1) Some divisibility rules

A number is divisible by

• 2 if and only if its last digit is even,

• 3 if and only if the sum of its digits is divisible by 3,

• 4 if and only if its two last digits form a number divisible by 4,

• 5 if and only if its last digit is either 0 or 5,

• 6 if and only if it is divisible by both 2 and 3,

• 7 if and only if taking the last digit, doubling it, and subtracting the result from the rest of the number gives the answer which is divisible by 7 (including 0),

• 8 if and only if its three last digits form a number divisible by 8,

• 9 if and only if its sum of the digits is divisible by 9,

• 10 if and only if it ends with 0,

• 11 if and only if alternately adding and subtracting the digits from left to right, the result (including 0) is divisible by 11,

• 12 if and only if it is divisible by both 3 and 4,

• 13 if and only if deleting the last digit from the number, then subtracting 9 times the deleted digit from the remaining number gives the answer which is is divisible by 13.

2) Prime numbers

• If $p > 1$ is not divisible by any prime number whose square is less than p, then p is a prime number.

• There are infinitely many prime numbers.

• Every positive integer greater than one has a unique prime factorization. The standard form of this decomposition is as follows:

$$n = p_1^{\alpha_1}.p_2^{\alpha_2}\dots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are prime numbers and $\alpha_1, \alpha_2, \dots, \alpha_k$ are positive integers (not necessarily distinct).

3) The greatest common divisor (g.c.d) and least common multiple (l.c.m.)

• The greatest common divisor of two integers a and b is written as $\gcd(a, b)$, or simply as (a, b). A number d is a common divisor of a and b if and only if it is a divisor of (a, b).

• Two numbers a and b are called *co-prime* or *relatively prime* if $(a, b) = 1$.

• The least common multiple, or lowest common multiple, or smallest common multiple of two integers a and b is written as $[a, b]$. A number m is a common multiple of a and b if and only if it is a multiple of $[a, b]$.

• For any two positive integers a and b there always holds: $(a, b)\cdot[a, b] = ab$.

• If n is divisible by both a and b, with $(a, b) = 1$, then n is divisible by ab.

4) Euclidean algorithm

It is possible to find the greatest common divisor of two numbers, without decomposition into prime factors. This is the Euclidean algorithm, also called Euclid's algorithm, the key property of which is the following result: *"If r is the remainder in the division of a by b, that is, $a = bq + r$, then $(a, b) = (b, r)$"*. The algorithm can be described as follows:

$$a = bq_1 + r_1 \Longrightarrow b = r_1 q_2 + r_2 \Longrightarrow \cdots .$$

As a result we obtain a decreasing sequence of positive numbers

$$a > b > r_1 > r_2 > \cdots .$$

Since there are finitely many positive integers less than a, the last nonzero r_k is (a, b).

2.3.2 Modulo operation

1) Definition

Given a positive integer m. If two integers a and b have the same remainder when divided by m (that is, $a-b$ is divisible by m), then we say that a and b are *congruent modulo* m, and write $a \equiv b \pmod{m}$.

2) Properties

1. - $a \equiv a \pmod{m}$
   - $a \equiv b \pmod{m} \Rightarrow b \equiv a \pmod{m}$
   - $a \equiv b \pmod{m}, b \equiv c \pmod{m} \Rightarrow a \equiv c \pmod{m}$.
2. - $a \equiv b \pmod{m}, c \equiv d \pmod{m} \Rightarrow a \pm c \equiv b \pm d \pmod{m}$
   - $a \equiv b \pmod{m}, c \equiv d \pmod{m} \Rightarrow ac \equiv bd \pmod{m}$.
3. - $a \equiv b \pmod{m} \Rightarrow a \pm c \equiv b \pm c \pmod{m}$
   - $a + c \equiv b \pmod{m} \Rightarrow a \equiv b - c \pmod{m}$
   - $a \equiv b \pmod{m} \Rightarrow ac \equiv bc \pmod{m}$
   - $a \equiv b \pmod{m} \Rightarrow a^n \equiv b^n \pmod{m}$.

3) Chinese Remainder Theorem

Let $m_1, \ldots, m_k$ be positive pairwise co-prime integers, $a_1, \ldots, a_k$ integers, such that $(a_1, m_1) = \cdots = (a_k, m_k) = 1$. For any integers $c_1, \ldots, c_k$, the system

$$\begin{cases} a_1 x \equiv c_1 \pmod{m_1} \\ \cdots \quad \cdots \\ a_k x \equiv c_k \pmod{m_k} \end{cases}$$

has a unique solution modulo $m_1 \ldots m_k$.

2.3.3 Fermat and Euler theorems

1) Fermat (Little) Theorem

If p is a prime number, then $n^p \equiv n \pmod{p}$, that is, $n^p - n$ is divisible by p, for all integers $n \geq 1$. In particular, if $(p, n) = 1$ then $n^{p-1} - 1$ is divisible by p.

2) Euler Theorem

Denote by $\varphi(m)$ the number of positive integers less than m and co-prime with m. If $n = p_1^{\alpha_1}.p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is a factorization of n into primes, then

$$\varphi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

There always holds: $n^{\varphi(m)} \equiv 1 \pmod{m}$, for all n with $(m, n) = 1$.

The Fermat Theorem is a special case of Euler Theorem when m is a prime number.

2.3.4 Numeral systems

1) Definition

This is a system for representing numbers of a given set in a consistent manner. For example, 10 is the binary numeral of *two*, the decimal numeral of *ten*, or other numbers in different bases.

2) Decimal system

Use digits $0, 1, \dots, 9$ to represent numbers. So $123 = 1 \cdot 10^2 + 2 \cdot 10 + 3$.

3) Other numeral systems

For a positive integer $g > 1$ any positive integer n can be represented uniquely in a base g as follows:

$$n = \overline{a_k a_{k-1} \dots a_1 a_0}_g = a_k \cdot g^k + a_{k-1} \cdot g^{k-1} + \cdots + a_1 \cdot g + a_0,$$

where $0 < a_k < g, 0 \le a_{k-1}, \dots, a_0 < g$ are integers.

2.4 Combinatorics

2.4.1 Counting

1) Permutations

A permutation of n elements is a linear arrangement of these elements in some order. The number of all distinct permutations of n elements is

$$P_n = n! := 1 \cdot 2 \cdot \cdots \cdot n.$$

Here as a convention, $0! := 1$.

2) Arrangements

An arrangement of n elements taken k at a time is an ordered arrangement of k elements from n given ones. The number of arrangements of n taken k is

$$A_n^k = \frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1),\ 0 \le k \le n.$$

3) Combinations

A combination of n elements taken k at a time is an arrangement of k elements from n given ones. The numbers of combinations of n taken k is

$$C_n^k := \binom{n}{k} = \frac{n!}{k!(n-k)!},\ 0 \le k \le n.$$

The following hold:

- $\binom{n}{k} = \binom{n}{n-k}$.
- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$.

2.4.2 Newton binomial formula

For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ we have

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

2.4.3 Dirichlet (or Pigeonhole) principle

1) Principle

It is impossible to have 7 pigeons in 3 holes so that in each hole there are at most 2 pigeons.

2) Some applications

• From any $n+1$ positive integers we can choose two so that their difference is divisible by n.

• If vertices of a triangle are in a rectangle (including the case they are on its sides), then the triangle's area is at most half of the rectangle's area.

2.4.4 Graph

1) Definitions

A graph is a set of a finite number of points called *vertices* and links connecting some pairs of vertices called *edges*.

Vertices of a graph is usually denoted by $A_1, \ldots, A_n$, while its edges denoted by $u_1, \ldots, u_m$; each edge u connecting two vertices A_i and A_j is denoted by $u = A_iA_j$.

A edge $u = A_iA_j$ is called a *circuit* if $A_i \equiv A_j$. Two or more edges connecting the same pair of vertices are called *multiple* edges. A *single* graph is a graph having neither circuits nor multiple edges.

The degree of a vertex A is the number of edges connecting to A and denoted by $d(A)$.

A sequence of vertices $A_1, \ldots, A_n$ of a single graph is called a *path* if:

1. A_iA_{i+1} $(1 \leq i \leq n-1)$ are edges of the graph, and

2. $A_i \neq A_j$ if $i \neq j$.

The *length* of a path is the number of edges through which it passes: $A_1, A_2, \ldots, A_n$ has length $n-1$. In particular, if A_nA_1 is an edge of the graph, then we have a *cycle*.

A graph is said to be *connected*, if there is a path between any two vertices.

Each non-connected graph is divided into connected subgraphs called *connected component* with property that no vertex is connected with any vertex of other subgraphs.

2) Some problems

1) There are $n \geq 2$ people in a party. Prove that the number of participants knowing odd numbers of participants is even.

If we consider each participant as a vertex of a graph and two familiar people are "connected" by an edge, we can represent the given problem as the follows: "for a single graph of $n \geq 2$ vertices the number of vertices with odd degree is even".

2) There are $2n$ students joining a tour. Each student has the addresses of at least n other students, and we assume that if A has the address of B then B also has the address of A. Prove that all students can inform each other.

We have the following problem: A single graph with $2n$ vertices, each of degree $\geq n$, is a connected graph.

2.5 Geometry

2.5.1 Trigonometric relationship in a triangle and a circle

1) Right triangle

Let ABC be a right triangle with $\angle A = 90°$, h be the length of the altitude from A, and b', c' be the lengths of the perpendicular projections of AB, AC to the hypothenuse BC.

1. The Pythagorean Theorem: $a^2 = b^2 + c^2$.
2. $b^2 = ab',\ c^2 = ac',\ bc = ah$.
3. $h^2 = b'c'$.
4. $\dfrac{1}{h^2} = \dfrac{1}{b^2} + \dfrac{1}{c^2}$.

2) Sine, cosine, tangent and cotangent laws

1. The law of sines:
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}(= 2R),$$
where R is a radius of the circum-circle of the triangle.
2. The law of cosines:
$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = c^2 + a^c - 2ca\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C. \end{cases}$$
3. The law of tangents:
$$\frac{a-b}{a+b} = \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}}.$$
4. The law of cotangents:
$$\cot A = \frac{b^2 + c^2 - a^2}{4S},$$
where S is the area of the triangle.

3) Formulas of triangle area

Let ABC be a triangle, S the area, R, r radii of the circum-circle, the in-circle respectively, and p the semi-perimeter.

1. $S = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$, where h_a, h_b, h_c are the altitudes of the triangle drawn from the vertices A, B, C respectively.
2. $S = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$.
3. $S = \frac{abc}{4R}$.
4. $S = pr$.
5. The Heron's formula: $S = \sqrt{p(p-a)(p-b)(p-c)}$.

4) The power

The power of a point M with respect to a circle centered at O of radius R is defined as

$$\mathcal{P}_M = OM^2 - R^2.$$

This is positive, negative or *zero* if M is outside, inside or on the circle, respectively.

For any line passing through M that intersects a circle at A, B (including $A = B$, when the line is a tangent) there holds:

$$\mathcal{P}_M = \overrightarrow{MA} \cdot \overrightarrow{MB}.$$

2.5.2 Trigonometric formulas

1) Addition

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}.$$

2) Double, triple angles

$$\sin 2a = 2\sin a \cos a,\ \sin 3a = 3\sin a - 4\sin^3 a$$
$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a,$$
$$\cos 3a = 4\cos^3 a - 3\cos a$$
$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a},\ \tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2 a}.$$

3) Sum-to-product, Product-to-sum

1)

$$\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$
$$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$
$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)].$$

2)

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$$
$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$$
$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$$
$$\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}.$$

4) Rationalization

If $t = \tan\frac{a}{2}$, then

$$\sin a = \frac{2t}{1+t^2},\ \cos a = \frac{1-t^2}{1+t^2},\ \tan a = \frac{2t}{1-t^2}\ (t \neq \pm 1).$$

2.5.3 Some important theorems

1) Thales' theorem

If two lines AA' and BB' intersect at a point O $(O \neq A', B')$, then

$$AB//A'B' \text{ if and only if } \frac{\overrightarrow{OA}}{\overrightarrow{OA'}} = \frac{\overrightarrow{OB}}{\overrightarrow{OB'}}.$$

(Here $\dfrac{\overrightarrow{a}}{\overrightarrow{b}}$ denotes the ratio of two nonzero collinear vectors).

2) Menelaus' theorem

Let ABC be a triangle and M, N, P be points on lines BC, CA, AB respectively, distinct from A, B, C. Then points M, N, P are collinear if and only if

$$\frac{\overrightarrow{MB}}{\overrightarrow{MC}} \cdot \frac{\overrightarrow{NC}}{\overrightarrow{NA}} \cdot \frac{\overrightarrow{PA}}{\overrightarrow{PB}} = 1.$$

3) Ceva's theorem

Let ABC be a triangle and M, N, P be points on lines BC, CA, AB respectively, distinct from A, B, C. Then lines AM, BN, CP are congruent if and only if

$$\frac{\overrightarrow{MB}}{\overrightarrow{MC}} \cdot \frac{\overrightarrow{NC}}{\overrightarrow{NA}} \cdot \frac{\overrightarrow{PA}}{\overrightarrow{PB}} = -1.$$

4) Euler formula

Let (O, R) and (I, r) be circum-circle and in-circle of a triangle ABC respectively. Then

$$d^2 = R^2 - 2Rr, \text{ where } d = OI.$$

Consequently, there always holds $R \geq 2r$.

2.5.4 Dihedral and trihedral angles

Two half-planes (A) and (B) passing through the same straight line PQ form a *dihedral angle*. The straight line is called an *edge* of a dihedral angle, the half-planes its faces. The third plane (C) perpendicular to the edge PQ forms, in its intersection with the half-planes (A) and (B), the *linear angle* $\theta \in (0°, 180°)$ of a dihedral angle. This linear angle is a measure of its dihedral angle.

The angle between two (distinct non-parallel) planes is the smallest linear angle φ among four dihedral angles formed by these planes and so $0° < \varphi < 90°$.

If we draw through the point O a set of planes (AOB), (BOC), (COD) etc., which are consequently intersected one with another along the straight lines OB, OC, OD etc. (the last of them (ZOA) intersects the first (AOB) along the straight line OA), then we receive a figure, called a polyhedral angle. The point O is called a *vertex* of a polyhedral angle. Planes, forming the polyhedral angle $(AOB, BOC, COD, \ldots, ZOA)$, are called its *faces*; straight lines, along which the consequent faces intersect $(OA, OB, OC, \ldots, OZ)$ are called *edges* of a polyhedral angle. Angles $\widehat{AOB}, \widehat{BOC}, \widehat{COD}, \ldots, \widehat{EOA}$ are called its *plane angles*.

The minimal number of faces of a polyhedral angle is 3, this is the *trihedral angle*.

The sum of any two face angles of a trihedral angle is greater than the third face angle.

2.5.5 Tetrahedra

A polyhedron is said to be regular if all its faces are congruent regular polygons and the same number of faces join in each its vertex. It is known only five convex regular polyhedrons and four non-convex regular polyhedrons. The regular convex polyhedrons are the following: a tetrahedron (4 faces), a hexahedron (6 faces) well known to us as a cube; an octahedron (8 faces); a dodecahedron (12 faces); an icosahedron (20 faces).

It is possible to inscribe a sphere into any regular polyhedron and to circumscribe a sphere around any regular polyhedron.

2.5.6 Prism, parallelepiped, pyramid

A prism can be triangular, quadrangular, pentagonal, hexagonal and so on, depending on the form of the polygon in its base. If lateral edges of a prism are perpendicular to a base plane, this prism is a *right prism*; otherwise it is an *oblique prism.* If a base of a right prism is a regular polygon, this prism is also called a *regular* one.

A parallelepiped is said to be *right*, if the four lateral faces of the parallelepiped are rectangles. A right parallelepiped is called right-angled, if all its six faces are rectangles.

A pyramid can be triangular, quadrangular, pentagonal, hexagonal and so on, depending on the form of the polygon in its base. A triangular pyramid is a tetrahedron, a quadrangular one is a pentahedron etc. A pyramid is called *regular* if its base is a regular polygon and the orthogonal projection of its vertex on the base coincides with the center of the base. All lateral edges of a regular pyramid are equal; all lateral faces are equal isosceles triangles. A height of lateral face is called an *apothem* of a regular pyramid.

If one draws two planes which are parallel to the base of the pyramid, then the body of the pyramid, concluded between these planes, is called a *truncated* pyramid. A truncated pyramid is called regular if the pyramid from which it was received is regular. All lateral faces of a regular truncated pyramid are equal isosceles trapezoids. The height of a lateral face is called an apothem of a regular truncated pyramid.

2.5.7 Cones

Conic surface is formed by the motion of a straight line, that passes through a fixed point, which is called a *vertex*, and intersects with the given line, which is called a *directrix*. Straight lines, corresponding to different positions of the straight line at its motion, are called *generatrices* of a conic surface.

A cone is a body, limited by one part of a conic surface (with a closed directrix) and a plane, intersecting it and which does not go through a vertex. A part of this plane, placed inside of the conic surface, is called a *base* of cone. The perpendicular, drawn from a vertex to a base, is called a *height* of cone. A cone is *circular*, if its base is a circle. The straight line, joining a cone vertex with a center of a base, is called an *axis* of a cone. If a height of circular cone coincides with its axis, then this cone is called a *round* cone.

Conic sections. The sections of circular cone, parallel to its base, are circles. The section, crossing only one part of a circular cone and not parallel to single its generatrix, is an *ellipse*. The section, crossing only one part of a circular cone and parallel to one of its generatrices, is a *parabola*. In a general case the section, crossing both parts of a circular cone, is a *hyperbola*, consisting of two branches. Particularly, if this section is going through the cone axis, then we receive a pair of intersecting straight lines.

Chapter 3

Problems

3.1 Algebra

3.1.1 (1962)

Prove that

$$\frac{1}{\frac{1}{a}+\frac{1}{b}}+\frac{1}{\frac{1}{c}+\frac{1}{d}} \le \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}},$$

for all positive real numbers a, b, c, d.

3.1.2 (1964)

Given an arbitrary angle α, compute

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right)$$

and

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right).$$

Generalize this result and justify your answer.

3.1.3 (1966)

Let x, y and z be nonnegative real numbers satisfying the following conditions:

(1) $x + cy \le 36$,

(2) $2x + 3z \leq 72$,

where c is a given positive number.
Prove that if $c \geq 3$ then the maximum of the sum $x + y + z$ is 36, while if $c < 3$, the maximum of the sum is $24 + \dfrac{36}{c}$.

3.1.4 (1968)

Let a and b satisfy $a \geq b > 0,\ a + b = 1$.

1) Prover that if m and n are positive integers with $m < n$, then $a^m - a^n \geq b^m - b^n > 0$.

2) For each positive integer n, consider a quadratic function

$$f_n(x) = x^2 - b^n x - a^n.$$

Show that $f(x)$ has two roots that are in between -1 and 1.

3.1.5 (1969)

Consider $x_1 > 0, y_1 > 0, x_2 < 0, y_2 > 0, x_3 < 0, y_3 < 0, x_4 > 0, y_4 < 0$. Suppose that for each $i = 1, \ldots, 4$ we have $(x_i - a)^2 + (y_i - b)^2 \leq c^2$. Prove that $a^2 + b^2 < c^2$.

Restate this fact in the form of geometric result in plane geometry.

3.1.6 (1970)

Prove that for an arbitrary triangle ABC

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} < \frac{1}{4}.$$

3.1.7 (1972)

Let α be an arbitrary angle and let $x = \cos\alpha,\ y = \cos n\alpha\ (n \in \mathbb{Z})$.

1) Prove that to each value $x \in [-1,1]$ corresponds one and only one value of y. Thus we can write y as a function of x, $y = T_n(x)$. Compute $T_1(x), T_2(x)$ and prove that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

From this it follows that $T_n(x)$ is a polynomial of degree n.

2) Prove that the polynomial $T_n(x)$ has n distinct roots in $[-1,1]$.

3.1.8 (1975)

Without solving the cubic equation $x^3 - x + 1 = 0$, compute the sum of the eighth powers of all roots of the equation.

3.1.9 (1975)

Find all real x which satisfy

$$\frac{x^3+m^3}{(x+m)^3} + \frac{x^3+n^3}{(x+n)^3} + \frac{x^3+p^3}{(x+p)^3} + \frac{3}{2}\frac{(x-m)(x-n)(x-p)}{(x+m)(x+n)(x+p)} = \frac{3}{2}.$$

3.1.10 (1976)

Find all integer solutions of the system

$$\begin{cases} x^{x+y} = y^{12}, \\ y^{y+x} = x^3. \end{cases}$$

3.1.11 (1976)

Let k and n be positive integers and $x_1, \ldots, x_k$ positive real numbers satisfying $x_1 + \cdots + x_k = 1$. Prove that

$$x_1^{-n} + \cdots + x_k^{-n} \geq k^{n+1}.$$

3.1.12 (1977)

Solve the inequality

$$\sqrt{x-\frac{1}{x}}-\sqrt{1-\frac{1}{x}}>\frac{x-1}{x}.$$

3.1.13 (1977)

Consider real numbers $a_0, a_1, \ldots, a_{n+1}$ that satisfy

$$a_0 = a_{n+1} = 0,\ |a_{k-1} - 2a_k + a_{k+1}| \le 1\ (k = 1, \ldots, n).$$

Prove that

$$|a_k| \le \frac{k(n-k+1)}{2},\ \forall k = 0, 1, \ldots, n+1.$$

3.1.14 (1978)

Find all values of m for which the following system has a unique solution:

$$\begin{cases} x^2 = 2^{|x|} + |x| - y - m, \\ x^2 = 1 - y^2. \end{cases}$$

3.1.15 (1978)

Find three irreducible fractions $\frac{a}{d}, \frac{b}{d}$ and $\frac{c}{d}$, that form an arithmetic progression, if

$$\frac{b}{a} = \frac{1+a}{1+d},\ \frac{c}{b} = \frac{1+b}{1+d}.$$

3.1.16 (1979)

An equation $x^3 + ax^2 + bx + c = 0$ has three (not necessarily distinct) real roots t, u, v. For what values of a, b, c are the numbers t^3, u^3, v^3 roots of an equation $x^3 + a^3x^2 + b^3x + c^3 = 0$?

3.1.17 (1979)

Find all values of α for which the equation

$$x^2 - 2x[x] + x - \alpha = 0$$

has two distinct nonnegative roots (here $[x]$ denotes the greatest integer less than or equal to a real number x).

3.1.18 (1980)

Denote by $\overline{m}$ the average of the positive numbers $m_1, \ldots, m_k$. Prove that

$$\left(m_1 + \frac{1}{m_1}\right)^2 + \cdots + \left(m_k + \frac{1}{m_k}\right)^2 \geq k\left(\overline{m} + \frac{1}{\overline{m}}\right)^2.$$

3.1.19 (1980)

Can the equation

$$z^3 - 2z^2 - 2z + m = 0$$

have three distinct rational roots? Justify your answer.

3.1.20 (1980)

Let $n > 1$ be an integer, $p > 0$ a real number. Find the maximum value of

$$\sum_{i=1}^{n-1} x_i x_{i+1},$$

when the x_i's run over nonnegative values with $\sum\limits_{i=1}^{n} x_i = p$.

3.1.21 (1981)

Solve the system

$$\begin{cases} x^2 + y^2 + z^2 + t^2 = 50, \\ x^2 - y^2 + z^2 - t^2 = -24, \\ xz = yt, \\ x - y + z + t = 0. \end{cases}$$

3.1.22 (1981)

Let $t_1, \ldots, t_n$ be real numbers with $0 < p \le t_k \le q$ $(k = 1, \ldots, n)$. Let $\bar{t} = \dfrac{1}{n}(t_1 + \cdots + t_n)$ and $T = \dfrac{1}{n}(t_1^2 + \cdots + t_n^2)$. Prove that

$$\frac{\bar{t}^2}{T} \ge \frac{4pq}{(p+q)^2}.$$

When does equality occur?

3.1.23 (1981)

Without using a calculator, compute

$$\frac{1}{\cos^2 10^\circ} + \frac{1}{\sin^2 20^\circ} + \frac{1}{\sin^2 40^\circ} - \frac{1}{\cos^2 45^\circ}.$$

3.1.24 (1981)

Let $n \ge 2$ be a positive integer. Solve the system

$$\begin{cases} 2t_1 - t_2 & = a_1, \\ -t_1 + 2t_2 - t_3 & = a_2, \\ -t_2 + 2t_3 - t_4 & = a_3 \\ \cdots \quad \cdots \quad \cdots & \cdots \\ -t_{n-2} + 2t_{n-1} - t_n & = a_{n-1}, \\ -t_{n-1} + 2t_n & = a_n. \end{cases}$$

3.1.25 (1982)

Find a quadratic equation with integer coefficients whose roots are $\cos 72^\circ$ and $\cos 144^\circ$.

3.1.26 (1982)

Let p be a positive integer, q and s real numbers. Suppose that $q^{p+1} \le s \le 1,\ 0 < q < 1$. Prove that

$$\prod_{k=1}^{p} \left| \frac{s - q^k}{s + q^k} \right| \le \prod_{k=1}^{p} \left| \frac{1 - q^k}{1 + q^k} \right|.$$

3.1.27 (1983)

Compare

$$S_n = \sum_{k=1}^{n} \frac{k}{(2n-2k+1)(2n-k+1)}$$

and

$$T_n = \sum_{k=1}^{n} \frac{1}{k}.$$

3.1.28 (1984)

Find the polynomial of the lowest degree with integer coefficients such that one of its roots is $\sqrt{2} + \sqrt[3]{3}$.

3.1.29 (1984)

Solve the equation

$$\sqrt{1+\sqrt{1-x^2}}\left(\sqrt{(1+x)^3} - \sqrt{(1-x)^3}\right) = 2 + \sqrt{1-x^2}.$$

3.1.30 (1984)

Find all positive values of t satisfying the equation

$$0.9t = \frac{[t]}{t-[t]},$$

where $[t]$ denotes the greatest integer less than or equal to t.

3.1.31 (1985)

Find all values of m for which the equation

$$16x^4 - mx^3 + (2m+17)x^2 - mx + 16 = 0$$

has four distinct roots forming a geometric progression.

3.1.32 (1986)

Consider n inequalities

$$4x^2 - 4a_ix + (a_i - 1)^2 \leq 0,$$

where $a_i \in [\frac{1}{2}, 5]$ $(i = 1, \ldots, n)$. Let x_i be an arbitrary solution corresponding to a_i. Prove that

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2} \leq \frac{1}{n}\sum_{i=1}^{n} x_i + 1.$$

3.1.33 (1986)

Find all integers $n > 1$ so that the inequality

$$\sum_{i=1}^{n} x_i^2 \geq x_n \sum_{i=1}^{n-1} x_i$$

is satisfied for all x_i $(i = 1, \ldots, n)$.

3.1.34 (1987)

Let $a_i > 0$ $(i = 1, \ldots, n)$ and $n \geq 2$. Put $S = \sum_{i=1}^{n} a_i$. Prove that

$$\sum_{i=1}^{n} \frac{(a_i)^{2^k}}{(S - a_i)^{2^t - 1}} \geq \frac{S^{1+2^k-2^t}}{(n-1)^{2^t-1} n^{2^k-2^t}},$$

for k, t nonnegative integers and $k \geq t$. When does equality occur?

3.1.35 (1988)

Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with $n \geq 3$. Suppose that $P(x)$ has n real roots and $a_0 = 1, a_1 = -n, a_2 = \frac{n^2-n}{2}$. Determine coefficients a_i for $i = 3, \ldots, n$.

3.1.36 (1989)

Consider two positive integers N and n. Prove that for all nonnegative $\alpha \leq N$ and real x the following inequality holds

$$\left|\sum_{k=0}^{n} \frac{\sin(\alpha+k)x}{N+k}\right| \leq \min\left\{(n+1)|x|, \frac{1}{N\left|\sin\frac{x}{2}\right|}\right\}.$$

3.1.37 (1990 B)

Prove that

$$\sqrt[3]{\frac{2}{1}}+\sqrt[3]{\frac{3}{2}}+\cdots+\sqrt[3]{\frac{996}{995}}-\frac{1989}{2}<\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{8961}.$$

3.1.38 (1991 B)

Suppose that a polynomial $P(x) = x^{10} - 10x^9 + 39x^8 + a_7x^7 + \cdots + a_1x + a_0$ with certain values of $a_7, \ldots, a_0$ has 10 real roots. Prove that all roots of $P(x)$ are in between -2.5 and 4.5.

3.1.39 (1992 B)

Given $n > 2$ real numbers $x_1, \ldots, x_n$ in $[-1, 1]$ with the sum $x_1 + \cdots + x_n = n - 3$, prove that

$$x_1^2 + \cdots + x_n^2 \leq n - 1.$$

3.1.40 (1992 B)

Prove that for any positive integer $n > 1$

$$\sqrt[n]{1+\frac{\sqrt[n]{n}}{n}}+\sqrt[n]{1-\frac{\sqrt[n]{n}}{n}}<2.$$

3.1.41 (1992)

Consider a polynomial

$$P(x) = 1 + x^2 + x^9 + x^{n_1} + \cdots + x^{n_s} + x^{1992},$$

where $n_1, \cdots, n_s$ are positive integers satisfying $9 < n_1 < \cdots < n_s < 1992$. Prove that a root of $P(x)$ (if any) must be at most $\dfrac{1-\sqrt{5}}{2}$.

3.1.42 (1994 B)

For real numbers x, y, u, v satisfying

$$\begin{cases} 2x^2 + 3y^2 = 10, \\ 3u^2 + 8v^2 = 6, \\ 4xv + 3yu \geq 2\sqrt{15}, \end{cases}$$

find the maximum and minimum values of $S = x + y + u$.

3.1.43 (1994)

Does there exist polynomials $P(x), Q(x), T(x)$ satisfying the following conditions?

(1) All coefficients of the polynomials are positive integers.

(2) $T(x) = (x^2 - 3x + 3)P(x) = \left(\frac{x^2}{20} - \frac{x}{15} + \frac{1}{12}\right) Q(x)$.

3.1.44 (1995)

Solve the equation

$$x^3 - 3x^2 - 8x + 40 - 8\sqrt[4]{4x+4} = 0.$$

3.1.45 (1996)

Solve the system of equations

$$\begin{cases} \sqrt{3x}\left(1 + \frac{1}{x+y}\right) = 2, \\ \sqrt{7y}\left(1 - \frac{1}{x+y}\right) = 4\sqrt{2}. \end{cases}$$

3.1.46 (1996)

Let four nonnegative numbers a, b, c and d satisfy

$$2(ab+ac+ad+bc+bd+cd)+abc+abd+acd+bcd=16.$$

Prove that

$$a+b+c+d \geq \frac{2}{3}(ab+ac+ad+bc+bd+cd).$$

3.1.47 (1997)

1. Find all polynomials $P(x)$ of the lowest degree with rational coefficients that satisfy $P(\sqrt[3]{3}+\sqrt[3]{9})=3+\sqrt[3]{3}$.

2. Does there exist a polynomial $P(x)$ with integer coefficients that satisfies $P(\sqrt[3]{3}+\sqrt[3]{9})=3+\sqrt[3]{3}$?

3.1.48 (1998 B)

Positive numbers $x_1, \ldots, x_n$ $(n \geq 2)$ satisfy

$$\frac{1}{x_1+1998}+\frac{1}{x_2+1998}+\cdots+\frac{1}{x_n+1998}=\frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 \ldots x_n}}{n-1} \geq 1998.$$

3.1.49 (1998)

Find all positive integers n for which there is a polynomial $P(x)$ with real coefficients satisfying

$$P(x^{1998}-x^{-1998})=x^n-x^{-n},\ \forall x \neq 0.$$

3.1.50 (1999)

Solve the system

$$\begin{cases} (1+4^{2x-y}).5^{1-2x+y} = 1+2^{2x-y+1}, \\ y^3+4x+1+\log(y^2+2x) = 0. \end{cases}$$

3.1.51 (1999)

Find the maximum value of

$$P = \frac{2}{a^2+1} - \frac{2}{b^2+1} + \frac{3}{c^2+1},$$

where $a, b, c > 0$ and $abc + a + c = b$.

3.1.52 (2001 B)

Positive numbers x, y, z satisfy

$$\begin{cases} \frac{2}{5} \le z \le \min\{x, y\}, \\ xz \ge \frac{4}{15}, \\ yz \ge \frac{1}{5}. \end{cases}$$

Find the maximum value of

$$P = \frac{1}{x} + \frac{2}{y} + \frac{3}{z}\cdot$$

3.1.53 (2002)

Let a, b, c be real numbers such that the polynomial $P(x) = x^3+ax^2+bx+c$ has three real (not necessarily distinct) roots. Prove that

$$12ab + 27c \le 6a^3 + 10(a^2-2b)^{3/2}.$$

When does equality occur?

3.1.54 (2003)

Given polynomials

$$P(x) = 4x^3 - 2x^2 - 15x + 9, \quad Q(x) = 12x^3 + 6x^2 - 7x + 1.$$

Prove that

1) Each of the two polynomials has three distinct real roots.
2) If a and b are the largest roots of P and Q respectively, then $a^2+3b^2 = 4$.

3.1.55 (2004 B)

Solve the system

$$\begin{cases} x^3 + 3xy^2 = -49, \\ x^2 - 8xy + y^2 = 8y - 17x. \end{cases}$$

3.1.56 (2004)

Solve the system

$$\begin{cases} x^3 + x(y-z)^2 = 2, \\ y^3 + y(z-x)^2 = 30, \\ z^3 + z(x-y)^2 = 16. \end{cases}$$

3.1.57 (2004)

Find the maximum and minimum values of

$$P = \frac{a^4 + b^4 + c^4}{(a+b+c)^4},$$

where $a, b, c > 0$ and $(a+b+c)^3 = 32abc$.

3.1.58 (2005)

Find the maximum and minimum values of

$$P = x + y,$$

if $x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$.

3.1.59 (2006 B)

Solve the system

$$\begin{cases} x^3+3x^2+2x-5=y, \\ y^3+3y^2+2y-5=z, \\ z^3+3z^2+2z-5=x. \end{cases}$$

3.1.60 (2006 B)

Find the greatest value of a real number k so that for any positive numbers a,b,c with $abc=1$ the following inequality holds

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+3k \geq (k+1)(a+b+c).$$

3.1.61 (2006)

Find all polynomials $P(x)$ with real coefficients that satisfy the equation

$$P(x^2)+x\left[3P(x)+P(-x)\right]=[P(x)]^2+2x^2,\ \forall x.$$

3.1.62 (2007)

Solve the system

$$\begin{cases} \left(1-\frac{12}{y+3x}\right)\sqrt{x}=2, \\ \left(1+\frac{12}{y+3x}\right)\sqrt{y}=6. \end{cases}$$

3.1.63 (2008)

Let x,y,z be distinct nonnegative real numbers. Prove that

$$(xy+yz+zx)\left(\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\right) \geq 4.$$

When does equality occur?

3.2 Analysis

3.2.1 (1965)

1) Two nonnegative real numbers x, y have constant sum a. Find the minimum value of $x^m + y^m$, where m is a given positive integer.

2) Let m, n be positive integers and k a positive real number. Consider nonnegative real numbers $x_1, x_2, \ldots, x_n$ having constant sum k. Prove that the minimum value of the quantity $x_1^m + \cdots + x_n^m$ occurs when $x_1 = x_2 = \cdots = x_n$.

3.2.2 (1975)

Prove that the sum of the (local) maximum and minimum values of the function

$$y = \frac{\cot^3 x}{\cot 3x}, \ 0 < x < \frac{\pi}{2},$$

is a rational number.

3.2.3 (1980)

Let $\alpha_1, \ldots, \alpha_n$ be real numbers in the interval $[0, \pi]$ such that $\displaystyle\sum_{i=1}^{n}(1+\cos\alpha_i)$ is an odd integer. Show that $\displaystyle\sum_{i=1}^{n}\sin\alpha_i \geq 1$.

3.2.4 (1983)

1) Show that $\sqrt{2}(\sin x + \cos x) \geq 2\sqrt[4]{\sin 2x}, \ 0 \leq x \leq \frac{\pi}{2}$.

2) Find all $y \in (0, \pi)$ such that

$$1 + 2\frac{\cot 2y}{\cot y} \geq \frac{\tan 2y}{\tan y}.$$

3.2.5 (1984)

Given a sequence (u_n) with $u_1 = 1, u_2 = 2, u_{n+1} = 3u_n - u_{n-1}$ $(n \geq 2)$. A sequence (v_n) is defined by $v_n = \sum_{i=1}^{n} \cot^{-1} u_i,\ n = 1, 2, \ldots$. Compute $\lim_{n\to\infty} v_n$.

3.2.6 (1984)

Let a, b be real numbers with $a \neq 0$. Find a polynomial $P(x)$ such that

$$xP(x-a) = (x-b)P(x),\ \forall x.$$

3.2.7 (1985)

Denote by M a set of functions defined on integers with real values satisfying the following two conditions:

(1) $f(x)f(y) = f(x+y) + f(x-y)$, $\forall$ integers x, y,

(2) $f(0) \neq 0$.

Find $f \in M$ such that $f(1) = \frac{5}{2}$.

3.2.8 (1986)

Let $M(y)$ be a polynomial of degree n such that

$$M(y) = 2^y,\ \forall y = 1, 2, \ldots, n+1.$$

Find $M(n+2)$.

3.2.9 (1986)

A sequence of positive integers is defined as follows. The first term is 1. Then take the next two even numbers $2, 4$. Then take the next three odd numbers $5, 7, 9$. Then take the next four even numbers $10, 12, 14, 16$, and so on. Find the nth term of the sequence.

3.2.10 (1987)

Given an arithmetic progression consisting of 1987 terms such that the first term $u_1 = \frac{\pi}{1987}$ and the difference $d = \frac{\pi}{3974}$. Compute the sum of the 2^{1987} terms $\cos(\pm u_1 \pm u_2 \pm \cdots \pm u_{1987})$.

3.2.11 (1987)

A function $f(x)$ is defined and differentiable on $[0, +\infty)$, and satisfies

(1) $|f(x)| \le 5$,

(2) $f(x)f'(x) \ge \sin x$.

Does $\lim\limits_{x \to +\infty} f(x)$ exist?

3.2.12 (1988)

Given a bounded sequence (x_n) with $x_n + x_{n+1} \ge 2x_{n+2}$, $\forall n \ge 1$. Does it necessarily converge?

3.2.13 (1989)

A sequence of polynomials is defined by

$$P_0(x) = 0, P_{n+1}(x) = P_n(x) + \frac{x - P_n^2(x)}{2},\ n = 0, 1, \ldots$$

Prove that for any $x \in [0, 1]$ and $n \ge 0$

$$0 \le \sqrt{x} - P_n(x) \le \frac{2}{n+1}.$$

3.2.14 (1990 B)

Consider a sequence

$$u_1 = a \cdot 1^{1990}, u_2 = a \cdot 2^{1990}, \ldots, u_{2000} = a \cdot 2000^{1990},$$

where a is a real number. From this sequence form a second one by

$$v_1 = u_2 - u_1, v_2 = u_3 - u_2, \ldots, v_{1999} = u_{2000} - u_{1999}.$$

From the second sequence form a third one in the same way, and so on. Prove that all terms of the 1991-th sequence are equal $a \cdot 1990!$.

3.2.15 (1990)

A sequence (x_n) is defined as follows:

$$|x_1| < 1, x_{n+1} = \frac{-x_n + \sqrt{3(1 - x_n^2)}}{2}.$$

1) Find necessary and sufficient conditions on x_1 for all terms of the sequence to be positive.
2) Is the sequence periodic? Justify your answer.

3.2.16 (1990)

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ be a polynomial with real coefficients such that $a_0 \neq 0$ and

$$f(x).f(2x^2) = f(2x^3 + x),\ \forall x.$$

Prove that $f(x)$ has no real root.

3.2.17 (1991)

Find all real functions $f(x)$ satisfying

$$\frac{1}{2}f(xy) + \frac{1}{2}f(xz) - f(x)f(yz) \geq \frac{1}{4},$$

for all real numbers x, y, z.

3.2.18 (1991)

Prove the following inequality

$$\frac{x^2y}{z} + \frac{y^2z}{x} + \frac{z^2x}{y} \geq x^2 + y^2 + z^2,\ \forall x \geq y \geq z > 0.$$

3.2.19 (1992 B)

Suppose that a real-valued function $f(x)$ of real numbers satisfies

$$f(x + 2xy) = f(x) + 2f(xy),$$

for all real x, y, and that $f(1991) = a$, where a is a real number. Compute $f(1992)$.

3.2.20 (1992)

Let a, b, c be positive numbers. Three sequences $(a_n), (b_n), (c_n)$ are defined by

(1) $a_0 = a, b_0 = b, c_0 = c$.

(1) $a_{n+1} = a_n + \dfrac{2}{b_n + c_n}$, $b_{n+1} = b_n + \dfrac{2}{c_n + a_n}$, $c_{n+1} = c_n + \dfrac{2}{a_n + b_n}$, $\forall n \geq 0$.

Prove that (a_n) tends to infinity.

3.2.21 (1993 B)

Find all values of a so that the following inequality holds for all $x \geq 0$:

$$\log(1 + x) \geq x - ax^2.$$

3.2.22 (1993)

Find the maximum and minimum values of the function

$$f(x) = x(1993 + \sqrt{1995 - x^2})$$

on its domain of definition.

3.2.23 (1993)

Two sequences $(a_n), (b_n)$ are defined by

$$a_0 = 2, b_0 = 1, a_{n+1} = \frac{2a_n b_n}{a_n + b_n}, b_{n+1} = \sqrt{a_{n+1} b_n},\ \forall n \geq 0.$$

Prove that both sequences converge to the same limit, and find this value.

3.2.24 (1994 B)

Solve the system

$$\begin{cases} x^2 + 3x + \log(2x + 1) = y, \\ y^2 + 3y + \log(2y + 1) = x. \end{cases}$$

3.2.25 (1994 B)

A sequence (x_n) is defined by

$$x_0 = a, x_n = \sqrt[3]{6(x_{n-1} - \sin x_{n-1})},\ \forall n \geq 1,$$

where a is a real number. Prove that the sequence converges and find the limit.

3.2.26 (1994)

Solve the system

$$\begin{cases} x^3 + 3x - 3 + \log(x^2 - x + 1) = y, \\ y^3 + 3y - 3 + \log(y^2 - y + 1) = z, \\ z^3 + 3z - 3 + \log(z^2 - z + 1) = x. \end{cases}$$

3.2.27 (1994)

A sequence (x_n) is defined by

$$x_0 = a \in (0, 1), x_n = \frac{4}{\pi^2}\left(\cos^{-1} x_{n-1} + \frac{\pi}{2}\right) \sin^{-1} x_{n-1},\ \forall n \geq 1.$$

Prove that the sequence converges and find the limit.

3.2.28 (1995 B)

A sequence (a_n) is defined by

$$a_0 = 2, a_{n+1} = 5a_n + \sqrt{24a_n^2 - 96},\ \forall n \geq 0.$$

Find a formula for a general term a_n and prove that $a_n \geq 2 \cdot 5^n$ for all n.

3.2.29 (1995 B)

For an integer $n \in [2000, 2095]$ put

$$a = \frac{1}{1995} + \frac{1}{1996} + \cdots + \frac{1}{n}, \quad b = \frac{n+1}{1995}.$$

Find the integral part of $b^{1/a}$.

3.2.30 (1995)

Determine all polynomials $P(x)$ satisfying the following conditions: for each $a > 1995$ the number of real roots (counted with multiplicities) of the equation $P(x) = a$ is equal to the degree of $P(x)$ and all these roots are strictly greater than 1995.

3.2.31 (1996 B)

Determine the number of real roots of the system

$$\begin{cases} x^3y - y^4 = a^2, \\ x^2y + 2xy^2 + y^3 = b^2, \end{cases}$$

where a, b are real parameters.

3.2.32 (1996)

Determine all functions defined on positive integers $f(n)$ satisfying the equality

$$f(n) + f(n+1) = f(n+2) \cdot f(n+3) - 1996, \ \forall n \geq 1.$$

3.2.33 (1997 B)

Let n and k be positive integers with $n \geq 7$ and $2 \leq k < n$. Prove that $k^n > 2n^k$.

3.2.34 (1997)

Let $n > 1$ be a positive integer which is not divisible by 1997. Define two sequences $(a_i), (b_j)$ by

$$a_i = i + \frac{ni}{1997} \ (i = 1, 2, \ldots, 1996), \quad b_j = j + \frac{1997j}{n} \ (j = 1, 2, \ldots, n-1).$$

Writing all terms of the two sequences in the increasing order, we get the sequence

$$c_1 \leq c_2 \leq \cdots \leq c_{1995+n}.$$

Prove that $c_{k+1} - c_k < 2$ for all $k = 1, 2, \ldots, 1994 + n$.

3.2.35 (1998 B)

A sequence (x_n) is defined by

$$x_1 = a, x_{n+1} = \frac{x_n(x_n^2+3)}{3x_n^2+1}, \ \forall n \geq 1,$$

where a is a real number. Prove that the sequence converges and find the limit.

3.2.36 (1998 B)

Let a, b be integers. A sequence of integers (a_n) is defined by

$$a_0 = a, \ a_1 = b, \ a_2 = 2b - a + 2,$$

$$a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n, \ \forall n \geq 0.$$

Determine a formula for the general term a_n and find all integers a, b for which a_n is a square for all $n \geq 1998$.

3.2.37 (1998)

A sequence (x_n) is defined by

$$x_1 = a, \ x_{n+1} = 1 + \log\left(\frac{x_n^2}{1+\log x_n}\right), \ \forall n \geq 1,$$

where a is a real number ≥ 1. Prove that the sequence converges and find the limit.

3.2.38 (1998)

Prove that there does not exist an infinite sequence of real numbers (x_n) that satisfies the following conditions:

(1) $|x_n| \leq 0.666, \ \forall n \geq 1,$

(2) $|x_n - x_m| \geq \dfrac{1}{n(n+1)} + \dfrac{1}{m(m+1)}, \ \forall m \neq n.$

3.2.39 (1999 B)

A sequence (u_n) is defined by

$$u_1 = 1, u_2 = 2, u_{n+2} = 3u_{n+1} - u_n,\ \forall n \geq 1.$$

Prove that

$$u_{n+2} + u_n \geq 2 + \frac{u_{n+1}^2}{u_n},\ \forall n \geq 1.$$

3.2.40 (1999 B)

Let a, b be real numbers so that the equation $ax^3 - x^2 + bx - 1 = 0$ has three positive roots (not necessarily distinct). Find the minimum value of

$$P = \frac{5a^2 - 3ab + 2}{a^2(b-a)}$$

for such a and b.

3.2.41 (1999 B)

A function $f(x)$, defined and continuous on $[0,1]$, satisfies the conditions:

$$\begin{aligned} &f(0) = f(1) = 0, \\ &2f(x) + f(y) = 3f\left(\frac{2x+y}{3}\right), \end{aligned}$$

for all $x, y \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

3.2.42 (2000 B)

Find all real functions $f(x)$ on real numbers satisfying

$$x^2 f(x) + f(1-x) = 2x - x^4,$$

for all real x.

3.2.43 (2000)

A sequence (x_n) is defined by

$$x_{n+1} = \sqrt{c - \sqrt{c + x_n}},\ \forall n \geq 0,$$

where c is a given positive number. Determine all values of c so that for any initial value $x_0 \in (0, c)$ the sequence (x_n) is well defined and converges.

3.2.44 (2000)

Given $\alpha \in (0, \pi)$.

1) Prove that there exists a unique quadratic function $f(x) = x^2 + ax + b$ with a, b real, such that for any $n > 2$ the polynomial

$$P_n(x) = x^n \sin\alpha - x\sin(n\alpha) + \sin(n-1)\alpha$$

is divisible by $f(x)$.

2) Prove that there does not exist a linear function $g(x) = x + c$ with c real, such that for any $n > 2$ the polynomial $P_n(x)$ mentioned above is divisible by $g(x)$.

3.2.45 (2001 B)

A sequence (x_n) is defined by

$$x_1 = \frac{2}{3},\ x_{n+1} = \frac{x_n}{2(2n+1)x_n + 1},\ \forall n \geq 1.$$

Compute $x_1 + x_2 + \cdots + x_{2001}$.

3.2.46 (2001)

Given real numbers a, b, a sequence (x_n) is defined by

$$x_0 = a,\ x_{n+1} = x_n + b\sin x_n,\ \forall n \geq 0.$$

1) Prove that if $b = 1$ then for any number a the sequence converges to a finite limit and compute this limit.

2) Prove that for any given number $b > 2$ there always exists a number a such that the sequence above diverges.

3.2.47 (2001)

Let $g(x) = \dfrac{2x}{1+x^2}$. Find all continuous functions $f(x)$ defined on $(-1,1)$ that satisfy the equation

$$(1-x^2)f\big(g(x)\big) = (1+x^2)^2 f(x),\ \forall x \in (-1,1).$$

3.2.48 (2002 B)

Find all real functions $f(x)$ on real numbers satisfying

$$f\big(y - f(x)\big) = f\big(x^{2002} - y\big) - 2001yf(x),$$

for all x, y real.

3.2.49 (2002 B)

For each positive integer n consider the equation

$$\frac{1}{2x} + \frac{1}{x-1^2} + \frac{1}{x-2^2} + \cdots + \frac{1}{x-n^2} = 0.$$

Prove that

1) The equation has a unique solution $x_n \in (0,1)$,

2) The sequence (x_n) converges.

3.2.50 (2002)

Consider the equation

$$\frac{1}{x-1} + \frac{1}{2^2x-1} + \cdots + \frac{1}{n^2x-1} = \frac{1}{2},\ n \in \mathbb{N}.$$

Prove that

1) For each positive integer n the equation has a unique solution $x_n > 1$,

2) The sequence (x_n) converges to 4.

3.2.51 (2003 B)

Find all polynomials $P(x)$ with real coefficients which satisfy

$$(x^3+3x^2+3x+2)P(x-1)=(x^3-3x^2+3x-2)P(x)$$

for all real x.

3.2.52 (2003 B)

For a real number $\alpha \neq 0$ define a sequence (x_n) by

$$x_1=0,\ x_{n+1}(x_n+\alpha)=\alpha+1,\ \forall n \geq 1.$$

1) Find a formula of a general term (x_n).

2) Prove that (x_n) converges and compute the limit.

3.2.53 (2003 B)

A function $f(x)$ defined on real numbers with real values satisfies

$$f(\cot x)=\sin 2x+\cos 2x,\ \forall x \in (0,\pi).$$

Find the maximum and minimum values of $g(x)=f(\sin^2 x)f(\cos^2 x)$ for all real x.

3.2.54 (2003)

Denote by $\mathcal{F}$ the set of all positive functions defined on positive real numbers, that satisfy

$$f(3x) \geq f\big(f(2x)\big)+x,\ \forall x>0.$$

Find the largest a such that for any $f \in \mathcal{F}$ we always have $f(x) \geq ax$, $\forall x>0$.

3.2.55 (2004)

Consider the sequence $(x_n)_{n=1}^{\infty}$ of real numbers defined by

$$x_1=1,\ x_{n+1}=\frac{(2+2\cos 2\alpha)x_n+\cos^2\alpha}{(2-2\cos 2\alpha)x_n+2-2\cos 2\alpha},\ n \geq 1,$$

where $\alpha \in \mathbb{R}$ is a parameter.
Determine all possible values of α such that the sequence $(y_n)_{n=1}^{\infty}$ defined by

$$y_n = \sum_{k=1}^{n} \frac{1}{2x_k + 1}, \ n \geq 1,$$

has a finite limit. Find the limit of (y_n) in these cases.

3.2.56 (2005)

Find all real functions $f(x)$ satisfying the equation

$$f\big(f(x-y)\big) = f(x)f(y) - f(x) + f(y) - xy,$$

for all x, y real.

3.2.57 (2006 B)

Find all real continuous functions $f(x)$ satisfying

$$f(x-y)f(y-z)f(z-x) + 8 = 0,$$

for all real x, y, z.

3.2.58 (2006)

Solve the system

$$\begin{cases} \sqrt{x^2 - 2x + 6} \cdot \log_3(6-y) = x, \\ \sqrt{y^2 - 2y + 6} \cdot \log_3(6-z) = y, \\ \sqrt{z^2 - 2z + 6} \cdot \log_3(6-x) = z. \end{cases}$$

3.2.59 (2007)

Given a positive number b, find all real functions $f(x)$ on real numbers that satisfy

$$f(x+y) = f(x) \cdot 3^{b^y + f(y) - 1} + b^x\left(3^{b^y + f(y) - 1} - b^y\right),$$

for all real numbers x, y.

3.2.60 (2007)

For $a > 2$ consider $f_n(x) = a^{10}x^{n+10} + x^n + \cdots + x + 1$ $(n = 1, 2, \ldots)$. Prove that for each positive integer n the equation $f_n(x) = a$ has a unique solution $x_n \in (0, +\infty)$ and the sequence (x_n) converges.

3.2.61 (2008)

Given a real number $a \geq 17$, determine the number of pairs (x, y) solving the following system:

$$\begin{cases} x^2 + y^3 = a, \\ \log_3 x \cdot \log_2 y = 1. \end{cases}$$

3.2.62 (2008)

A sequence (x_n) is defined by

$$x_1 = 0,\ x_2 = 2,\ x_{n+2} = 2^{-x_n} + \frac{1}{2},\ \forall n \geq 1.$$

Prove that (x_n) converges and find its limit.

3.3 Number Theory

3.3.1 (1963)

Three students A, B and C, walking on the street, witnessed a car violating a traffic regulation. No one remembered the licence number, but each got some particular aspect of it. A remembered that the first two digits are equal, B noted that the last two digits are also equal, and C said that it is a four-digit number and is a perfect square. What is the licence number of the car?

3.3.2 (1970)

Find all positive integers which divide $1890 \cdot 1930 \cdot 1970$ and are not divisible by 45.

3.3.3 (1971)

Consider positive integers $m < n, p < q$ such that $(m, n) = 1, (p, q) = 1$ and satisfy the condition that if $\frac{m}{n} = \tan\alpha$ and $\frac{p}{q} = \tan\beta$, then $\alpha + \beta = 45°$.

1) Given m, n, find p, q.

2) Given n, q, find m, p.

3) Given m, q, find n, p.

3.3.4 (1972)

For any positive integer N, let $f(N) = \sum(-1)^{\frac{d-1}{2}}$, where the sum is taken over all odd d dividing N. Prove that

1) $f(2) = 1$, $f(2^r) = 1$ (r is an integer).

2) If $p > 2$ is a prime number, then $f(p) = \begin{cases} 2, & \text{if } p = 4k+1 \\ 0, & \text{if } p = 4k-1, \end{cases}$

$$f(p^r) = \begin{cases} 1+r, & \text{if } p = 4k+1 \\ 1, & \text{if } p = 4k-1, r \text{ is even} \\ 0, & \text{if } p = 4k-1, r \text{ is odd.} \end{cases}$$

3) If M, N are co-prime, then $f(M \cdot N) = f(M) \cdot f(N)$. Use this to compute $f(5^4 \cdot 11^{28} \cdot 17^{19})$ and $f(1980)$. Derive a general rule for computing $f(N)$.

3.3.5 (1974)

1) Find all positive integers n for which a number

$$\underbrace{11\ldots1}_{2n \text{ times}} - \underbrace{77\ldots7}_{n \text{ times}}$$

is a square.

2) Replace 7 by an integer $b \in [1, 9]$ and solve the same problem.

3.3.6 (1974)

1) How many integers n are there such that n is divisible by 9 and $n+1$ is divisible by 25?

2) How many integers n are there such that n is divisible by 21 and $n+1$ is divisible by 165?

3) How many integers n are there such that n is divisible by 9, $n+1$ is divisible by 25, and $n+2$ is divisible by 4?

3.3.7 (1975)

Find all terms of the arithmetic progression $-1, 18, 37, \ldots$, that have 5 in all their digits.

3.3.8 (1976)

Find all three-digit integers $n = \overline{abc}$ such that $2n = 3a!b!c!$.

3.3.9 (1977)

Let $P(x)$ be a real polynomial of degree three. Find necessary and sufficient conditions on its coefficients so that $P(n)$ is an integer for every integer n.

3.3.10 (1978)

Find all three-digit numbers $\overline{abc}$ such that $2\overline{abc} = \overline{bca} + \overline{cab}$.

3.3.11 (1981)

Find all integral values of m such that $g(x) = x^3 + 2x + m$ divides $f(x) = x^{12} - x^{11} + 3x^{10} + 11x^3 - x^2 + 23x + 30$.

3.3.12 (1982)

Find all positive integer solutions to the equation $2^x + 2^y + 2^z = 2336$ $(x < y < z)$.

3.3.13 (1983)

For which positive integers a and b with $b > 2$ does the number $2^b - 1$ divides $2^a + 1$?

3.3.14 (1983)

Is it possible to represent 1 in the form

1) $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_6}$,

2) $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_9}$,

where a_i's are distinct odd positive numbers?
Generalize the problem.

3.3.15 (1984)

Find the minimum value of $A = |5x^2 + 11xy - 5y^2|$, where x, y are integers not both zeros.

3.3.16 (1985)

Find all integer solutions to the equation

$$x^3 - y^3 = 2xy + 8.$$

3.3.17 (1985)

For three positive integers a, b and m, prove that there is a positive integer n such that m divides $(a^n - 1)b$ if and only if $(ab, m) = (b, m)$.

3.3.18 (1987)

Two sequences $(x_n), (y_n)$ are defined by

$$x_0 = 365,\ x_{n+1} = x_n(x_n^{1986} + 1) + 1622,\ \forall n \geq 0$$

and

$$y_0 = 16,\ y_{n+1} = y_n(y_n^3 + 1) - 1952,\ \forall n \geq 0.$$

Prove that $|x_n - y_k| > 0,\ \forall n, k \geq 1$.

3.3.19 (1989)

Consider the Fibonacci sequence

$$a_1 = a_2 = 1,\ a_{n+2} = a_{n+1} + a_n,\ \forall n \geq 1.$$

Let $f(n) = 1985n^2 + 1956n + 1960$.

1) Show that there are infinitely many terms F in the sequence such that $f(F)$ is divisible by 1989.
2) Does there exist a term G in the sequence such that $f(G) + 2$ is divisible by 1989?

3.3.20 (1989)

Are there integers x, y both not divisible by 5 such that $x^2 + 19y^2 = 198 \cdot 10^{1989}$?

3.3.21 (1990)

Let $A = \{1, 2, 3, \ldots, 2n-1\}$. Remove at least $n-1$ numbers from A by the following rule:

(1) If the number $a \in A$ is removed and $2a \in A$, then $2a$ must be removed,

(2) If the numbers $a, b \in A$ are removed and $a+b \in A$, then $a+b$ must be removed.

What numbers must removed so that the sum of the remaining numbers is maximum?

3.3.22 (1991)

Let $k > 1$ be an odd integer. For each positive integer n, denote by $f(n)$ the greatest positive integer such that $k^n - 1$ is divisible by $2^{f(n)}$. Determine $f(n)$ in terms of k and n.

3.3.23 (1992)

Let n be a positive integer. Denote by $f(n)$, the number of divisors of n which end with the digits 1 or 9, and by $g(n)$, the number of divisors of n which end with digits 3 or 7. Prove that $f(n) \geq g(n)$.

3.3.24 (1995)

A sequence (a_n) is defined by

$$a_0 = 1, a_1 = 3, a_{n+2} = \begin{cases} a_{n+1} + 9a_n, & \text{if } n \text{ is even} \\ 9a_{n+1} + 5a_n, & \text{if } n \text{ is odd.} \end{cases}$$

Prove that

1) $\sum_{k=1995}^{2000} a_k^2$ is divisible by 20,

2) a_{2n+1} is not a square for every positive integer n.

3.3.25 (1996 B)

Find all functions defined on integers $f(n)$ with integer values such that $f(1995) = 1996$ and for each integer n, if $f(n) = m$ then $f(m) = n$ and $f(m+3) = n - 3$.

3.3.26 (1997 B)

A sequence of integers (a_n) is defined by

$$a_0 = 1, a_1 = 45, a_{n+2} = 45a_{n+1} - 7a_n, \ \forall n \geq 0.$$

1) Determine the number of positive divisors of $a_{n+1}^2 - a_n a_{n+2}$ in terms of n.

2) Prove that $1997a_n^2 + 4 \cdot 7^{n+1}$ is a square for each n.

3.3.27 (1997)

Prove that for any positive integer n there always exists a positive integer k such that $19^k - 97$ is divisible by 2^n.

3.3.28 (1999 B)

Two sequences $(x_n), (y_n)$ are defined by

$$x_1 = 1,\ y_1 = 2,\ x_{n+1} = 22y_n - 15x_n,\ y_{n+1} = 17y_n - 12x_n,\ \forall n \geq 1.$$

1) Prove that both sequences $(x_n), (y_n)$ have only nonzero terms, infinitely many positive terms, and infinitely many negative terms

2) Are the 1999^{1945}-th terms of both sequences divisible by 7? Justify the answer.

3.3.29 (1999)

Find all functions f, defined on nonnegative integers n with values from the set $T = \{0, 1, \ldots, 1999\}$ such that

(1) $f(n) = n,\ \forall n \in T,$

(2) $f(m+n) = f\big(f(m) + f(n)\big),\ \forall m, n \geq 0.$

3.3.30 (2001)

Let n be a positive integer and $a > 1,\ b > 1$ be two co-prime integers. Suppose that $p > 1,\ q > 1$ are two odd divisors of $a^{6^n} + b^{6^n}$, find the remainder when $p^{6^n} + q^{6^n}$ is divided by $6 \cdot (12)^n$.

3.3.31 (2002 B)

Let $\mathcal{S}$ be the set of all integers in the interval $[1, 2002]$, and $\mathcal{T}$ be the collection of all nonempty subsets of $\mathcal{S}$. For each $X \in \mathcal{T}$ denote $m(X)$ the arithmetic mean of all numbers in X. Compute

$$\frac{1}{|\mathcal{T}|} \sum m(X),$$

where the sum is taken over all X in $\mathcal{T}$, and $|\mathcal{T}|$ is the number of elements in $\mathcal{T}$.

3.3.32 (2002 B)

Find all positive integers n satisfying

$$\binom{2n}{n} = (2n)^k,$$

where k is the number of prime divisors of $\binom{2n}{n}$.

3.3.33 (2002)

Find all positive integers n for which the equation

$$x + y + u + v = n\sqrt{xyuv}$$

has positive integer solutions.

3.3.34 (2003)

Find the largest positive integer n such that the equations

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+n)^2 + y_n^2$$

have an integer solution $(x, y_1, \ldots, y_n)$.

3.3.35 (2004 B)

Solve the following equation for positive integers x, y and z.

$$(x+y)(1+xy) = 2^z.$$

3.3.36 (2004)

Find the smallest positive integer k for which in any subset of $\{1, 2, \ldots, 16\}$ with k elements, there are two distinct numbers a, b such that $a^2 + b^2$ is prime.

3.3.37 (2004)

Denote by $S(n)$ the sum of all digits of a positive integer n. Find the smallest value of $S(m)$, where m's are positive multiple of 2003.

3.3.38 (2005 B)

Find all positive integers x, y, n such that

$$\frac{x! + y!}{n!} = 3.$$

3.3.39 (2005)

Find all nonnegative integer (x, y, n) such that

$$\frac{x! + y!}{n!} = 3^n.$$

3.3.40 (2006)

Let S be a set consisting of 2006 integers. A subset T of S has the property that for any two integers in T (possibly equal) their sum is not in T.

1) Prove that if S consists of the first 2006 positive integers, then T has at most 1003 elements.
2) Prove that if S consists of 2006 arbitrary positive integers, then there exists a set T with 669 elements.

3.3.41 (2007)

Let $x \neq -1, y \neq -1$ be integers such that $\dfrac{x^4 - 1}{y + 1} + \dfrac{y^4 - 1}{x + 1}$ is an integer. Prove that $x^4 y^{44} - 1$ is divisible by $x + 1$.

3.3.42 (2008)

Let $m = 2007^{2008}$. Determine the number of positive integers n with $n < m$ such that $n(2n + 1)(5n + 2)$ is divisible by m.

3.4 Combinatorics

3.4.1 (1969)

A graph G has $n+k$ vertices. Let A be a subset of n vertices of the graph G, and B be a subset of other k vertices. Each vertex of A is joined to at least $k-p$ vertices of B. Prove that if $np<k$ then there is a vertex in B that can be joined to all vertices of A.

3.4.2 (1977)

Into how many regions do n circles divide the plane, if each pair of circles intersects at two points and no point lies on three circles?

3.4.3 (1987)

Given 5 rays in the space starting from the same point, show that we can always find two with an angle between them of at most $90°$.

3.4.4 (1990)

Some children sit in a circle. Each has an even number of sweets (larger than 0, maybe equal, maybe different). One child gives half his sweets to the child on his right. Then this child does the same, and so on. If a child about to give sweets has an odd number, then the teacher gives him an extra sweet. Show that after several steps there will be a moment when if a child gives half of sweets not to the next friend, but to the teacher, then all children have the same number of sweets.

3.4.5 (1991)

A group of 1991 students sit in a circle, consecutively counting numbers 1, 2 and 3 and repeating. Starting from some student A with number 1, and counting clockwise round the remaining students, students that count numbers 2 and 3 must leave the circle until only one remains. Determine who is the last student?

3.4.6 (1992)

Given a rectangle consisting of 1991×1992 squares denoted by (m, n) with $1 \leq m \leq 1991, 1 \leq n \leq 1992$. Color all squares by the following rule: first three squares $(r, s), (r+1, s+1), (r+2, s+1)$ for some $1 \leq r \leq 1989, 1 \leq s \leq 1991$. Subsequently color three consecutive uncolored squares in the same row or the same column. Is it possible to color all squares in the rectangle?

3.4.7 (1993)

Arrange points $A_1, A_2, \ldots, A_{1993}$ in a circle. Each point is labeled $+1$ or -1 (not all points with the same sign). Each time relabel simultaneously all points by the rules:

(1) If signs at A_i and A_{i+1} are the same, then the sign at A_i is changed into plus $(+)$,

(2) If signs at A_i and A_{i+1} are different, then the sign at A_i is changed into minus $(-)$.

(A convention: $A_{1994} = A_1$).
Prove that there exists an integer $k \geq 2$ such that after k consecutively changes of signs, the sign at each A_i $(i = 1, 2, \ldots, 1993)$ coincides with the sign at this point itself after the first change of signs.

3.4.8 (1996)

Let n be a positive integer. Find the number of ordered k-tuples $(a_1, a_2, \ldots, a_k)$, $k \leq n$ from $(1, 2, \ldots, n)$ satisfying at least one of the following conditions:

(1) There exist $s, t \in \{1, 2, \ldots, k\}$ such that $s < t$ and $a_s > a_t$,

(2) There exists $s \in \{1, 2, \ldots, k\}$ such that $a_s - s$ is an odd number.

3.4.9 (1997)

Suppose that there are 75 points inside a unit cube such that no three points are collinear. Prove that it is possible to choose three points from those given above which form a triangle with the area at most $\frac{7}{12}$.

3.4.10 (2001)

Given a positive integer n, let $(a_1, a_2, \ldots, a_{2n})$ be a permutation of $(1, 2, \ldots, 2n)$ such that the numbers $|a_{i+1} - a_i|$ $(i = 1, 2, \ldots, 2n-1)$ are distinct. Prove that $a_1 - a_{2n} = n$ if and only if $1 \le a_{2k} \le n$ for all $k = 1, 2, \ldots, n$.

3.4.11 (2004 B)

Let $n \ge 2$ be an integer. Prove that for each integer k with $2n - 3 \le k \le \frac{n(n-1)}{2}$, there exist n distinct real numbers $a_1, \ldots, a_n$ such that among all numbers of the form $a_i + a_j$ $(1 \le i < j \le n)$ there are exactly k distinct numbers.

3.4.12 (2005)

Let $A_1A_2 \ldots A_8$ be an octagon such that no three diagonals have a common point. Denote by S the set of the intersections of the diagonals of the octagon. Let $T \subset S$ and i, j be two numbers satisfying $1 \le i < j \le 8$. Denote by $S(i, j)$ the number of quadrilaterals with vertices in $\{A_1, A_2, \ldots, A_8\}$ such that A_i, A_j are vertices and the intersection of its diagonals is the intersection of diagonals of the given octagon. Assume that $S(i, j)$ is the same number for all $1 \le i < j \le 8$. Determine the smallest possible $|T|$ for all such subsets T of S.

3.4.13 (2006)

Suppose we have a table $m \times n$ of unit squares, where $m, n \ge 3$. We are allowed to put each time, 4 balls into 4 squares of the following forms (see Fig. 3.1):

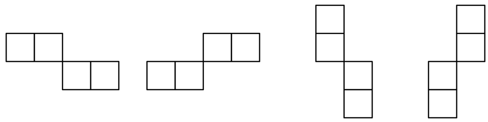

Figure 3.1:

Is it possible to have all squares having the same number of balls, if

1) $m = 2004, n = 2006$?

2) $m = 2005, n = 2006$?

3.4.14 (2007)

Given a regular polygon with 2007 vertices, find the smallest positive number k satisfying the property that for any choice of k vertices there always exists 4 vertices forming a convex quadrilateral whose 3 sides are sides of the given polygon.

3.4.15 (2008)

Determine the number of positive integers, each of which satisfies the following properties:

(1) It is divisible by 9,

(2) It has not more than 2008 digits,

(3) There are at least two digits 9.

3.5 Geometry

Plane Geometry

3.5.1 (1963)

The triangle ABC has half-perimeter p. Find the length of the side a and the area S in terms of $\angle A, \angle B$ and p. In particular, find S if $p = 23.6,\ \angle A = 52°42,\ \angle B = 46°16$.

3.5.2 (1965)

At a time $t = 0$, a navy ship is at a point O, while an enemy ship is at a point A cruising with speed v perpendicular to $OA = a$. The speed and direction of the enemy ship do not change. The strategy of the navy ship is to travel with constant speed u at a angle $0 < \varphi < \frac{\pi}{2}$ to the line OA.

1) Let φ be chosen. What is the minimum distance between the two ships? Under what conditions will the distance vanish?

2) If the distance does not vanish, what is the choice of φ to minimize the distance? What are directions of the two ships when their distance is minimum?

3.5.3 (1968)

Let (I, r) be a circle centered at I of radius r, x and y be two parallel lines on the plane with a distance h apart. A variable triangle ABC with A on x, B and C on y has (I, r) as its in-circle.

1) Given (I, r), α and x, y, construct a triangle ABC so that $\angle A = \alpha$.

2) Calculate angles $\angle B$ and $\angle C$ in terms of h, r and α.

3) If the in-circle touches the side BC at D, find a relation between DB and DC.

3.5.4 (1974)

Let ABC be a triangle with $A = 90°$, AH the altitude, P, Q the feet of the perpendiculars from H to AB, AC respectively. Let M be a variable point on the line PQ. The line through M perpendicular to MH meets the lines AB, AC at R, S respectively.

1) Prove that a circum-circle of ARS always passes the fixed point H.

2) Let M_1 be another position of M with corresponding points R_1, S_1. Prove that the ratio $\frac{RR_1}{SS_1}$ is constant.

3) The point K is symmetric to H with respect to M. The line through K perpendicular to the line PQ meets the line RS at D. Prove that $\angle BHR = \angle DHR, \angle DHS = \angle CHS$.

3.5.5 (1977)

Show that there are 1977 non-similar triangles whose angles A, B, C satisfy the following conditions:

(1) $\dfrac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \dfrac{12}{7}$,

(2) $\sin A \sin B \sin C = \frac{12}{25}$.

3.5.6 (1979)

Let ABC be a triangle with sides that are not equal. Find point X on BC such that

$$\frac{\text{area } \Delta ABX}{\text{area } \Delta ACX} = \frac{\text{perimeter } \Delta ABX}{\text{perimeter } \Delta ACX}.$$

3.5.7 (1982)

Let ABC be a triangle. Consider equilateral triangles $A'BC, A''BC$, where A' is on the different side and A'' is on the same side of BC as A; other points B', B'', C', C'' are defined similarly. Denote by Δ and Δ' the triangles whose vertices are centers of equilateral triangles $A'BC, B'CA, C'AB$ and $A''BC, B''CA, C''AB$ respectively. Prove that $S_{ABC} = S_{\Delta} - S_{\Delta'}$.

3.5.8 (1983)

Let M be a variable point inside a triangle ABC, and D, E and F be the feet of the perpendiculars from M to the sides of the triangle. Find the locus of M such that the area of a triangle DEF is constant.

3.5.9 (1989)

Let $ABCD$ be a square of side 2. The segment AB is moving continuously until it coincides with the segment CD ($A \equiv C, B \equiv D$). Denote by S the area of the figure that AB passed over. Show that we can have $S < \frac{5}{6}\pi$ (note that if some area is passed over twice, then it is counted once only).

3.5.10 (1990)

Let ABC be a triangle in the plane and M be a variable point. Denote by A', B' and C' the feet of the perpendiculars from M to the lines BC, CA and AB respectively. Find the locus of M such that

$$MA \cdot MA' = MB \cdot MB' = MC \cdot MC'.$$

3.5.11 (1991)

Let ABC be a triangle with centroid G and circum-radius R. The lines AG, BG and CG meet the circum-circle again at D, E and F, respectively. Prove that

$$\frac{3}{R} \le \frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \le \sqrt{3}\left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right).$$

3.5.12 (1992)

Let $\mathcal{H}$ be a rectangle in which the angle between its diagonals is not greater than $45°$. Rectangle $\mathcal{H}$ is rotated around its center at an angle θ, $0° \le \theta \le 360°$, to get a rectangle $\mathcal{H}_\theta$. Find θ such that the common area between $\mathcal{H}$ and $\mathcal{H}_\theta$ attains its minimum value.

3.5.13 (1994)

Let ABC be a triangle in the plane. Let A', B', C' be the reflections of the vertices A, B, C with respect to the sides BC, CA, AB respectively. Find the necessary and sufficient conditions on the nature of ABC so that the triangle $A'B'C'$ is equilateral.

3.5.14 (1997)

Suppose in the plane we have a circle with center O, radius a. let P be a point lying inside this circle ($OP = d < a$). Among all convex quadrilaterals $ABCD$ inscribed in the circle such that their diagonals AC and BD are perpendicular at P, determine the quadrilateral having the greatest perimeter and the quadrilateral having the smallest perimeter. Calculate the perimeters in terms of a and d.

3.5.15 (1999)

Let ABC be a triangle. Denote by A', B', C' the midpoints of the arcs BC, CA, AB of the circum-circle that do not contain A, B, C respectively. The sides BC, CA, AB of the triangle intersect the pairs of the segments $A'C', A'B'; B'A', B'C'; C'B', C'A'$ at M, N, P, Q, R, S. Prove that $MN = PQ = RS$ if and only if ABC is equilateral.

3.5.16 (2001)

Suppose two circles (O_1) and (O_2) in the plane intersect at two points A and B, and P_1P_2 is a common tangent to these circles, $P_1 \in (O_1), P_2 \in (O_2)$. The orthogonal projections of P_1, P_2 on the line O_1O_2 are denoted by M_1, M_2 respectively. The line AM_i intersects again (O_i) at the second point N_i $(i = 1, 2)$. Prove that three points N_1, B and N_2 are collinear.

3.5.17 (2003)

Let two fixed circles (O_1, R_1) and (O_2, R_2) be given in the plane. Suppose (O_1) and (O_2) intersect at a point M and $R_2 > R_1$. Consider a point A lying on the circle (O_2) such that three points O_1, O_2, A are non-collinear. From A draw the tangents AB and AC to the circle (O_1) $(B, C$ are points of tangency). The lines MB, MC intersect again the circle (O_2) at E, F. The point of intersection of EF and the tangent at A of the circle (O_2) is D. Prove that D moves on a fixed line when A moves on the circle (O_2) such that three points O_1, O_2, A are non-collinear.

3.5.18 (2004 B)

Given an acute triangle ABC, with the orthocenter H, inscribed in a circle (O). On the arc BC not containing A of the circle (O) take a point P different from B, C. Let D satisfy $\overrightarrow{AD} = \overrightarrow{PC}$ and K be the orthocenter of the triangle ACD. Denote by E, F the feet of perpendiculars from K to the lines BC, AB respectively. Prove that the line EF passes the midpoint of the segment HK.

3.5.19 (2005)

Given a circle (O) centered at O of radius R and two fixed points A, B on the circle such that A, B, O are not collinear. Let C be a variable point on (O), different from A, B. The circle (O_1) passes through A and is tangent to the line BC at C, and the circle (O_2) passes through B and is tangent to the line AC at C. These two circles intersect, besides C, at the second point D. Prove that

1) $CD \leq R$,

2) The line CD always passes through a fixed point when C moves on the circle (O) not coinciding with A and B.

3.5.20 (2006 B)

Given an isosceles trapezoid $ABCD$ (CD is the longest base). A variable point M is moving on the line CD so that it does not coincide with C nor D. Let N be another intersection of two circles passing triples B, C, M and D, A, M. Prove that

1) The point N is always on the fixed circle,

2) The line MN always passes through a fixed point.

3.5.21 (2007)

Given a triangle ABC, where the two vertices B, C are fixed and a vertex A varies. Let H and G be the orthocenter and centroid of ΔABC, respectively. Find the locus of A, if the midpoint K of HG belongs to the line BC.

3.5.22 (2007)

Let $ABCD$ be a trapezoid $ABCD$ with the bigger base BC inscribed in the circle (O) centered at O. Let P be a variable point on the line BC and outside of the segment BC such that PA is not tangent of (O). A circle of diameter PD meets (O) at E different from D. Denote by M the intersection of BC and DE, and by N the second intersection of PA and (O). Prove that the line MN passes the fixed point.

3.5.23 (2008)

Let ABC be a triangle and E be the midpoint of the side AB. Let M be a point on the ray EC such that $\widehat{BME} = \widehat{ECA}$. Compute the ratio $\frac{MC}{AB}$ in terms of $\alpha = \widehat{BEC}$.

Solid Geometry

3.5.24 (1962)

Given a pyramid $SABCD$ such that the base $ABCD$ is a square with the center O, and $SO \perp ABCD$. The height SO is h and the angle between

SAB and $ABCD$ is α. The plane passing through the edge AB is perpendicular to the opposite face SCD. Find the volume of the prescribed pyramid. Analyze the formula obtained.

3.5.25 (1963)

The tetrahedron $SABC$ has the perpendicular faces SBC and ABC. The three angles at S are all $60°$ and $SB = SC = 1$. Find the volume of the tetrahedron.

3.5.26 (1964)

Let P be a plane and two points $A \in (P), O \notin (P)$. For each line in (P) through A, let H be the foot of the perpendicular from O to the line. Find the locus (c) of H.
Denote by $(\mathcal{C})$ the oblique cone with peak O and base (c). Prove that all planes, either parallel to (P) or perpendicular to OA, intersect $(\mathcal{C})$ by circles.
Consider the two symmetric faces of $(\mathcal{C})$ that intersect $(\mathcal{C})$ by the angles α and β respectively. Find a relation between α and β.

3.5.27 (1970)

A plane (P) passes through a vertex A of a cube $ABCDEFGH$ and the three edges AB, AD, AE make equal angles with (P).

1) Compute the cosine of that common angle and find the perpendicular projection of the cube onto the plane.

2) Find some relationships between (P) and lines passing through two vertices of the cube and planes passing through three vertices of the cube.

3.5.28 (1972)

Let $ABCD$ be a regular tetrahedron with side a. Take E, E' on the edge AB, F, F' on the edge AC and G, G' on the edge AD so that $AE = a/6, AE' = 5a/6; AF = a/4, AF' = 3a/4; AG = a/3, AG' = 2a/3$. Compute the volume of $EFGE'F'G'$ in term of a and find the angles between the lines AB, AC, AD and the plane EFG.

3.5.29 (1975)

Let $ABCD$ be a tetrahedron with $BA \perp AC, DB \perp (BAC)$. Denote by O the midpoint of AB, and K the foot of the perpendicular from O to DC. Suppose that $AC \neq BD$. Prove that

$$\frac{V_{KOAC}}{V_{KOBD}} = \frac{AC}{BD}$$

if and only if $2AC \cdot BD = AB^2$.

3.5.30 (1975)

In the space given a fixed line Δ and a fixed point $A \notin \Delta$, a variable line d passes through A. Denote by MN the common perpendicular between d and Δ $(M \in d, N \in \Delta)$. Find the locus of M and the locus of the midpoint I of MN.

3.5.31 (1978)

Given a rectangular parallelepiped $ABCDA'B'C'D'$ with the bases $ABCD$, $A'B'C'D'$, the edges AA', BB', CC', DD' and $AB = a, AD = b, AA' = c$. Show that there exists a triangle with the sides equal to the distances from A, A', D to the diagonal BD' of the parallelepiped. Denote those distances by m_1, m_2, m_3. Find the relationship between a, b, c, m_1, m_2, m_3.

3.5.32 (1984)

Let $Sxyz$ be a trihedral angle with $\widehat{xSy} = \widehat{ySz} = \widehat{zSx} = 90°$, O be a fixed point on Sz with $SO = a$. Consider two variable points $M \in Sx, N \in Sy$ with $SM + SN = a$.

(1) Prove that $\widehat{SOM} + \widehat{SON} + \widehat{MON}$ is constant.

(2) Find the locus of the circum-sphere of $OSMN$.

3.5.33 (1985)

Let $OABC$ be a tetrahedron with base ABC of area S. The altitudes from A, B and C are at least half of $OB + OC, OC + OA$ and $OA + OB$, respectively. Find the volume of the tetrahedron.

3.5.34 (1986)

Let $ABCD$ be a square, and ABM be an equilateral triangle in the plane perpendicular to $ABCD$. Let further, E be the midpoint of AB, O the midpoint of CM. A variable point S on the side AB is of a distance x from B.

(1) Find the locus P of the food of the perpendicular from M to the side CS.

(2) Find the maximum and minimum values of SO.

3.5.35 (1990)

Let $ABCD$ be a tetrahedron of volume V. We wish to make three plane cuts to obtain a parallelepiped three of whose faces and all of whose vertices belong to the surface of the tetrahedron.

1) Is it possible to have a parallelepiped whose volume is $\dfrac{9V}{40}$? Justify your answer.

2) Find the intersection of the three planes so that the volume of the parallelepiped is $\dfrac{11V}{50}$.

3.5.36 (1990 B)

Let $SABC$ and $RDEF$ be two equilateral triangle pyramids that satisfy the following properties: the two vertices R and S are centroids of ABC and DEF respectively; and each pair of the edges AB and EF, AC and DE, BC and DF are parallel and equal.

1) How to construct a common part of the two pyramids?

2) Compare the volumes of the common part and the pyramid $SABC$.

3.5.37 (1991)

Let $Oxyz$ be a right trihedral angle and A, B, C be three fixed points on Ox, Oy, Oz, respectively. A variable sphere $\mathcal{S}$ passes through A, B, C and meets Ox, Oy, Oz at A', B', C', respectively. Let M, M' be the centroids of $A'BC, AB'C'$, respectively. Find the locus of the midpoint S of MM'.

3.5.38 (1991 B)

Let two concentric spheres of radii R and r with $R > r > 0$ be given. Find conditions for R, r under which we can construct an equilateral tetrahedron $SABC$ so that the three vertices A, B, C are on the bigger sphere and the three faces SAB, SBC, SCA are tangent to the smaller sphere.

3.5.39 (1992)

A tetrahedron $ABCD$ has the following properties:

(1) $\widehat{ACD} + \widehat{BCD} = 180°$,

(2) The sum of the three plane angles at A equals to the sum of the three plane angles at B, and both sums are equal to $180°$.

Compute the surface area of the tetrahedron $ABCD$ in terms of $AC+CB = k$ and $\widehat{ACB} = \alpha$.

3.5.40 (1993)

A variable tetrahedron $ABCD$ is inscribed in a given sphere. Show that the sum

$$AB^2 + AC^2 + AD^2 - BC^2 - CD^2 - DB^2$$

attains its minimal value if and only if the trihedral angle at the vertex A is rectangular.

3.5.41 (1995 B)

Consider a sphere centered at I, a fixed point P inside the sphere and another fixed point Q different from I. For each variable tetrahedron $ABCD$ inscribed in the sphere with the centroid P, let A' be the projection of Q on the plane tangent with the sphere at A. Prove that the centroid of the tetrahedron $A'BCD$ is always on the fixed sphere.

3.5.42 (1996)

Given a trihedral angle $Sxyz$, a plane (P), not passing through S, meets Sx, Sy, Sz at A, B, C, respectively. In (P) there are three triangles DAB, EBC, FCA outside of ABC such that $\Delta DAB = \Delta SAB$, $\Delta EBC =$

ΔSBC, $\Delta FCA = \Delta SCA$. Consider a sphere $\mathcal{S}$ satisfying the following conditions:

(1) $\mathcal{S}$ tangents $(SAB), (SBC), (SCA)$ and (P),

(2) $\mathcal{S}$ is inside the trihedral angle $Sxyz$ and outside of the tetrahedron $SABC$.

Prove that $\mathcal{S}$ and (P) are tangent at the circum-center of DEF.

3.5.43 (1996 B)

A tetrahedron $ABCD$ with $AB = AC = AD$ is inscribed in a sphere centered at O. Let G be the centroid of ACD, let E be the midpoint of BG and let F be the midpoint of AE. Prove that $OF \perp BG$ if and only if $OD \perp AC$.

3.5.44 (1998)

Let O be the center of the circum-sphere of the tetrahedron $ABCD$, AA_1, BB_1, CC_1 and DD_1 the diameters of this sphere. Let A_0, B_0, C_0 and D_0 be centroids of the triangles BCD, CDA, DAB and ABC respectively. Show that

1) The lines A_0A_1, B_0B_1, C_0C_1 and D_0D_1 meet at a point, called the point F,

2) The line passing through F and the midpoint of a side of the tetrahedron is perpendicular to the opposite side of the tetrahedron.

3.5.45 (1998 B)

Let P be a point on the sphere $\mathcal{S}$ of a radius R. Consider all pyramids $PABC$ with the right trihedral angles at the vertex P and A, B, C are points on the sphere. Prove that the plane (ABC) always passes through a fixed point and find the maximum value of the area of ΔABC.

3.5.46 (1999)

Consider fours rays Ox, Oy, Oz and Ot in space such that the angles between any two rays are equal.

1) Determine the value of the angle between any two rays.

2) Let Or be a variable ray and $\alpha, \beta, \gamma, \delta$ the angles between Or and other three rays. Prove that $\cos\alpha + \cos\beta + \cos\gamma + \cos\delta$ and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ are constants.

3.5.47 (2000 B)

For the tetrahedron $ABCD$, the radii of the circum-circles of ABC, ACD, ABD and BCD are equal. Prove that $AB = CD, AC = BD$ and $AD = BC$.

3.5.48 (2000)

Find all positive integers $n > 3$ such that there exist n points in space satisfying the following conditions:

(1) No three points are collinear,

(2) No fours points are concyclic,

(3) The circles passing through any three points all have the same radius.

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Chapter 4

Solutions

4.1 Algebra

4.1.1

We prove that

$$\frac{1}{\frac{1}{a+c}+\frac{1}{b+d}}-\left(\frac{1}{\frac{1}{a}+\frac{1}{b}}+\frac{1}{\frac{1}{c}+\frac{1}{d}}\right)\geq 0.$$

A straightforward summing up and simplification show that this is equivalent to

$$\frac{a^2d^2-2abcd+b^2c^2}{(a+b+c+d)(a+b)(c+d)}\geq 0,$$

which is always true, as the denominator is positive, and the numerator is $(ad-bc)^2 \geq 0$.
The equality occurs if and only if $ad = bc$.

4.1.2

Using $\cos x+\cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$, we have

$$\cos\left(\alpha+\frac{2\pi}{3}\right)+\cos\left(\alpha+\frac{4\pi}{3}\right)=2\cos(\alpha+\pi)\cos\frac{\pi}{3}=\cos(\alpha+\pi)=-\cos\alpha,$$

and hence the first sum is 0.

Similarly, using $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$, we find that the second sum is also 0.

Generally, for any angle α and positive integer n, we always have

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{n}\right) + \cdots + \cos\left(\alpha + \frac{2\pi(n-1)}{n}\right) = 0$$

and

$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{n}\right) + \cdots + \sin\left(\alpha + \frac{2\pi(n-1)}{n}\right) = 0.$$

This can be seen by using vector geometry. On the unit circle, consider n vectors $\overrightarrow{OA_1}, \overrightarrow{OA_2}, \ldots, \overrightarrow{OA_n}$ that form with the x-axis angles $\pi, \pi + \frac{2\pi}{n}, \ldots, \pi + \frac{2(n-1)\pi}{n}$, so that $A_1A_2\ldots A_n$ is a regular polygon. Due to symmetry, $\overrightarrow{OA} = \overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n} = \overrightarrow{0}$, and hence $\mathrm{pr}_x\overrightarrow{OA} = \mathrm{pr}_y\overrightarrow{OA} = 0$, where $\mathrm{pr}_x\overrightarrow{a}$ and $\mathrm{pr}_y\overrightarrow{a}$ are the projections of $\overrightarrow{a}$ on the x-axis and y-axis, respectively. Using $\mathrm{pr}_x(\overrightarrow{a} + \overrightarrow{b}) = \mathrm{pr}_x\overrightarrow{a} + \mathrm{pr}_x\overrightarrow{b}$, $\mathrm{pr}_y(\overrightarrow{a} + \overrightarrow{b}) = \mathrm{pr}_y\overrightarrow{a} + \mathrm{pr}_y\overrightarrow{b}$, we get the desired results.

4.1.3

For $c \geq 3$ we have

$$\begin{cases} x + cy \leq 36, \\ 2x + 3z \leq 72, \end{cases}$$

and hence $3x + 3z + cy \leq 108$, or $3(x + y + z) \leq 108 - (c-3)y$.

Note that since $c \geq 3,\ y \geq 0$, we get $108 - (c-3)y \leq 108$, and therefore $x + y + z \leq 36$.

For $c < 3$ we first notice that $cy \leq 36 - x$, or

$$3y \leq \frac{108 - 3x}{c} \text{ (as } c > 0).$$

Also,

$$3x + 3z \leq 72 + x.$$

From the last two inequalities it follows that

$$3(x + y + z) \leq \frac{108}{c} + 72 - \frac{3-c}{c}x \leq \frac{108}{c} + 72,$$

or equivalently, $x + y + z \leq \frac{36}{c} + 24$. In both cases the equality obviously occurs.

4.1.4

1) Let $k = n - m \in (0, n)$. Consider $a^m b - ab^m = ab(a^{m-1} - b^{m-1})$ with $m - 1 \geq 0$. Since $a \geq b > 0$, we have $a^{m-1} \geq b^{m-1}$. Hence

$$a^m b \geq ab^m. \tag{1}$$

On the other hand, notice that $a \geq b, a^2 \geq b^2, \ldots, a^{k-1} \geq b^{k-1}$, which implies

$$1 + a + a^2 + \cdots + a^{k-1} \geq 1 + b + b^2 + \cdots + b^{k-1}. \tag{2}$$

From (1) and (2) it follows that

$$a^m b(1 + a + a^2 + \cdots + a^{k-1}) \geq ab^m(1 + b + b^2 + \cdots + b^{k-1}),$$

which can be written as

$$a^m(1-a)(1 + a + a^2 + \cdots + a^{k-1}) \geq b^m(1-b)(1 + b + b^2 + \cdots + b^{k-1}),$$

or equivalently, $a^m(1 - a^k) \geq b^m(1 - b^k)$. That is, $a^m - a^n \geq b^m - b^n$.

It remains to prove that $b^m - b^n > 0$. Indeed, $b^m - b^n = b^m(1 - b^k) > 0$, as $0 < b < 1$.

The equality occurs if and only if $a = b = 1/2$.

2) Since $\Delta = b^{2n} + 4a^n > 0$, $f_n(x)$ has two distinct real roots $x_1 \neq x_2$. Also, note that if $a, b \in (0, 1)$ then

$$\begin{cases} f_n(1) = 1 - b^n - a^n = a + b - b^n - a^n = (a - a^n) + (b - b^n) \geq 0, \\ f_n(-1) = 1 + b^n - a^n = (1 - a^n) + b^n \geq 0, \\ \frac{S}{2} = \frac{x_1 + x_2}{2} = \frac{b^n}{2} \in (-1, 1). \end{cases}$$

We conclude that $x_1, x_2 \in [-1, 1]$.

4.1.5

From $(x_i - a)^2 + (y_i - b)^2 \leq c^2$ it follows that $a^2 + b^2 - 2ax_i - 2by_i \leq c^2$, as $x_i^2 + y_i^2 > 0$. In particular, for $i = 1, 2$ we have

$$\begin{cases} a^2 + b^2 - 2ax_1 - 2by_1 \leq c^2, \\ a^2 + b^2 - 2ax_2 - 2by_2 \leq c^2. \end{cases}$$

Choose k so that $kx_1 + (1-k)x_2 = 0$. That is, $k = \dfrac{x_2}{x_2 - x_1}$. Since $x_1 x_2 < 0$, we have $k \in (0, 1)$. Sum up the last two inequalities, after first multiplying the first by k and the second by $(1 - k)$. We get

$$a^2 + b^2 - 2a[kx_1 + (1-k)x_2] - 2b[ky_1 + (1-k)y_2] \leq c^2,$$

or equivalently,

$$a^2 + b^2 - 2b[ky_1 + (1-k)y_2] \leq c^2.$$

Since $y_1, y_2 > 0$, we get $Y_1 = ky_1 + (1-k)y_2 > 0$ with $a^2 + b^2 - 2bY_1 \leq c^2$.

Similarly, for $i = 3, 4$ by choosing m so that $mx_3 + (1-m)x_4 = 0$, we obtain that $a^2 + b^2 - 2bY_2 \leq c^2$ with $Y_2 = my_3 + (1-m)y_4 < 0$.

Finally, choosing n such that $nY_1 + (1-n)Y_2 = 0$, and repeating the same argument, we find

$$a^2 + b^2 - 2b[nY_1 + (1-n)Y_2] = a^2 + b^2 \leq c^2.$$

Geometrically, this specifies a circle of radius c centered at (a, b) in the rectangular system of coordinates Oxy, and four points (x_i, y_i) in fours quadrants, respectively. The assumptions of the problem show that all points (x_i, y_i) are inside of the circle, while the conclusion says that the origin of coordinates is inside the circle. So we can restate the problem as follows: *Let two straight-lines, perpendicular at a point O, and four points which are in four performed quadrants, be given. Then a circle containing all fours given points must contain O.*

4.1.6

On the one hand, since $A + B + C = \pi$, there is at least one angle, say C, with $C \leq \frac{\pi}{3}$. Then $\sin\frac{C}{2} \leq \sin\frac{\pi}{6} = \frac{1}{2}$.

On the other hand, $\frac{A}{2} = \frac{\pi}{2} - \frac{B+C}{2} < \frac{\pi}{2} - \frac{B}{2} < \frac{\pi}{2}$, and hence $\sin\frac{A}{2}\sin\frac{B}{2} < \sin\left(\frac{\pi}{2} - \frac{B}{2}\right)\sin\frac{B}{2} = \cos\frac{B}{2}\sin\frac{B}{2} = \frac{1}{2}\sin B$. Thus $\sin\frac{A}{2}\sin\frac{B}{2} < \frac{1}{2}$.

The desired inequality follows.

Remark. In fact, we can prove that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$ always holds, and the equality occurs for the regular triangle.

4.1.7

1) For each $x \in [-1, 1]$ there exists a unique $\alpha_0 \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ such that $\cos\alpha_0 = x$. The other solutions of $\cos\alpha = x_0$ are $\alpha = \pm\alpha_0 + 2k\pi$ $(k \in \mathbb{Z})$. Then

$$y = \cos n\alpha = \cos[n(\pm\alpha_0 + 2k\pi)] = \cos(\pm n\alpha_0 + 2nk\pi) = \cos n\alpha_0,$$

which shows that for any α satisfying $\cos\alpha = x$ we always have $y = \cos n\alpha_0$. Thus y is defined uniquely.

We have

$$T_1(x) = \cos\alpha = x,$$
$$T_2(x) = \cos 2\alpha = 2\cos^2\alpha - 1 = 2x^2 - 1.$$

Furthermore,

$$\begin{aligned} T_{n-1}(x) + T_{n+1}(x) &= \cos(n-1)\alpha + \cos(n+1)\alpha \\ &= 2\cos n\alpha \cos\alpha \\ &= 2xT_n(x). \end{aligned}$$

We notice that T_1 is a polynomial of degree 1 with the leading coefficient 1 and T_2 is of degree 2 with the leading coefficient 2. Therefore, from this equation, by induction, we can show that T_n is a polynomial of degree n with leading coefficient 2^{n-1}.

2) We have $T_n(x) = 0$ if and only if $\cos n\alpha = 0$, or $n\alpha = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$, or equivalently,

$$\alpha = \frac{\pi}{2n} + \frac{2k\pi}{2n} := \alpha_k \ (k \in \mathbb{Z}).$$

These angles α_k give n distinct roots $x_k = \cos\alpha_k$ of $T_n(x)$ in the interval $[-1, 1]$, corresponding to $k = 0, 1, \ldots, n-1$.

4.1.8

If x_1, x_2, x_3 are roots of the give cubic equation then, by the Viète formula, we have

$$\begin{cases} x_1 + x_2 + x_3 = 0, \\ x_1x_2 + x_2x_3 + x_3x_1 = -1, \\ x_1x_2x_3 = -1. \end{cases}$$

Furthermore, from $x_i^3 - x_i + 1 = 0$ it follows that

$$x_i^3 = x_i - 1,$$
$$x_i^5 = x_i^3.x_i^2 = (x_i - 1)x_i^2 = x_i^3 - x_i^2 = -x_i^2 + x_i - 1,$$
$$x_i^8 = x_i^5.x_i^3 = (-x_i^2 + x_i - 1)(x_i - 1) = -x_i^3 + 2x_i^2 - 2x_i + 1 = 2x_i^2 - 3x_i + 2.$$

Then

$$x_1^8 + x_2^8 + x_3^8 = 2(x_1^2 + x_2^2 + x_3^2) - 3(x_1 + x_2 + x_3) + 6.$$

But

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1) = 2,$$

and so $x_1^8 + x_2^8 + x_3^8 = 4 - 0 + 6 = 10$.

4.1.9

The equation is defined if $x \neq -m, -n, -p$.

Note that
$$\frac{x^3+s^3}{(x+s)^3} = \frac{1}{4} + \frac{3}{4}\frac{(x-s)^2}{(x+s)^2},\ x \neq -s.$$

The equation can be rewritten as

$$\frac{1}{4} + \frac{3}{4}a^2 + \frac{1}{4} + \frac{3}{4}b^2 + \frac{1}{4} + \frac{3}{4}c^2 - \frac{3}{2} + \frac{3}{2}abc = 0,$$

where
$$a = \frac{x-m}{x+m}, b = \frac{x-n}{x+n}, c = \frac{x-p}{x+p}.$$

Simplifying the equation yields $(c+ab)^2 = (1-a^2)(1-b^2)$, or equivalently,
$$x^2[(x^2+mn-mp-np)^2 - 4mn(x+p)^2] = 0.$$

We can write the last equation as

$$x^2[x^2+mn-mp-np+2(x+p)\sqrt{mn}]\cdot[x^2+mn-mp-np-2(x+p)\sqrt{mn}] = 0,$$

which gives

(i) $x^2 = 0$, that is, $x_{1,2} = 0$.

(ii) $x^2+mn-mp-np+2(x+p)\sqrt{mn} = 0$, or $x+\sqrt{mn} = \pm(\sqrt{mp}-\sqrt{np})$, and hence $x_{3,4} = \pm(\sqrt{mp}-\sqrt{np}) - \sqrt{mn}$.

(iii) $x^2+mn-mp-np-2(x+p)\sqrt{mn} = 0$, or $x_{5,6} = \sqrt{mn}\pm(\sqrt{np}-\sqrt{mn})$.

Removing those values of x which satisfy $(x+m)(x+n)(x+p) = 0$, we get the solutions of the given equation.

4.1.10

• If $y = 0$, then (1) takes the from $x^x = 0$, and there is no solution.

• If $y = 1$, then (2) takes the form $x^3 = 1$, which gives $x = 1$. So we have $(x_1, y_1) = (1, 1)$.

• If $y = -1$, then (1) takes the form $x^{x-1} = 1$, which gives $x = \pm 1$, and we have $(x_2, y_2) = (1, -1)$, as the other pair $(-1, -1)$ does not satisfy (2).

Now consider $y \neq 0, \pm 1$. From (2) it follows that

$$x = y^{\frac{x+y}{3}}.$$

Substitute this into (1) we obtain

$$y^{\frac{(x+y)^2}{3}} = y^{12},$$

which gives $(x+y)^2 = 36$, or equivalently, $x + y = \pm 6$.

(i) If $x + y = 6$ then (2) gives $y^6 = x^3$, or equivalently, $x = y^2$. So $y^2 + y = 6$ gives $y = 2, -3$ and hence $(x_3, y_3) = (4, 2)$ and $(x_4, y_4) = (9, -3)$.

(ii) If $x + y = -6$ then (2) gives $y^3 + 6y^2 + 1 = 0$. By the theorem on integer solutions of polynomials with integer coefficients, the only possible integer solutions should be ± 1, and both of ± 1 do not satisfy the equation.

Thus there are four integer solutions.

4.1.11

Since $x_i > 0$, for all positive integers n we have $x_i^{-n} > 0$. By the Arithmetic-Geometric Mean inequality

$$x_1^{-n} + \cdots + x_k^{-n} \geq k \sqrt[k]{\frac{1}{x_1^n \cdots x_k^n}}$$

and

$$x_1 \cdots x_k \leq \left(\frac{x_1 + \cdots + x_k}{k}\right)^k = \left(\frac{1}{k}\right)^k.$$

These inequalities give

$$x_1^{-n} + \cdots + x_k^{-n} \geq k \sqrt[k]{k^{kn}} = k^{n+1}.$$

The equality occurs if and only if $x_1 = \cdots = x_k = \frac{1}{k}$.

4.1.12

Note that $x = 1$ is not a solution. So the inequality is defined for $-1 \leq x < 0$ and $x > 1$.

As $\sqrt{\frac{x-1}{x}} > 0$, we can divide both sides by $\sqrt{\frac{x-1}{x}}$ to obtain

$$\sqrt{x+1} > 1 + \sqrt{\frac{x-1}{x}}. \tag{1}$$

For $-1 \le x < 0$ the left-hand side of (1) is less than 1, while the right-hand side is greater than 1. So we consider $x > 1$. In this case, by squaring (1), we have

$$x - 1 + \frac{1}{x} > 2\sqrt{\frac{x-1}{x}}. \tag{2}$$

By the Arithmetic-Geometric Mean inequality, the left-hand side of (2) is greater than or equal to the right-hand side. The equality occurs if and only if $x - 1 = \frac{1}{x}$, that is $x = \frac{1 \pm \sqrt{5}}{2}$; however, the smaller value is not in the interval $(1, \infty)$.

Thus the solutions are all $x > 1$ $(x \ne \frac{1+\sqrt{5}}{2})$.

4.1.13

Let $b_k = \dfrac{k(n-k+1)}{2}$, we can verify that

$$b_0 = b_{n+1} = 0,$$

and that

$$b_{k-1} - 2b_k + b_{k+1} = -1 \quad (k = 1, \dots, n).$$

Assume that there is an index i such that $a_i > b_i$. Then the sequence

$$a_0 - b_0, a_1 - b_1, \dots, a_{n+1} - b_{n+1} \tag{1}$$

contains at least one positive term. Let $a_j - b_j$ be the biggest term in (1), and choose j such that $a_{j-1} - b_{j-1} < a_j - b_j$.

If either $j = 0$, or $j = n + 1$, the required inequalities are obvious.

For $1 \le j \le n$ we have

$$(a_{j-1} - b_{j-1}) + (a_{j+1} - b_{j+1}) < 2\,(a_j - b_j)\,. \tag{2}$$

By the assumption, $a_{k-1} - 2a_k + a_{k+1} \ge -1$. Substituting $-1 = b_{k-1} - 2b_k + b_{k+1}$ into this, we obtain

$$(a_{k-1} - b_{k-1}) - 2\,(a_k - b_k) + (a_{k+1} - b_{k-1}) \ge 0, \ \forall k = 1, \dots, n.$$

In particular, for $k = j$ we get

$$(a_{j-1} - b_{j-1}) + (a_{j+1} - b_{j+1}) \ge 2\,(a_j - b_j)\,,$$

which contradicts (2). Thus $a_k \le b_k$ for all $0 \le k \le n + 1$.

The inequalities $a_k \ge -b_k$ are proved in a similar way. This completes the proof.

4.1.14

Necessity. Note that if (x, y) is a solution then $(-x, y)$ is also a solution. Therefore, we should have $x = -x$, or $x = 0$. Substituting $x = 0$ into the system, we get

$$\begin{cases} m = 1 - y, \\ y^2 = 1, \end{cases}$$

which gives $m = 0, 2$.

Sufficiency. For $m = 0$ we have

$$\begin{cases} x^2 = 2^{|x|} + |x| - y, \\ x^2 + y^2 = 1. \end{cases}$$

From the second equation it follows that $|x|, |y| \leq 1$. Then, expressing the first equation as $x^2 - |x| + y = 2^{|x|}$, we note that $|x|^2 - |x| = |x|(|x| - 1) \leq 0$ and hence the left-hand side satisfies $x^2 - |x| + y \leq y \leq 1$, while the right-hand side satisfies $2^{|x|} \geq 2^0 \geq 1$. Therefore, the system is equivalent to $x^2 - |x| + y = 2^{|x|} = 1$, which gives $x = 0, y = 1$.

For $m = 2$ we note that the system

$$\begin{cases} x^2 = 2^{|x|} + |x| - y - 2, \\ x^2 + y^2 = 1, \end{cases}$$

has at least the two solutions $(0, -1)$ and $(1, 0)$.

Thus the only value is $m = 0$, with the unique solution $(x, y) = (0, 1)$.

4.1.15

We have

$$\frac{a}{d} - \frac{b}{d} = \frac{b}{d} - \frac{c}{d} \Longleftrightarrow 2b = a + c. \tag{1}$$

Also

$$\frac{b}{a} : \frac{c}{b} = \frac{1+a}{1+d} : \frac{1+b}{1+d} \Longleftrightarrow \frac{b^2}{ac} = \frac{1+a}{1+b}. \tag{2}$$

Substituting $c = 2b - a$ from (1) into (2), we obtain

$$\frac{b^2}{a(2b-a)} = \frac{1+a}{1+b} \Longleftrightarrow b^2(1+b) = a(1+a)(2b-a),$$

or equivalently,

$$(a-b)(a^2-ab-b^2+a-b)=0.$$

If $a=b$, (1) gives $b=c$, and hence $a=b=c=d$. This is impossible. So $a\neq b$ and

$$a^2-ab-b^2+a-b=0,$$

which is equivalent to

$$b^2+(a+1)b-a^2-a=0.$$

This equation with respect to b has roots

$$b=\frac{-(a+1)\pm\sqrt{5a^2+6a+1}}{2}.$$

Since b is an integer, $t=\sqrt{5a^2+6a+1}$ is a rational number. Then $a=\dfrac{-3\pm\sqrt{5t^2+4}}{5}$, which means that $\sqrt{5t^2+4}=s$ must be rational. The last equation is written as

$$s^2-5t^2=4\Longleftrightarrow\left(\frac{s}{2}\right)^2-5\left(\frac{t}{2}\right)^2=1,$$

or equivalently,

$$s_1^2-5t_1^2=1,$$

where $s_1=\dfrac{s}{2},\ t_1=\dfrac{t}{2}$.

The smallest numbers satisfying this equation are $s_1=9, t_1=4$, which give $s=18, t=8$, and so $a=3;\ b=2,-6;\ c=1,-15;\ d=5,-3$. From this we get the fractions $\dfrac{3}{5},\dfrac{2}{5},\dfrac{1}{5}$.

4.1.16

Let $P(x)=x^3+ax^2+bx+c$, with roots t,u,v, and $Q(x)=x^3+a^3x^2+b^3x+c^3$, whose roots are t^3,u^3,v^3, respectively. By the Viète formula, we have

$$\begin{cases} t+u+v=-a,\\ tu+uv+vt=b,\\ tuv=-c, \end{cases}$$

and

$$\begin{cases} t^3 + u^3 + v^3 = -a^3, \\ (tu)^3 + (uv)^3 + (vt)^3 = b^3, \\ (tuv)^3 = -c^3. \end{cases}$$

Note that

$$(t + u + v)^3 = t^3 + u^3 + v^3 + 3(t + u + v)(tu + uv + vt) - 3tuv,$$

which gives $-a^3 = -a^3 - 3ab + 3c$, or equivalently, $c = ab$. In this case $Q(x)$ has the form

$$Q(x) = x^3 + a^3x^2 + b^3x + (ab)^3 = (x + a^3)(x^2 + b^3).$$

This polynomial has a root $x = -a$, and for the other two roots we should have $b \leq 0$. Thus the conditions are

$$\begin{cases} ab = c, \\ b \leq 0. \end{cases}$$

4.1.17

Note that for any real number x we always have

$$[x] \leq x < [x] + 1.$$

Then putting $[x] = y$ and $x - [x] = z$, we have

$$z^2 + z - y^2 + y - \alpha = 0,$$

where y is an integer and $z \in [0, 1)$.

Expressing z in terms of y yields

$$z = \frac{-1 \pm \sqrt{\Delta}}{2}, \ \Delta = 1 + 4(y^2 - y + \alpha).$$

Since $z \geq 0$, we have

$$z = \frac{-1 + \sqrt{\Delta}}{2}. \tag{1}$$

So $0 \leq \frac{-1+\sqrt{\Delta}}{2} < 1$, or, equivalently,

$$0 \leq y^2 - y + \alpha < 2. \tag{2}$$

If $x_1 > x_2$ are two distinct nonnegative roots of the given equation, then $y_1 > y_2$. Indeed, since $[x_i] = y_i$ and $x_i - [x_i] = z_i$ $(i = 1, 2)$, we have $y_1 \geq y_2$. Assume that $y_1 = y_2$. In this case, by (1), $z_1 = z_2$, and so $x_1 = x_2$. This is impossible.

Thus $y_1 > y_2$. From (2) it follows that

$$|y_1^2 - y_1 - y_2^2 + y_2| < 2,$$

or equivalently,

$$(y_1 - y_2)|y_1 + y_2 - 1| < 2.$$

Note that y_1, y_2 are integer, and so $y_1 - y_2 \geq 1$. Then the last inequality shows that $|y_1 + y_2 - 1| = 0,\ 1$.

For $|y_1 + y_2 - 1| = 0$: $y_1 + y_2 = 1$ and hence $y_1 = 1,\ y_2 = 0$.

For $|y_1 + y_2 - 1| = 1$: $y_1 + y_2 = 2$ and so $y_1 = 2,\ y_2 = 0$. But these values do not satisfy $(y_1 - y_2)|y_1 + y_2 - 1| < 2$.

Thus we see that if the given equation has two nonnegative distinct roots $x_1 > x_2$, then $[x_1] = 1,\ [x_2] = 0$. Hence,

$$\begin{cases} x_1 = \frac{\sqrt{1+4\alpha}+1}{2}, \\ x_2 = \frac{\sqrt{1+4\alpha}-1}{2}. \end{cases}$$

Obviously, this equation cannot have more than two distinct roots.

Finally, from (2) it follows that the possible range of α is $0 \leq \alpha < 2$.

4.1.18

By the Arithmetic-Mean, Geometric-Mean and Harmonic-Mean inequalities, we have

$$m_1^2 + \cdots + m_k^2 \geq \left(\frac{m_1 + \cdots + m_k}{k}\right)^2 = \overline{m}^2,$$

and

$$\left(\frac{1}{m_1}\right)^2 + \cdots + \left(\frac{1}{m_k}\right)^2 \geq \left(\frac{\frac{1}{m_1} + \cdots + \frac{1}{m_k}}{k}\right)^2 \geq \left(\frac{k}{m_1 + \cdots + m_k}\right)^2 = \frac{1}{\overline{m}^2}.$$

The desired inequality follows.

4.1.19

Suppose that three solutions are $\frac{u}{t}, \frac{v}{t}, \frac{w}{t}$ with integers u, v, w, t not all even. By the Viète formula

$$\begin{cases} u+v+w=2t, \\ uv+vw+wu=-2t, \end{cases}$$

which implies that $u^2+v^2+w^2=4t(t+1)$ is divisible by 8. Then u, v, w must all be even and therefore t is odd.

However,

$$\frac{t}{2}=-\frac{u}{2}\cdot\frac{v}{2}-\frac{v}{2}\cdot\frac{w}{2}-\frac{w}{2}\cdot\frac{u}{2}$$

is also an integer, so t is even, which is a contradiction.

Thus the given equation cannot have three distinct rational roots.

4.1.20

Let $x_k=\max\{x_1,\ldots,x_n\}$. Then

$$\begin{aligned} \sum_{i=1}^{n-1} x_i x_{i+1} &= \sum_{i=1}^{k-1} x_i x_{i+1} + \sum_{i=k}^{n-1} x_i x_{i+1} \\ &\le x_k \sum_{i=1}^{k-1} x_i + x_k \sum_{i=k}^{n-1} x_{i+1} \\ &= x_k(p-x_k) \le \frac{p^2}{4}. \end{aligned}$$

The equality occurs if say $x_1=x_2=\frac{p}{2},\ x_3=\cdots=x_n=0$. Thus the answer is $\frac{p^2}{4}$.

4.1.21

Put $xz=yt=u,\ x+z=y-t=v$. Summing up and subtracting the first and the second equations of the system, we obtain

$$x^2+z^2=13,\ y^2+t^2=37.$$

Hence

$$(x+z)^2 = 13 + 2u,\ (y-t)^2 = 37 - 2u. \tag{1}$$

That is, we have

$$\begin{cases} v^2 = 13 + 2u, \\ v^2 = 37 - 2u, \end{cases}$$

which give $u = 6, v = \pm 5$. From this it follows that

$$(x-z)^2 = x^2 + z^2 - 2xz = 13 - 2u = 1 \Longrightarrow x - z = \pm 1, \tag{2}$$

and

$$(y+t)^2 = y^2 + t^2 + 2yt = 37 + 2u = 49 \Longrightarrow y + t = \pm 7. \tag{3}$$

Also, (1) becomes

$$(x+z)^2 = 25 \Longleftrightarrow x + z = \pm 5, \tag{4}$$

and

$$(y-t)^2 = 25 \Longleftrightarrow y - t = \pm 5. \tag{5}$$

Combining (2) – (5) yields eight solutions of the given system:

$$(3,6,2,1);\ (2,6,3,1);\ (3,-1,2,-6);\ (2,-1,3,-6);$$

$$(-3,-6,-2,-1);\ (-2,-6,-3,-1);\ (-3,1,-2,6);\ (-2,1,-3,6).$$

4.1.22

Since $p \le t_k \le q$, we have $(t_k - p)(t_k - q) \le 0,\ \forall k = 1, \ldots, n$. Then

$$\sum_{k=1}^{n} (t_k - p)(t_k - q) \le 0,$$

or equivalently,

$$\sum_{k=1}^{n} t_k^2 - (p+q) \sum_{k=1}^{n} t_k + npq \le 0.$$

That is $T - (p+q)\overline{t} + pq \le 0$. From this it follows that

$$\frac{T}{\overline{t}^2} \le \frac{-pq}{\overline{t}^2} + \frac{p+q}{\overline{t}} = -pq\left(\frac{1}{\overline{t}} - \frac{p+q}{2pq}\right)^2 + \frac{(p+q)^2}{4pq} \le \frac{(p+q)^2}{4pq}.$$

The equality occurs if and only if

$$(t_k - p)(t_k - q) = 0,\ \text{for all } k = 1, \ldots, n, \text{ and } \overline{t} = \frac{2pq}{p+q}.$$

4.1.23

Note that $\cos^2 45^\circ = \dfrac{1}{2}$, and

$$\frac{1}{\cos^2 10^\circ} = \frac{1}{\sin^2 80^\circ}$$

$$\frac{1}{\sin^2 20^\circ} = \frac{4\cos^2 20^\circ}{\sin^2 40^\circ} = \frac{2(1+\cos 40^\circ)}{\sin^2 40^\circ} = \frac{2(1+\cos 40^\circ)\cdot 4\cos^2 40^\circ}{\sin^2 80^\circ}$$

$$\frac{1}{\sin^2 40^\circ} = \frac{4\cos^2 40^\circ}{\sin^2 80^\circ}.$$

Consequently,

$$\begin{aligned}\frac{1}{\cos^2 10^\circ} + \frac{1}{\sin^2 20^\circ} + \frac{1}{\sin^2 40^\circ} &= \frac{1 + 2(1+\cos 40^\circ)\cdot 4\cos^2 40^\circ + 4\cos^2 40^\circ}{\sin^2 80^\circ} \\ &= \frac{1 + (3 + 2\cos 40^\circ)\cdot 4\cos^2 40^\circ}{\cos^2 10^\circ}.\end{aligned}$$

Furthermore,

$$\begin{aligned}1+(3+2\cos 40^\circ)\cdot 4\cos^2 40^\circ &= 1+(3+2\cos 40^\circ)\cdot 2(1+\cos 80^\circ) \\ &= 1+(6+4\cos 40^\circ+6\cos 80^\circ+4\cos 40^\circ\cos 80^\circ) \\ &= 1+[6+4\cos 40^\circ+6\cos 80^\circ+2(\cos 120^\circ+\cos 40^\circ)] \\ &= 6+6\cos 40^\circ+6\cos 80^\circ \\ &= 6+6\cdot 2\cos 60^\circ\cos 20^\circ \\ &= 6(1+\cos 20^\circ) = 12\cos^2 10^\circ.\end{aligned}$$

Thus,

$$\frac{1}{\cos^2 10^\circ} + \frac{1}{\sin^2 20^\circ} + \frac{1}{\sin^2 40^\circ} - \frac{1}{\cos^2 45^\circ} = 12 - 2 = 10.$$

4.1.24

From the first equation it follows that

$$t_1 - t_2 = a_1 - t_1.$$

Substituting this into the second equation, we get

$$t_2 - t_3 = a_2 + (t_1 - t_2) = a_1 + a_2 - t_1.$$

Next, substituting this into the third equation yields

$$t_3 - t_4 = a_3 + (t_2 - t_3) = a_1 + a_2 + a_3 - t_1,$$

and so on.

Finally, the last two equations are as follows:

$$t_{n-1} - t_n = a_{n-1} + (t_{n-2} - t_{n-1}) = a_1 + a_2 + \cdots + a_{n-1} - t_1,$$

and

$$t_n = a_n + (t_{n-1} - t_n) = a_1 + a_2 + \cdots + a_n - t_1.$$

Hence,

$$\begin{aligned}
&t_n = a_1 + \cdots + a_n - t_1,\\
&t_{n-1} = 2a_1 + \cdots + 2a_{n-1} + a_n - 2t_1,\\
&t_{n-2} = 3a_1 + \cdots + 3a_{n-2} + 2a_{n-1} + a_n - 3t_1,\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
&t_2 = (n-1)a_1 + (n-1)a_2 + (n-2)a_3 + \cdots + a_n - (n-1)t_1,\\
&t_1 = na_1 + (n-1)a_2 + \cdots + 2a_{n-1} + a_n - nt_1.
\end{aligned}$$

Therefore, the solution of the equation is

$$\begin{aligned}
&t_1 = \frac{n}{n+1}a_1 + \frac{n-1}{n+1}a_2 + \cdots + \frac{2}{n+1}a_{n-1} + \frac{1}{n+1}a_n,\\
&t_2 = \frac{n-1}{n+1}a_1 + \frac{2(n-1)}{n+1}a_2 + \cdots + \frac{4}{n+1}a_{n-1} + \frac{2}{n+1}a_n,\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
&t_{n-1} = \frac{2}{n+1}a_1 + \frac{4}{n+1}a_2 + \cdots + \frac{2(n-1)}{n+1}a_{n-1} + \frac{n-2}{n+1}a_n,\\
&t_n = \frac{1}{n+1}a_1 + \frac{2}{n+1}a_2 + \cdots + \frac{n-1}{n+1}a_{n-1} + \frac{n}{n+1}a_n.
\end{aligned}$$

4.1.25

We have

$$\begin{aligned} S = \cos 144^\circ + \cos 72^\circ &= 2\cos 108^\circ \cos 36^\circ \\ &= -2\cos 72^\circ \cos 36^\circ \\ &= \frac{-2\cos 72^\circ \cos 36^\circ \sin 36^\circ}{\sin 36^\circ} \\ &= \frac{-\cos 72^\circ \sin 72^\circ}{\sin 36^\circ} \\ &= \frac{-\sin 144^\circ}{2\sin 36^\circ} \\ &= -\frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} P = \cos 144^\circ \cdot \cos 72^\circ &= -\cos 72^\circ \cos 36^\circ \\ &= \frac{-\cos 72^\circ \cos 36^\circ \sin 36^\circ}{\sin 36^\circ} \\ &= \frac{-\cos 72^\circ \sin 72^\circ}{2\sin 36^\circ} \\ &= \frac{-\sin 144^\circ}{4\sin 36^\circ} \\ &= -\frac{1}{4}. \end{aligned}$$

Therefore, the equation is $x^2 + \frac{1}{2}x - \frac{1}{4} = 0$, or, equivalently, $4x^2+2x-1 = 0$.

4.1.26

We have $1 > q > q^2 > \cdots > q^{p+1} > 0$. Since $q^{p+1} \le s \le 1$, there exists t such that $q^{t+1} \le s \le q^t$. Then, for $i \ge t+1$, we have $s \ge q^i$, and therefore,

$$\left|\frac{s-q^i}{s+q^i}\right| = \frac{s-q^i}{s+q^i}.$$

Noticing that

$$\frac{s-q^i}{s+q^i} - \frac{1-q^i}{1+q^i} = \frac{2q^i(s-1)}{(s+q^i)(s-q^i)} \le 0,$$

we find

$$\prod_{k=t+1}^{p}\left|\frac{s-q^k}{s+q^k}\right| \le \prod_{k=t+1}^{p}\left|\frac{1-q^k}{1+q^k}\right|. \tag{1}$$

On the other hand, for each $k = 1, \ldots, t$, we have

$$\left|\frac{s-q^{t-(k-1)}}{s+q^{t-(k-1)}}\right| = \frac{q^{t-(k-1)}-s}{q^{t-(k-1)}+s},$$

which implies

$$\frac{q^{t-(k-1)}-s}{q^{t-(k-1)}+s} - \frac{1-q^k}{1+q^k} = \frac{2(q^{t+1}-s)}{q^{t-(k-1)}+s(1+q^k)} \le 0,$$

and hence

$$\prod_{k=1}^{t}\left|\frac{s-q^k}{s+q^k}\right| \le \prod_{k=1}^{t}\left|\frac{1-q^k}{1+q^k}\right|. \tag{2}$$

From (1) and (2) the desired inequality follows.

4.1.27

Since

$$\frac{1}{2n-2k+1} - \frac{1}{2n-k+1} = \frac{k}{(2n-2k+1)(2n-k+1)},$$

we have

$$S_n = \sum_{k=1}^{n} \frac{k}{(2n-2k+1)(2n-k+1)} = \sum_{k=1}^{n}\frac{1}{2n-2k+1} - \sum_{k=1}^{n}\frac{1}{2n-k+1}$$

$$= \left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right) - \left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right).$$

Then

$$T_n - S_n = \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right)$$

$$-\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n-1}\right) = \frac{1}{2}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) = \frac{1}{2}T_n,$$

and hence $T_n = 2S_n$.

4.1.28

Put $x = \sqrt{2} + \sqrt[3]{3}$. Then

$$x^2 = 2 + 2\sqrt{2}\sqrt[3]{3} + \sqrt[3]{9}, \tag{1}$$

and also

$$x^3 = 2\sqrt{2} + 6\sqrt[3]{3} + 3\sqrt{2}\sqrt[3]{9} + 3. \tag{2}$$

Since $\sqrt[3]{3} = x - \sqrt{2}$, from (1) it follows that

$$\begin{aligned}\sqrt[3]{9} &= x^2 - 2 - 2\sqrt{2}\sqrt[3]{3}\\ &= x^2 - 2 - 2\sqrt{2}(x - \sqrt{2})\\ &= x^2 + 2 - 2x\sqrt{2}.\end{aligned}$$

Substituting two expressions for $\sqrt[3]{3}$ and $\sqrt[3]{9}$ into (2), we get

$$x^3 = 2\sqrt{2} + 6(x - \sqrt{2}) + 3\sqrt{2}(x^2 + 2 - 2x\sqrt{2}) + 3,$$

or equivalently,

$$x^3 + 6x - 3 = \sqrt{2}(3x^2 + 2).$$

Squaring both sides yields

$$x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1 = 0,$$

which is the required polynomial.

4.1.29

Since the domain of definition of the given equation is $|x| \le 1$, we can set $x = \cos y$ where $y \in [0, \pi]$. In this case the left-hand side is written as

$$\begin{aligned}&\sqrt{1+\sin y}\left(\sqrt{(1+\cos y)^3} - \sqrt{(1-\cos y)^3}\right)\\ &= \sqrt{\left(\sin\frac{y}{2} + \cos\frac{y}{2}\right)^2}\left(\sqrt{\left(2\cos^2\frac{y}{2}\right)^3} - \sqrt{\left(2\sin^2\frac{y}{2}\right)^3}\right)\\ &= 2\sqrt{2}\left(\sin\frac{y}{2} + \cos\frac{y}{2}\right)\left(\cos^3\frac{y}{2} - \sin^3\frac{y}{2}\right)\\ &= 2\sqrt{2}\left(\sin\frac{y}{2} + \cos\frac{y}{2}\right)\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)\left(\cos^2\frac{y}{2} + \sin^2\frac{y}{2} + \sin\frac{y}{2}\cos\frac{y}{2}\right)\\ &= 2\sqrt{2}\left(\sin\frac{y}{2} + \cos\frac{y}{2}\right)\left(\cos\frac{y}{2} - \sin\frac{y}{2}\right)\left(1 + \frac{1}{2}\sin y\right)\\ &= 2\sqrt{2}\left(\cos^2\frac{y}{2} - \sin^2\frac{y}{2}\right)\left(1 + \frac{1}{2}\sin y\right)\\ &= 2\sqrt{2}\cos y\left(1 + \frac{1}{2}\sin y\right).\end{aligned}$$

Therefore, the given equation is equivalent to

$$2\sqrt{2}\cos y(1+\frac{1}{2}\sin y) = 2(1+\frac{1}{2}\sin y).$$

Since $1+\dfrac{1}{2}\sin y \neq 0$, the last equation gives $\cos y = \dfrac{1}{\sqrt{2}}$; that is, $x = \dfrac{\sqrt{2}}{2}$.

4.1.30

From the given equation we observe that $t - [t] \neq 0$. Also note that $[t] = 0$ does not satisfy the equation. Thus t must not be an integer, and $[t] \neq 0$. Then $t \in (n, n+1)$ for some positive integer n. In this case the equations becomes

$$0.9t = \frac{n}{t-n},$$

or equivalently,

$$t^2 - nt - \frac{10}{9}n = 0.$$

The last equation has the solutions

$$t_1 = \frac{n}{2} - \frac{1}{6}\sqrt{9n^2+40},\ t_2 = \frac{n}{2} + \frac{1}{6}\sqrt{9n^2+40}.$$

Note that $t_1 < 0$, which does not satisfy the equation. When t_2 is greater than n we must have $t_2 < n+1$, or equivalently, $n < 9$.

Thus the given equation has eight solutions

$$t = \frac{n}{2} + \frac{1}{6}\sqrt{9n^2+40},\ n = 1, 2, \ldots, 8.$$

4.1.31

Note that $x = 0$ is not a root of the equation. Then we can write it as

$$16x^2 - mx + (2m+17) - \frac{m}{x} + \frac{16}{x^2} = 0,$$

or equivalently,

$$16t^2 - mt + (2m-15) = 0,$$

where $t = x + \frac{1}{x}$, and hence $|t| \geq 2$.

From this it follows that the given equation has four distinct real roots if and only if the quadratic function $f(t) = 16t^2 - mt + (2m-15)$ has two distinct real roots $t_1,\ t_2$ which are not in $[-2,2]$, as the equation $|t|=2$ give two equal roots.

First, in order to have two distinct real roots, we must have

$$\Delta = m^2-64(2m-15)>0 \Longleftrightarrow (m-8)(m-120)>0 \Longleftrightarrow m<8,\ m>120.$$

Next, we note that $f(2)=16\cdot 4-15>0$, so either $t_1<t_2<-2$ or $2<t_1<t_2$. The first case cannot happen. Indeed, if it does then by the Viète formula

$$\frac{m}{16}=t_1+t_2<-4 \Longleftrightarrow m<-64 \Longrightarrow t_1t_2=\frac{2m-15}{16}<0,$$

which is impossible.

Thus we get $2<t_1<t_2$, and each of the two equations

$$x+\frac{1}{x}=t_1,\ x+\frac{1}{x}=t_2,$$

has two real positive distinct roots, which we denote by x_1,x_1' and x_2,x_2', respectively.

Note that $x_1x_1'=x_2x_2'=1$. We can assume that $1<x_1<x_2$, which implies that $1>x_1'>x_2'$. Then we have $x_2',\ x_1',\ x_1,\ x_2$ form an increasing geometric progression. Therefore,

$$x_2=(x_1)^3,\ x_2'=(x_1')^3,$$

which implies that

$$\begin{aligned}
t_2 &= x_2+\frac{1}{x_2}=x_2+x_2' \\
&= (x_1)^3+(x_1')^3=(x_1+x_1')\left[(x_1)^2-x_1x_1'+(x_1')^2\right] \\
&= (x_1+x_1')\left[(x_1+x_1')^2-3x_1x_1'\right] \\
&= t_1\left[(t_1)^2-3\right].
\end{aligned}$$

Then

$$\frac{m}{16}=t_1+t_2=t_1\left[(t_1)^2-2\right],$$

and hence

$$\begin{aligned}
m &= 16t_1\left[(t_1)^2-2\right]=t_1\left[16(t_1)^2-32\right] \\
&= t_1\left[(mt_1-2m+15)-32\right]=m(t_1)^2-(2m+17)t_1,
\end{aligned}$$

which gives

$$m=\frac{17t_1}{(t_1)^2-2t_1-1}.$$

Substituting this value of m into the equation $f(t_1) = 16(t_1)^2 - mt_1 + 2m - 15 = 0$ we obtain

$$16(t_1)^4 - 31(t_1)^3 - 48(t_1)^2 + 64t_1 + 15 = 0.$$

Denote $y = 2t_1$, we have

$$y^4 - 4y^3 - 12y^2 + 32y + 15 = 0 \Longleftrightarrow (y-5)(y+3)(y^2 - 2y - 1) = 0.$$

From this it follows that the unique possible value of y for which $t_1 > 2$ is $y = 5$. Hence $t_1 = \frac{5}{2}$, and so $m = 170$.

Conversely, for $m = 170$ the equation $16x^4 - 170x^3 + 357x^2 - 170x + 6 = 0$ has four distinct roots $\dfrac{1}{8}, \dfrac{1}{2}, 2, 8$ which obviously form a geometric progression with the ratio $r = 4$.

Thus the only solution to the problem is $m = 170$.

4.1.32

For each $i = 1, \ldots, n$ we have discriminants $\Delta_i' = (2a_i)^2 - 4(a_i - 1)^2 = 4(2a_i - 1) \geq 0$ as $a_i \geq \frac{1}{2}$. So the inequalities have solutions

$$\frac{a_i - \sqrt{2a_i - 1}}{2} \leq x \leq \frac{a_i + \sqrt{2a_i - 1}}{2}. \tag{1}$$

Note that

$$\max_{\frac{1}{2} \leq a_i \leq 5} \frac{a_i + \sqrt{2a_i - 1}}{2} = 4 \quad \text{and} \quad \min_{\frac{1}{2} \leq a_i \leq 5} \frac{a_i - \sqrt{2a_i - 1}}{2} = 0.$$

Then from (1) it follows that $0 \leq x_i \leq 4$, and so $x_i^2 - 4x_i \leq 0$. Therefore,

$$\sum_{i=1}^{n} x_i^2 - 4\sum_{i=1}^{n} x_i \leq 0,$$

which gives

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq \frac{4}{n}\sum_{i=1}^{n} x_i.$$

On the other hand,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i - 1\right)^2 \geq 0,$$

or equivalently,

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i + 1\right)^2 \geq \frac{4}{n}\sum_{i=1}^{n} x_i.$$

Thus

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq \left(\frac{1}{n}\sum_{i=1}^{n} x_i + 1\right)^2$$

and the desired inequality follows.

4.1.33

For a particular case $x_1 = \cdots = x_{n-1} = 1$ and $x_n = 2$ we get $(n-1)+4 \geq 2(n-1)$, which implies that $n \leq 5$.

We rewrite the inequality as a quadratic function with respect to x_n:

$$x_n^2 - (x_1 + \cdots + x_{n-1})x_n + (x_1^2 + \cdots + x_{n-1}^2) \geq 0, \ \forall x_n,$$

which is equivalent to

$$\Delta = (x_1 + \cdots + x_{n-1})^2 - 4(x_1^2 + \cdots + x_{n-1}^2) \leq 0. \tag{2}$$

By Cauchy-Schwarz inequality for $x_1, \ldots, x_{n-1}$ and $\underbrace{1, \ldots, 1}_{(n-1)\text{ times}}$ we have

$$(1^2 + \cdots + 1^2)(x_1^2 + \cdots + x_{n-1}^2) \geq (1 \cdot x_1 + \cdots + 1 \cdot x_{n-1})^2,$$

or equivalently,

$$(n-1)(x_1^2 + \cdots + x_{n-1}^2) \geq (x_1 + \cdots + x_{n-1})^2,$$

as $n - 1 \leq 4$. So (2) is proven.

Thus $n = 2, 3, 4, 5$.

4.1.34

Note that for $k = t = 0$ the equality occurs.

Put

$$A_{t,k} = \sum_{i=1}^{n} \frac{(a_i)^{2^k}}{(S - a_i)^{2^t - 1}}.$$

By Cauchy-Schwarz inequality

$$
\begin{aligned}
&(n-1)S \cdot A_{t,k} \ge A_{t-1,k-1}^2, && (1)\\
&(n-1)S \cdot A_{t-1,k-1} \ge A_{t-2,k-2}^2, && (2)\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&(n-1)S \cdot A_{1,k-t+1} \ge A_{0,k-t}^2. && (t)
\end{aligned}
$$

Take the power of degree 2^{s-1} both sides of the s-th inequality ($1 \le s \le t$) and multiply all results, after some simplifications, we obtain

$$(n-1)^{\sum_{s=0}^{t-1} 2^s} \cdot S^{\sum_{s=0}^{t-1} 2^s} \cdot A_{t,k} \ge (A_{0,k-t})^{2^t}.$$

Furthermore, by Arithmetic-Geometric Mean inequality

$$\sqrt[2^{k-t}]{\frac{\sum_{i=1}^n (a_i)^{2^{k-t}}}{n}} \ge \frac{S}{n}.$$

Notice that $A_{0,k-t} = \sum_{i=1}^n (a_i)^{2^{k-t}}$, we then have

$$(n-1)^{2^t-1} \cdot S^{2^t-1} \cdot A_{t,k} \ge \frac{S^{2^k}}{n^{2^k-2^t}},$$

and the desired inequality follows. The equality occurs if and only if $a_1 = \cdots = a_n$.

Thus the problem is proved, and the equality occurs when either $k = t = 0$ and for any (a_i), or $a_1 = \cdots = a_n$.

4.1.35

Let $x_1, \ldots, x_n$ be n real roots of the polynomial. By Viète formula we have

$$
\begin{aligned}
\sum_{i=1}^n x_i &= n,\\
\sum_{\substack{i,j=1\\ i<j}}^n x_i x_j &= \frac{n^2-n}{2}.
\end{aligned}
$$

Then

$$\sum_{i=1}^n x_i^2 = \left(\sum_{i=1}^n x_i\right)^2 - 2\sum_{\substack{i,j=1\\ i<j}}^n x_i x_j = n^2 - n^2 + n = n$$

and

$$\sum_{i=1}^{n}(x_i-1)^2=\sum_{i=1}^{n}x_i^2-2\sum_{i=1}^{n}x_i+n=n-2n+n=0.$$

The last equation shows that $x_i=1$ for all i, and $P(x)=(x-1)^n$. So coefficients are

$$a_k=(-1)^k\binom{n}{k},\ k=0,1,\ldots,n.$$

4.1.36

Note that $|\sin t|\le|t|$ for all real t. Then we have

$$\left|\sum_{k=0}^{n}\frac{\sin(\alpha+k)x}{N+k}\right|\le\sum_{k=0}^{n}\frac{|\sin(\alpha+k)x|}{N+k}\le\sum_{k=0}^{n}\frac{(\alpha+k)|x|}{N+k}\le\sum_{k=0}^{n}|x|=(n{+}1)|x|.$$

Now we prove that

$$\left|\sum_{k=0}^{n}\frac{\sin(\alpha+k)x}{N+k}\right|\le\frac{1}{N|\sin\frac{x}{2}|}.$$

Case 1: $x=2k\pi$ $(k\in\mathbb{Z})$, then $\sin\frac{x}{2}=0$, the inequality always holds.

Case 2: $x\ne 2k\pi$ $(k\in\mathbb{Z})$, then $\sin\frac{x}{2}\ne 0$. Using the following transformation

$$\sum_{i=0}^{n}a_ib_i=\sum_{i=0}^{n-1}A_i(b_i-b_{i+1})+A_nb_n,$$

where $A_i=a_0+\cdots+a_i$, we have

$$\sum_{k=0}^{n}\frac{\sin(\alpha+k)x}{N+k}=\sum_{k=0}^{n-1}S_k\left(\frac{1}{N+k}-\frac{1}{N+k+1}\right)+S_n\frac{1}{N+n},$$

where

$$S_k=\sum_{i=0}^{k}\sin(\alpha+i)x.$$

Then

$$\begin{aligned}\left|\sum_{k=0}^{n}\frac{\sin(\alpha+k)x}{N+k}\right| &= \left|\sum_{k=0}^{n-1}S_k\left(\frac{1}{N+k}-\frac{1}{N+k+1}\right)+S_n\frac{1}{N+n}\right| \\ &\le \max_{0\le i\le n}|S_k|\left(\sum_{k=0}^{n-1}\left(\frac{1}{N+k}-\frac{1}{N+k+1}\right)+\frac{1}{N+n}\right) \\ &= \frac{1}{N}\max_{0\le i\le n}|S_k|.\end{aligned}$$

Note also that

$$|S_k| = \left|\sum_{i=0}^{k}\sin(\alpha+i)x\right| = \left|\frac{\sin(\alpha x+\frac{kx}{2})\sin\frac{(k+1)x}{2}}{\sin\frac{x}{2}}\right| \le \frac{1}{|\sin\frac{x}{2}|},$$

for all $0\le k\le n$. Therefore, we obtain

$$\left|\sum_{k=0}^{n}\frac{\sin(\alpha+k)x}{N+k}\right| \le \frac{1}{N|\sin\frac{x}{2}|}.$$

This completes the proof.

4.1.37

Note that

$$\frac{3k+1}{3k}+\frac{3k+2}{3k+1}+\frac{3k+3}{3k+2} = 3+\frac{1}{3k}+\frac{1}{3k+1}+\frac{1}{3k+2}$$

and

$$\frac{3k+1}{3k}\cdot\frac{3k+2}{3k+1}\cdot\frac{3k+3}{3k+2} = \frac{k+1}{k},$$

by the Arithmetic-Geometric Mean inequality, we have

$$3+\frac{1}{3k}+\frac{1}{3k+1}+\frac{1}{3k+2} > 3\sqrt[3]{\frac{k+1}{k}}.$$

Then

$$3\sum_{k=1}^{995}\sqrt[3]{\frac{k+1}{k}} < \sum_{k=1}^{995}\left(3+\frac{1}{3k}+\frac{1}{3k+1}+\frac{1}{3k+2}\right),$$

and hence

$$\sum_{k=1}^{995}\sqrt[3]{\frac{k+1}{k}}-\frac{1989}{2} < 995-\frac{1989}{2}+\sum_{k=1}^{995}\frac{1}{3}\left(\frac{1}{3k}+\frac{1}{3k+1}+\frac{1}{3k+2}\right)$$

$$= \frac{1}{2} + \left(\frac{1}{9} + \frac{1}{12} + \cdots + \frac{1}{8961}\right) = \frac{1}{3} + \frac{1}{6} + \left(\frac{1}{9} + \frac{1}{12} + \cdots + \frac{1}{8961}\right).$$

4.1.38

Let $x_1, \ldots, x_{10}$ be 10 real roots of $P(x)$. By Viète formula

$$\sum_{i=1}^{10} x_i = 10,$$
$$\sum_{\substack{i,j=1\\i<j}}^{10} x_i x_j = 39.$$

Then, since

$$\left(\sum_{i=1}^{10} x_i\right)^2 = \sum_{i=1}^{10} x_i^2 + 2 \sum_{\substack{i,j=1\\i<j}}^{10} x_i x_j,$$

we get

$$\sum_{i=1}^{10} x_i^2 = 100 - 2 \cdot 39 = 22.$$

On the other hand,

$$\sum_{i=1}^{10} (x_i - 1)^2 = \sum_{i=1}^{10} x_i^2 - 2\sum_{i=1}^{10} x_i + \sum_{i=1}^{10} 1 = 22 - 2 \cdot 10 + 10 = 12,$$

which implies that $(x_i - 1)^2 \leq 12 < (3.5)^2$, $\forall 1 \leq i \leq 10$, and hence $-2.5 < x_i < 4.5$, for all $i = 1, \ldots, 10$.

4.1.39

Put $y_i = 1 - x_i$, we have $0 \leq y_i \leq 2$ and $\sum_{i=1}^{n} y_i = 3$. Then

$$x_1^2 + \cdots + x_n^2 \leq n - 1 \iff \sum_{i=1}^{n} y_i^2 \leq 5.$$

Note that there are at most two y_i's greater than 1. Then we have the following cases:

Case 1: $0 \le y_i \le 1,\ \forall i = 1, \dots, n$. In this case $y_i^2 \le y_i$ and hence

$$\sum_{i=1}^{n} y_i^2 \le \sum_{i=1}^{n} y_i = 3 < 5.$$

Case 2: There is only say $y_1 > 1$. In this case

$$\sum_{i=1}^{n} y_i^2 \le y_1^2 + \sum_{i=2}^{n} y_i = y_1(y_1 - 1) + \sum_{i=1}^{n} y_i \le 2 \cdot 1 + 3 = 5.$$

Case 3: There are say $y_1, y_2 > 1$. In this case

$$\sum_{i=1}^{n} y_i^2 \le y_1^2 + y_2^2 + \sum_{i=3}^{n} y_i = y_1(y_1 - 1) + y_2(y_2 - 1) + \sum_{i=1}^{n} y_i$$
$$\le 2(y_1 - 1) + 2(y_2 - 1) + 3 = 2(y_1 + y_2) - 1 \le 2 \cdot 3 - 1 = 5.$$

4.1.40

Note that $0 < \frac{\sqrt[n]{n}}{n} < 1$ for $n > 1$. By the Arithmetic-Geometric Mean inequality, we have

$$\frac{\underbrace{1 + \cdots + 1}_{(n-1)\text{ times}} + \left(1 + \frac{\sqrt[n]{n}}{n}\right)}{n} > \sqrt[n]{1 + \frac{\sqrt[n]{n}}{n}},$$

and

$$\frac{\underbrace{1 + \cdots + 1}_{(n-1)\text{ times}} + \left(1 - \frac{\sqrt[n]{n}}{n}\right)}{n} > \sqrt[n]{1 - \frac{\sqrt[n]{n}}{n}}.$$

Combining these results yields the desired inequality.

4.1.41

For $x \ge 0$ we see that $P(x) \ge 1 > 0$. So to prove the problem it suffices to show that $P(x) > 0$ for $x \in \left(\frac{1-\sqrt{5}}{2}, 0\right)$.

Indeed, for $x < 0, x \ne -1$ we always have

$$P(x) \ge 1 + x + x^3 + x^5 + \cdots + x^{1991} = 1 + x\frac{1 - x^{996}}{1 - x^2} = \frac{1 - x^2 + x - x^{997}}{1 - x^2}.$$

Note that if $x \in \left(\frac{1-\sqrt{5}}{2}, 0\right)$ then $1 - x^2 > 0,\ -x^{997} > 0,\ 1 - x^2 + x > 0$, and therefore, $P(x) > 0$. The desired conclusion follows.

4.1.42

By Cauchy-Schwarz inequality we have

$(4xv+3yu)^2 = (x\sqrt{2}\cdot 2v\sqrt{2}+y\sqrt{3}\cdot u\sqrt{3})^2 \le (2x^2+3y^2)(8v^2+3u^2) = 60,$

which gives

$$4xv+3yu \le 2\sqrt{15}.$$

Combining this and $4xv+3yu \ge 2\sqrt{15}$ yields $4xv+3yu = 2\sqrt{15}$. The equality occurs if and only if

$$\frac{x\sqrt{2}}{2v\sqrt{2}} = \frac{y\sqrt{3}}{u\sqrt{3}},$$

which shows that xv and yu must be positive. Then

$$\frac{2x^2}{8v^2} = \frac{3y^2}{3u^2} = \frac{2x^2+3y^2}{8v^2+3u^2} = \frac{10}{6},$$

and hence $x = \frac{2v\sqrt{5}}{\sqrt{3}}, y = \frac{u\sqrt{5}}{\sqrt{3}}$. Thus

$$S = x+y+u = \frac{2v\sqrt{5}}{\sqrt{3}} + \left(\frac{\sqrt{5}}{\sqrt{3}}+1\right)u.$$

Applying again Cauchy-Schwarz inequality, we have

$$S^2 = \left(\frac{\sqrt{5}}{\sqrt{6}}\cdot 2v\sqrt{2} + \frac{\sqrt{5}+\sqrt{3}}{3}\cdot u\sqrt{3}\right)^2$$

$$\le \left(\frac{5}{6}+\frac{8+2\sqrt{15}}{9}\right)\left(8v^2+3u^2\right) = \frac{31+4\sqrt{15}}{3},$$

which gives

$$-\sqrt{\frac{31+4\sqrt{15}}{3}} \le S \le \sqrt{\frac{31+4\sqrt{15}}{3}}.$$

Moreover, $S = \sqrt{\frac{31+4\sqrt{15}}{3}}$ at $u_1 = \sqrt{\frac{16+4\sqrt{15}}{23+2\sqrt{15}}},\ v_1 = \sqrt{\frac{45}{92+8\sqrt{15}}}$, that is, $x_1 = \frac{2v_1\sqrt{5}}{\sqrt{3}},\ y_1 = \frac{u_1\sqrt{5}}{\sqrt{3}}$.

Similarly, $S = -\sqrt{\frac{31+4\sqrt{15}}{3}}$ at $u_2 = -u_1,\ v_2 = -v_1$, that is, $x_2 = \frac{2v_2\sqrt{5}}{\sqrt{3}} = -\frac{2v_1\sqrt{5}}{\sqrt{3}},\ y_2 = \frac{u_2\sqrt{5}}{\sqrt{3}} = -\frac{u_1\sqrt{5}}{\sqrt{3}}$.

Thus the minimum and maximum values of S are $-\sqrt{\frac{31+4\sqrt{15}}{3}}$ and $\sqrt{\frac{31+4\sqrt{15}}{3}}$, respectively.

4.1.43

We rewrite the given equation as

$$60T(x) = 60(x^2 - 3x + 3)P(x) = (3x^2 - 4x + 5)Q(x).$$

Since polynomials $60(x^2-3x+3)$ and $3x^2-4x+5$ have no real roots, they are co-prime. Then the existence of polynomials $P(x), Q(x), T(x)$ that satisfy the problem, is equivalent to the existence of a polynomial $S(x)$ such that

$$60(x^2 - 3x + 3)S(x), (3x^2 - 4x + 5)S(x)$$

and

$$(3x^2 - 4x + 5)(x^2 - 3x + 3)S(x)$$

all are polynomials with positive integer coefficients.

We find $S(x)$ in combination with $(x + 1)^n$.

Choose n so that $P_1(x) = (3x^2 - 4x + 5)(x + 1)^n$ is a polynomial with positive integer coefficients. We have

$$\begin{aligned} P_1(x) \quad &= \quad 3x^{n+2} + \left[3\binom{n}{1} - 4\right] x^{n+1} \\ &\quad + \sum_{k=1}^{n-1}\left[3\binom{n}{k+1} - 4\binom{n}{k} + 5\binom{n}{k-1}\right] x^{n-k+1} \\ &\quad + \left[5\binom{n}{n-1} - 4\right] x + 5. \end{aligned}$$

Note that $3\binom{n}{1} - 4$ and $5\binom{n}{n-1} - 4$ are positive integers for all $n \geq 2$. Then coefficients of x^{n-k+1} are positive integers for all $1 \leq k \leq n-1$ if and only if

$$3\binom{n}{k+1} - 4\binom{n}{k} + 5\binom{n}{k-1} > 0,$$

that is,

$$12k^2 - 2(5n - 1)k + 3n^2 - n - 4 > 0, \ \forall k = 1, 2, \ldots, n - 1.$$

In this case we must have $\Delta = 11n^2 - 2n - 49 > 0$, which is true for all $n \geq 3$.

Similarly, $Q_1(x) = (x^2 - 3x + 3)(x + 1)^n$ is a polynomial with positive integer coefficients for all $n \geq 15$.

From the above discussion, we conclude that $S(x) = (x+1)^{18}$ satisfies the required conditions, and hence the desired polynomials can be chosen, for example, as follows.

$$\begin{aligned}
P(x) &= (3x^2 - 4x + 5)(x+1)^{18},\\
Q(x) &= 60(x^2 - 3x + 3)(x+1)^{18},\\
T(x) &= (3x^2 - 4x + 5)(x^2 - 3x + 3)(x+1)^{18}.
\end{aligned}$$

4.1.44

The domain of definition is $x \geq -1$. We rewrite the given equation as follows

$$x^3 - 3x^2 - 8x + 40 = 8\sqrt[4]{4x+4} = 0, \;\; x \geq -1.$$

For the right-hand side, by the Arithmetic-Geometric Mean inequality, we have

$$8\sqrt[4]{4x+4} = \sqrt[4]{2^4 \cdot 2^4 \cdot 2^4 \cdot (4x+4)} \leq \frac{2^4 + 2^4 + 2^4 + (4x+4)}{4} = x + 13.$$

The equality occurs if and only if $2^4 = 4x + 4 \Longleftrightarrow x = 3$.

We show that for the left-hand side the following inequality holds

$$x^3 - 3x^2 - 8x + 40 \geq x + 13, \;\; x \geq -1.$$

Indeed, this is equivalent to $(x-3)^2(x+3) \geq 0$, which is true. The equality occurs if and only if $x = 3$.

Thus

$$8\sqrt[4]{4x+4} \leq x + 13 \leq x^3 - 3x^2 - 8x + 40,$$

and the equalities occur if and only if $x = 3$. This is obviously the only solution of the given equation.

4.1.45

The domain of definition is $x > 0, y > 0$. The system is equivalent to

$$\begin{cases} 1 + \frac{1}{x+y} & = \frac{2}{\sqrt{3x}} \\ 1 - \frac{1}{x+y} & = \frac{4\sqrt{2}}{\sqrt{7y}} \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \frac{1}{x+y} & = \frac{1}{\sqrt{3x}} - \frac{2\sqrt{2}}{\sqrt{7y}} \\ 1 & = \frac{1}{\sqrt{3x}} + \frac{2\sqrt{2}}{\sqrt{7y}}. \end{cases}$$

Multiplying both equations yields

$$\frac{1}{x+y}=\frac{1}{3x}-\frac{8}{7y},$$

or equivalently, $(y-6x)(7y+4x)=0$, which give only $y=6x$, as $x,y>0$.

Substituting $y=6x$ into either of two equations of the system, we get $\sqrt{x}=\frac{2+\sqrt{7}}{\sqrt{21}}$. This gives a unique solution

$$x=\frac{11+4\sqrt{7}}{21}, \text{ and } y=\frac{22+8\sqrt{7}}{7}.$$

4.1.46

Put

$$\begin{cases} p=a+b+c+d,\\ q=ab+ac+ad+bc+bd+cd,\\ r=abc+bcd+cda+dab,\\ s=abcd. \end{cases}$$

By Viète formula, four nonnegative numbers a,b,c,d are roots of a polynomial $P(x)=x^4-px^3+qx^2-rx+s$. Then $P'(x)=4x^3-3px^2+2qx-r$ has three nonnegative roots α,β,γ.

Again by Viète formula we have

$$\begin{cases} \alpha+\beta+\gamma=\frac{3}{4}p,\\ \alpha\beta+\beta\gamma+\gamma\alpha=\frac{1}{2}q,\\ \alpha\beta\gamma=\frac{1}{4}r. \end{cases}$$

In this case, we can rewrite the assumption of the problem as

$$2(ab+ac+ad+bc+bd+cd)+abc+abd+acd+bcd=16 \Longleftrightarrow 2q+r=16,$$

or equivalently,

$$\alpha\beta+\beta\gamma+\gamma\alpha+\alpha\beta\gamma=4.$$

Then the problem is reduced to the inequality

$$\alpha+\beta+\gamma\geq\alpha\beta+\beta\gamma+\gamma\alpha.$$

We prove this inequality.

Without loss of generality we can assume that $\gamma = \min\{\alpha, \beta, \gamma\}$. Since $\alpha, \beta > 0$, we have

$$\alpha + \beta + \alpha\beta > 0,$$

and

$$\gamma = \frac{4 - \alpha\beta}{\alpha + \beta + \alpha\beta}.$$

Then

$$\begin{aligned}
\alpha + \beta + \gamma - (\alpha\beta + \beta\gamma + \gamma\alpha) &= \alpha + \beta + \gamma + \alpha\beta\gamma - (\alpha\beta + \beta\gamma + \gamma\alpha + \alpha\beta\gamma) \\
&= \alpha + \beta + \gamma + \alpha\beta\gamma - 4 \\
&= \alpha + \beta + \frac{4 - \alpha\beta}{\alpha + \beta + \alpha\beta} + \frac{\alpha\beta(4 - \alpha\beta)}{\alpha + \beta + \alpha\beta} - 4 \\
&= \frac{(\alpha + \beta)^2 - 4(\alpha + \beta) + 4 + \alpha\beta(\alpha + \beta - \alpha\beta - 1)}{\alpha + \beta + \alpha\beta} \\
&= \frac{(\alpha + \beta - 2)^2 - \alpha\beta(\alpha - 1)(\beta - 1)}{\alpha + \beta + \alpha\beta}.
\end{aligned}$$

Since $\alpha + \beta + \alpha\beta > 0$, it remains to prove that

$$M = (\alpha + \beta - 2)^2 - \alpha\beta(\alpha - 1)(\beta - 1) \geq 0.$$

There are two cases.

Case 1: If $(\alpha - 1)(\beta - 1) \leq 0$, then $M \geq 0$. The equality occurs if and only if $\alpha = \beta = 1$, and so $\gamma = 1$.

Case 2: If $(\alpha - 1)(\beta - 1) > 0$, then

$$(\alpha + \beta - 2)^2 = [(\alpha - 1) + (\beta - 1)]^2 \geq 4(\alpha - 1)(\beta - 1).$$

Also from

$$\gamma = \frac{4 - \alpha\beta}{\alpha + \beta + \alpha\beta}$$

it follows that $\alpha\beta \leq 4$. Then

$$(\alpha + \beta - 2)^2 \geq \alpha\beta(\alpha - 1)(\beta - 1),$$

that is, $M \geq 0$. The equality occurs if and only if $\gamma = 0$ and $\alpha = \beta = 2$.

4.1.47

Note that if $u\sqrt[3]{3}+v\sqrt[3]{9}=0$ with $u,v\in\mathbb{Q}$ then $u=v=0$.

1) Consider a polynomial $P(x)=ax+b$ with $a,b\in\mathbb{Q}$. If $P(x)$ satisfies the problem, that is, $a(\sqrt[3]{3}+\sqrt[3]{9})+b=3+\sqrt[3]{3}$, then $(a-1)\sqrt[3]{3}+a\sqrt[3]{9}=3-b\in\mathbb{Q}$ and hence $a=0=a-1$. This is impossible, and so there does not exist such a linear polynomial.

Now consider a quadratic $P(x)=ax^2+bx+c$ with $a,b,c\in\mathbb{Q}$. From $f(\sqrt[3]{3}+\sqrt[3]{9})=3+\sqrt[3]{3}$ which is equivalent to

$$(a+b)\sqrt[3]{9}+(3a+b)\sqrt[3]{3}+6a+c=3+\sqrt[3]{3},$$

it follows that

$$\begin{cases} a+b=0, \\ 3a+b=1, \\ 6a+c=3, \end{cases}$$

or equivalently, $a=\frac{1}{2}, b=-\frac{1}{2}, c=0$. Thus $P(x)=\frac{1}{2}(x^2-x)$ is only solution to the problem.

2) Put $s=\sqrt[3]{3}+\sqrt[3]{9}$ we have $s^3=9s+12$. This shows that the polynomial $G(x)=x^3-9x-12$ has s as its root.

Assume that there exists a polynomial $P(x)$ of degree $n\geq 3$ with integer coefficients satisfying $P(s)=3+\sqrt[3]{3}$. Then by long division, we have

$$P(x)=G(x)\cdot Q(x)+R(x),$$

where $Q(x)$ and $R(x)$ are polynomials with integer coefficients, and $\deg(R)<3$. In this case

$$P(s)=G(s)\cdot Q(s)+R(s)=0\cdot Q(s)+R(s)=R(s),$$

that is, $R(s)=3+\sqrt[3]{3}$. This is impossible, because by 1) there exists a unique polynomial of degree ≤ 2 with rational coefficients satisfying the given condition, which is $\frac{1}{2}(x^2-x)$.

So there does not exist such a polynomial $P(x)$.

4.1.48

Rewrite the given equation as

$$\frac{1998}{x_1+1998}+\frac{1998}{x_2+1998}+\cdots+\frac{1998}{x_n+1998}=1.$$

Putting $y_i = \dfrac{x_i}{1998}$ $(i = 1, \ldots, n)$ we get the equation

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} + \cdots + \frac{1}{1+y_n} = 1. \tag{1}$$

Note also that $x_i, y_i > 0$ for all i. Then we have

$$\frac{\sqrt[n]{x_1 \cdots x_n}}{n-1} \geq 1998 \Longleftrightarrow x_1 \cdots x_n \geq 1998^n (n-1)^n,$$

or equivalently,

$$y_1 \cdots y_n \geq (n-1)^n. \tag{2}$$

So the problem can be restated as follow: if n positive numbers $y_1, \ldots, y_n$ satisfy (1), then (2) holds.

Again put $z_i = \dfrac{1}{1+y_i}$ $(i = 1, \ldots, n)$, we have $\displaystyle\sum_{i=1}^{n} z_i = 1$, $0 < z_i < 1$.

Let $P = \displaystyle\prod_{i=1}^{n} z_i$, by the Arithmetic-Geometric Mean inequality, we have

$$1 - z_i = \sum_{j=1, j\neq i}^{n} z_j \geq (n-1) \sqrt[n-1]{\prod_{j=1, j\neq i}^{n} z_j} = (n-1) \left(\frac{P}{z_i}\right)^{\frac{1}{n-1}},$$

which implies that

$$\prod_{i=1}^{n} y_i = \prod_{i=1}^{n} \frac{1-z_i}{z_i} = \frac{1}{P} \prod_{i=1}^{n} (1-z_i) \geq \frac{1}{P} \prod_{i=1}^{n} \left[(n-1) \left(\frac{P}{z_i}\right)^{\frac{1}{n-1}}\right]$$

$$= \frac{1}{P} (n-1)^n \frac{P^{\frac{n}{n-1}}}{\prod_{i=1}^{n} z_i^{\frac{1}{n-1}}} = (n-1)^n \frac{P^{\frac{1}{n-1}}}{P^{\frac{1}{n-1}}} = (n-1)^n.$$

The equality occurs if and only if $z_1 = \cdots = z_n$, or $y_1 = \cdots = y_n$, that is $x_1 = \cdots = x_n = 1998(n-1)$.

4.1.49

We prove a general case, when 1998 is replaced by a positive integer k.

Let $P(x) = \displaystyle\sum_{i=0}^{m} a_i x^i$ $(a_m \neq 0)$. Then $P(x^k - x^{-k}) = x^n - x^{-n}$ is equivalent to

$$\sum_{i=0}^{m} a_i \frac{(x^{2k}-1)^i}{x^{ki}} = \frac{x^{2n}-1}{x^n}, \tag{1}$$

that is,

$$\sum_{i=0}^{m} a_i x^n (x^{2k}-1)^i x^{k(m-i)} = x^{km}(x^{2n}-1),\ \forall x \neq 0.$$

The polynomial on the left is of degree $n+2km$, while that on the right is of degree $2n+km$, and so $n = km$.

Now we prove that m must be odd. Indeed, assume that m is even. Then putting $y = x^k$, we can rewrite the given equation as follows

$$P\left(y - \frac{1}{y}\right) = y^m - \frac{1}{y^m},\ \forall y \neq 0. \tag{2}$$

Substituting $y = 2$ and then $y = -\frac{1}{2}$ into (2) we obtain

$$P\left(\frac{3}{2}\right) = 2^m - \frac{1}{2^m} > 0 \text{ and } P\left(\frac{3}{2}\right) = \frac{1}{2^m} - 2^m < 0,$$

which contradict each other.

Thus if there exists a polynomial satisfying requirements of the problem then $n = km$ with m odd.

We now prove that the converse is also true. Suppose that $n = km$ with m odd. Put again $y = x^k$, we show, by induction along m, that there exists a polynomial $P(x)$ that satisfies (2).

For $m = 1$, we can see that $P_1(y) = y$ satisfies (2). If $m = 3$ then $P_3(y) = y^3 + 3y$ satisfies (2). Suppose that we have $P_1, P_3, \ldots, P_m$ satisfy (2). Consider

$$P_{m+2}(x) = (x^2+2)P_m(x) - P_{m-2}(x).$$

By the inductive hypothesis, note that $y \neq 0$, we have

$$\begin{aligned}
P_{m+2}\left(y - \frac{1}{y}\right) &= \left[\left(y - \frac{1}{y}\right)^2 + 2\right] P_m\left(y - \frac{1}{y}\right) - P_{m-2}\left(y - \frac{1}{y}\right)\\
&= \left(y^2 + \frac{1}{y^2}\right)\left(y^m - \frac{1}{y^m}\right) - \left(y^{m-2} - \frac{1}{y^{m-2}}\right)\\
&= y^{m+2} - \frac{1}{y^{m+2}}\\
&\quad (y \neq 0).
\end{aligned}$$

By induction principle, the converse is proven.

Now back to the given problem, we conclude that $n = 1998m$ with m odd.

4.1.50

The domain of definition is $y^2+2x>0$. Put $z=2x-y$ the first equation is written as

$$(1+4^z)\cdot 5^{1-z}=1+2^{1+z},$$

or equivalently,

$$\frac{1+4^z}{5^z}=\frac{1+2^{z+1}}{5}.$$

Note that the left-hand side is decreasing, while the right-hand side is increasing, and $z=1$ is a solution, so it is the unique solution of the first equation.

Thus $2x-y=1$, that is, $x=\frac{y+1}{2}$. Substituting this into the second equation we get

$$y^3+2y+3+\log(y^2+y+1)=0.$$

The left-hand side is increasing, and $y=-1$ is a solution and therefore this is a unique solution of the equation.

We conclude that the only solution of the system is $(x,y)=(0,-1)$.

4.1.51

We write the given condition as follows

$$a\frac{1}{b}+\frac{1}{b}c+ca=1.$$

Since $a,b,c>0$, there exist $A,B,C\in(0,\pi)$ such that $A+B+C=\pi$ and $a=\tan\frac{A}{2}$, $\frac{1}{b}=\tan\frac{B}{2}$, $c=\tan\frac{C}{2}$.

Note that

$$1+\tan^2 x=\frac{1}{\cos^2 x},\ 1+\cos 2x=2\cos^2 x,\ 1-\cos 2x=2\sin^2 x,$$

we have

$$\begin{aligned}
P &= 2\cos^2\frac{A}{2}-2\sin^2\frac{B}{2}+3\cos^2\frac{C}{2}\\
&= (1+\cos A)-(1-\cos B)+3\left(1-\sin^2\frac{C}{2}\right)\\
&= -3\sin^2\frac{C}{2}+2\sin\frac{C}{2}\cos\frac{A-B}{2}+3\\
&= -3\left(\sin\frac{C}{2}-\frac{1}{3}\cos\frac{A-B}{2}\right)^2+\frac{1}{3}\cos^2\frac{A-B}{2}+3.
\end{aligned}$$

This shows that

$$P \le \frac{1}{3}\cos^2\frac{A-B}{2} + 3 \le \frac{1}{3} + 3 = \frac{10}{3},$$

the equality occurs when

$$\begin{cases} \sin\frac{C}{2} = \frac{1}{3}\cos\frac{A-B}{2} \\ \cos\frac{A-B}{2} = 1 \end{cases} \iff \begin{cases} A = B \\ \sin\frac{C}{2} = \frac{1}{3}. \end{cases}$$

4.1.52

We have

$$\frac{2}{5} \le z \le \min\{x, y\} \implies \frac{1}{z} \le \frac{5}{2};\ \frac{z}{x} \le 1.$$

Also

$$xz \ge \frac{4}{15} \implies \frac{1}{xz} \le \frac{15}{4},$$

or equivalently,

$$\frac{1}{\sqrt{x}} \le \frac{\sqrt{15}}{2}\sqrt{z}.$$

From these facts it follows that

$$\begin{aligned} \frac{1}{x} + \frac{1}{z} &= \frac{2}{x} + \frac{1}{z} - \frac{1}{x} \\ &= \frac{2}{\sqrt{x}}\frac{1}{\sqrt{x}} + \frac{1}{z}\left(1 - \frac{z}{x}\right) \\ &\le \frac{2}{\sqrt{x}}\frac{\sqrt{15}}{2}\sqrt{z} + \frac{5}{2}\left(1 - \frac{z}{x}\right). \end{aligned}$$

Furthermore, by the Arithmetic-Geometric Mean inequality, we have

$$\frac{2}{\sqrt{x}}\frac{\sqrt{15}}{2}\sqrt{z} = 2\sqrt{\frac{3}{2}\cdot\frac{5z}{2x}} \le \frac{3}{2} + \frac{5z}{2x}.$$

Then we obtain

$$\frac{1}{x} + \frac{1}{z} \le \frac{3}{2} + \frac{5z}{2x} + \frac{5}{2}\left(1 - \frac{z}{x}\right) = 4.$$

Similarly, we can prove that

$$\frac{1}{y} + \frac{1}{z} \le \frac{9}{2}.$$

Thus

$$P(x,y,z) = \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = \frac{1}{x} + \frac{1}{z} + 2\left(\frac{1}{y} + \frac{1}{z}\right) \le 4 + 9 = 13.$$

The equality occurs if and only if $x = \frac{2}{3}, y = \frac{1}{2}, z = \frac{2}{5}$.

4.1.53

Rewrite the inequality in a form

$$-6a(a^2 - 2b) \le -27c + 10(a^2 - 2b)^{3/2}. \tag{1}$$

Let α, β, γ be three real roots of the given polynomial. By Viète formula

$$\begin{cases} \alpha + \beta + \gamma = -a, \\ \alpha\beta + \beta\gamma + \gamma\alpha = b, \\ \alpha\beta\gamma = -c, \end{cases}$$

and therefore,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = a^2 - 2b.$$

Then the inequality is equivalent to

$$6(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2) \le 27\alpha\beta\gamma + 10(\alpha^2 + \beta^2 + \gamma^2)^{3/2}.$$

Consider two cases.

Case 1: $\alpha^2 + \beta^2 + \gamma^2 = 0$.
Then $\alpha = \beta = \gamma = 0$ and the inequality is obvious.

Case 2: $\alpha^2 + \beta^2 + \gamma^2 > 0$.
Then without loss of generality we can assume that $|\alpha| \le |\beta| \le |\gamma|$ and $\alpha^2 + \beta^2 + \gamma^2 = 9$. In this case the inequality is equivalent to

$$2(\alpha + \beta + \gamma) - \alpha\beta\gamma \le 10.$$

Note also that $3\gamma^2 \ge \alpha^2 + \beta^2 + \gamma^2$, which implies that $\gamma^2 \ge 3$.

We have

$$\begin{aligned} [2(\alpha + \beta + \gamma) - \alpha\beta\gamma]^2 &= [2(\alpha + \beta) + \gamma(2 - \alpha\beta)]^2 \\ &\le \left[(\alpha + \beta)^2 + \gamma^2\right] \cdot \left[4 + (2 - \alpha\beta)^2\right] \\ &\quad \text{(by Cauchy-Schwarz inequality)} \\ &= (9 + 2\alpha\beta)\left[8 - 4\alpha\beta + (\alpha\beta)^2\right] \\ &= 2(\alpha\beta)^3 + (\alpha\beta)^2 - 20(\alpha\beta) + 72 \\ &= (\alpha\beta + 2)^2(2\alpha\beta - 7) + 100. \end{aligned}$$

From $\gamma^2 \geq 3$ it follows that $2\alpha\beta \leq \alpha^2 + \beta^2 = 9 - \gamma^2 \leq 6$. Then

$$[2(\alpha + \beta + \gamma) - \alpha\beta\gamma]^2 \leq 100,$$

or equivalently,

$$2(\alpha + \beta + \gamma) - \alpha\beta\gamma \leq 10.$$

The equality occurs if and only if

$$\begin{cases} |\alpha| \leq |\beta| \leq |\gamma|, \\ \alpha^2 + \beta^2 + \gamma^2 = 9, \\ \frac{\alpha+\beta}{2} = \frac{\gamma}{2-\alpha\beta}, \\ \alpha\beta + 2 = 0, \\ 2(\alpha + \beta + \gamma) - \alpha\beta\gamma \geq 0, \end{cases}$$

which is equivalent to

$$\alpha = -1,\ \beta = \gamma = 2.$$

From both cases it follows that the equality in the problem occurs if and only if (a, b, c) is any permutation of $(-k, 2k, 2k)$ with $k \geq 0$.

4.1.54

1) Note that both $P(x)$ and $Q(x)$ have no zero root, and so $a,\ b \neq 0$. The equality $a^2 + 3b^2 = 4$ gives $a < 2$ and $b < 1.2$. We then have

$$P(-2) = -1,\ P(-1) = 18,\ P(1.5) = -4.5,\ P(1.9) = 0.716,$$

which shows that $P(x)$ has three distinct real roots and the largest is $a \in (1.5, 1.9)$.

Similarly, from

$$Q(-2) = -57,\ Q(-1) = 2,\ Q(0.3) = -0.236,\ Q(1) = 12,$$

it follows that $Q(x)$ has three distinct real roots and the largest is $b \in (0.3, 1)$.

2) We have $P(a) = 0 \Longleftrightarrow 4a^3 - 15a = 2a^2 - 9$. Squaring both sides of this equation we obtain

$$16a^6 - 124a^4 + 261a^2 - 81 = 0. \tag{1}$$

Note that $4 - a^2 > 0$, as $a \in (1.5, 1.9)$. We show that $x_0 = \frac{\sqrt{3(4-a^2)}}{3}$ is a root of $Q(x)$. Indeed,

$$\begin{aligned} Q(x_0) &= \frac{4}{3}(4-a^2)\sqrt{3(4-a^2)} + 2(4-a^2) - \frac{7}{3}\sqrt{3(4-a^2)} + 1 \\ &= \left(\frac{9-4a^2}{3}\right)\sqrt{3(4-a^2)} + 9 - 2a^2. \end{aligned}$$

As $2a^2 < 9 < 4a^2$, we have $9 - 2a^2 > 0$ and $4a^2 - 9 > 0$. Then the equality $Q(x_0) = 0$ is equivalent to

$$\begin{aligned} & \left(\frac{9-4a^2}{3}\right)\sqrt{3(4-a^2)} + 9 - 2a^2 = 0 \\ \Longleftrightarrow\quad & \left(\frac{4a^2-9}{3}\right)\sqrt{3(4-a^2)} = 9 - 2a^2 \\ \Longleftrightarrow\quad & (4a^2-9)^2\left[3(4-a^2)\right] = 9(9-2a^2)^2 \\ \Longleftrightarrow\quad & 3(16a^6 - 124a^4 + 261a^2 - 81) = 0, \end{aligned}$$

which is true, by (1).

Moreover, from $a \in (1.5, 1.9)$ it follows that $\sqrt{1.17} < 3x_0 < \sqrt{5.25}$, and so $0.3 < x_0 < 1$. Since $x = b$ is the unique root of $Q(x)$ in the interval $(0.3, 1)$, we conclude that $x_0 = b$, that is,

$$\frac{\sqrt{3(4-a^2)}}{3} = b,$$

or equivalently, $a^2 + 3b^2 = 4$.

4.1.55

Adding to the first equation the second one, multiplying by 3, we get

$$\begin{aligned} & x^3 + 3x^2 + 3xy^2 - 24xy + 3y^2 = 24y - 51x - 49 \\ \Longleftrightarrow\ & (x^3 + 3x^2 + 3x + 1) + 3y^2(x+1) - 24y(x+1) + 48(x+1) = 0 \\ \Longleftrightarrow\ & (x+1)\left[(x+1)^2 + 3y^2 - 24y + 48\right] = 0 \\ \Longleftrightarrow\ & (x+1)\left[(x+1)^2 + 3(y-4)^2\right] = 0. \end{aligned}$$

Case 1: If $x + 1 = 0$, then $x = -1$. Substituting this value into the first equation we obtain $y = \pm 4$.

Case 2: If $(x+1)^2 + 3(y-4)^2 = 0$, then $x = -1$ and $y = 4$.

Thus there are two solutions $(-1, \pm 4)$.

4.1.56

Note that $x, y, z \neq 0$. Then we can write equivalently the system as follows

$$\begin{cases} x(x^2+y^2+z^2)-2xyz=2 \\ y(x^2+y^2+z^2)-2xyz=30 \\ z(x^2+y^2+z^2)-2xyz=16 \end{cases} \Longleftrightarrow \begin{cases} x(x^2+y^2+z^2)-2xyz=2 \\ (y-z)(x^2+y^2+z^2)=14 \\ (z-x)(x^2+y^2+z^2)=14 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x(x^2+y^2+z^2)-2xyz=2 \\ (y-z)(x^2+y^2+z^2)=14 \\ y=2z-x \end{cases} \Longleftrightarrow \begin{cases} 2x^3-2x^2z+xz^2=2 \\ -2x^3+6x^2z-9xz^2+5z^3=14 \\ y=2z-x \end{cases}$$

$$\Longleftrightarrow \begin{cases} 2x^3-2x^2z+xz^2=2 \\ 5z^3-16xz^2+20x^2z-16x^3=0 \\ y=2z-x. \end{cases}$$

As $x, z \neq 0$, put $t = \dfrac{z}{x}$, we have

$$5t^3-16t^2+20t-16=0 \Longleftrightarrow (t-2)(5t^2-6t+8)=0 \Longleftrightarrow t=2.$$

Hence $z = 2x$ and the system is equivalent to

$$\begin{cases} 2x^3-2x^2z+xz^2=2 \\ z=2x \\ y=2z-x \end{cases} \Longleftrightarrow x=1, y=3, z=2.$$

4.1.57

Note that $P(a,b,c)$ is homogeneous, that is $P(ta,tb,tc)=P(a,b,c),\ \forall t>0$. Also note that if (a,b,c) satisfy conditions of the problem, then so does $(ta,tb,tc),\ t>0$. Therefore, w.l.o.g. we can assume that $a+b+c=4$, and hence $abc=2$. So the problem can be restated as follows: find the maximum and minimum values of

$$P=\frac{1}{256}(a^4+b^4+c^4),$$

if $a,b,c>0$ satisfying $a+b+c=4$ and $abc=2$.

Put $A = a^4 + b^4 + c^4$, $B = ab + bc + ca$, we have

$$\begin{aligned}
A &= (a^2 + b^2 + c^2)^2 - 2(a^2b^2 + b^2c^2 + c^2a^2)\\
&= [(a + b + c)^2 - 2(ab + bc + ca)]^2 - 2[(ab + bc + ca)^2 - 2abc(a + b + c)]\\
&= (16 - 2B)^2 - 2(B^2 - 16)\\
&= 2(B^2 - 32B + 144).
\end{aligned}$$

By conditions $a + b + c = 4$ and $abc = 2$, the inequality $(b + c)^2 \geq 4bc$ is equivalent to

$$\begin{aligned}
(4 - a)^2 \geq \frac{8}{a} \quad &\Longleftrightarrow \quad a^3 - 8a^2 + 16a - 8 \geq 0\\
&\Longleftrightarrow \quad (a - 2)(a^2 - 6a + 4) \geq 0\\
&\Longleftrightarrow \quad 3 - \sqrt{5} \leq a \leq 2 \text{ (as } a \in (0, 4)).
\end{aligned}$$

By symmetry, we also have $3 - \sqrt{5} \leq b \leq 2$ and $3 - \sqrt{5} \leq c \leq 2$.

Then we have

$$\begin{aligned}
(a - 2)(b - 2)(c - 2) \leq 0 &\Longleftrightarrow abc - 2(ab + bc + ca) + 4(a + b + c) - 8 \leq 0\\
&\Longleftrightarrow 10 - 2B \leq 0\\
&\Longleftrightarrow 5 \leq B.
\end{aligned}$$

Similarly, from $[a - (3 - \sqrt{5})][b - (3 - \sqrt{5})][c - (3 - \sqrt{5})] \geq 0$ we get

$$8\sqrt{5} - 14 - (3 - \sqrt{5})B \geq 0 \Longleftrightarrow B \leq \frac{5\sqrt{5} - 1}{2}.$$

Since $A = 2(B^2 - 32B + 144)$, as a quadratic function of B, is decreasing on $(0, 16) \supset \left[5, \dfrac{5\sqrt{5} - 1}{2}\right]$, we obtain

$$A_{\min} = A\Big|_{B=\frac{5\sqrt{5}-1}{2}} = 383 - 165\sqrt{5},\ A_{\max} = A\Big|_{B=5} = 18,$$

that is,

$$P_{\min} = \frac{383 - 165\sqrt{5}}{256},\ \text{occurs at say } a = 3 - \sqrt{5}, b = c = \frac{1 + \sqrt{5}}{2},$$

$$P_{\max} = \frac{9}{128},\ \text{occurs at say } a = 2, b = c = 1.$$

4.1.58

The domain of definition is $x \geq -1,\ y \geq -2$. We write the assumption of the problem as follows

$$x + y = 3(\sqrt{x+1} + \sqrt{y+2}).$$

Then m belongs to the range of $P = x + y$ if and only if the following system is solvable

$$\begin{cases} 3(\sqrt{x+1} + \sqrt{y+2}) = m, \\ x + y = m. \end{cases}$$

Put $u = \sqrt{x+1},\ v = \sqrt{y+2}$, we have

$$\begin{cases} 3(u+v) = m \\ u^2 + v^2 = m + 3 \end{cases} \iff \begin{cases} u + v = \frac{m}{3} \\ uv = \frac{1}{2}\left(\frac{m^2}{9} - m - 3\right) \\ u, v \geq 0. \end{cases}$$

The given system is solvable if and only the second system is solvable, that is, by Viète theorem, the quadratic equation

$$18t^2 - 6mt + m^2 - 9m - 27 = 0$$

has two nonnegative roots, or equivalently

$$\begin{cases} -m^2 + 18m + 54 \geq 0 \\ m \geq 0 \\ m^2 - 9m - 27 \geq 0 \end{cases} \iff \frac{9 + 3\sqrt{21}}{2} \leq m \leq 9 + 3\sqrt{15}.$$

Since nonnegative numbers $u,\ v$ exist, there are $x,\ y$ such that

$$P_{\min} = \frac{9 + 3\sqrt{21}}{2}, \text{ and } P_{\max} = 9 + 3\sqrt{15}.$$

4.1.59

Assume that $x = \max\{x, y, z\}$. Consider two cases:

Case 1: $x \geq y \geq z$. In this case we have

$$\begin{cases} x^3 + 3x^2 + 2x - 5 \leq x \\ z^3 + 3z^2 + 2z - 5 \geq z \end{cases} \iff \begin{cases} (x-1)[(x+2)^2 + 1] \leq 0 \\ (z-1)[(z+2)^2 + 1] \geq 0 \end{cases} \iff \begin{cases} x \leq 1 \\ z \geq 1. \end{cases}$$

Case 2: $x \geq z \geq y$. Similarly, we have

$$\begin{cases} x \leq 1, \\ y \geq 1. \end{cases}$$

Both cases give $x = y = z = 1$, which is a unique solution of the system.

4.1.60

Substituting $a = \dfrac{1}{\alpha^2}, b = c = \alpha \neq 0$ into the inequality we have

$$\begin{aligned} & \alpha^4 + \frac{2}{\alpha^2} + 3k \geq (k+1)\left(\frac{1}{\alpha^2} + 2\alpha\right) \\ \iff \quad & (2\alpha^3 - 3\alpha^2 + 1)k \leq \alpha^6 - 2\alpha^3 + 1. \end{aligned}$$

Notice that $k \leq 1$ as $\alpha \to 0$. So we show that $k = 1$ is the desired value, that is for any positive numbers a, b, c with $abc = 1$ we have to prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + 3 \geq 2(a + b + c). \tag{1}$$

Indeed, since $abc = 1$, there are two numbers say a, b such that either $a, b \geq 1$, or $a, b \leq 1$. In this case, due to the facts that

$$\frac{1}{ab} = c, \;\; a^2b^2 = \frac{1}{c^2},$$

the inequality (1) is equivalent to

$$\left(\frac{1}{a} - \frac{1}{b}\right)^2 + 2(a-1)(b-1) + (ab-1)^2 \geq 0,$$

which is obviously true.

4.1.61

It is obvious that $\deg P > 0$.

Case 1. If $\deg P = 1$, $P(x) = ax + b, \;\; a \neq 0$. Substituting it into the given equation we get

$$(a^2 - 3a + 2)x^2 + 2b(a-2)x + b^2 - b = 0, \;\; \forall x,$$

which gives $(a, b) = (1, 0), (2, 0)$ and $(2, 1)$. Thus we obtain $P(x) = x, P(x) = 2x$ and $P(x) = 2x + 1$.

Case 2. If $\deg P = n \geq 2$, put $P(x) = ax^n + Q(x),\ a \neq 0$, where Q is a polynomial with $\deg Q = k < n$. Substituting it into the equation we get

$$(a^2 - a)x^{2n} + [Q(x)]^2 - Q(x^2) + 2ax^n Q(x)$$

$$= [3 + (-1)^n]\, ax^{n+1} + [3Q(x) + Q(-x)]\, x - 2x^2,\ \forall x.$$

Notice that the degree of the right-hand side polynomial is $n + 1$, and $n + 1 < 2n$, we have $a^2 - a = 0$, that is $a = 1$. Then

$$2x^n Q(x) + [Q(x)]^2 - Q(x^2)$$

$$= [3 + (-1)^n]\, x^{n+1} + [3Q(x) + Q(-x)]\, x - 2x^2,\ \forall x.$$

Again notice that the degree of the left-hand side polynomial is $n + k$, while the degree of the right-hand side polynomial is $n + 1$, we get $k = 1$. Moreover, for $x = 0$ we have $[Q(0)]^2 - Q(0) = 0$, or equivalently, either $Q(0) = 0$ or $Q(0) = 1$. Thus $Q(x)$ is of the form either ax, or $ax + 1$.

Case 1. If $Q(x) = ax$, then

$$[3 + (-1)^n - 2a]\, x^{n+1} - (a^2 - 3a + 2)x^2 = 0,\ \forall x$$

$$\Longleftrightarrow \begin{cases} 3 + (-1)^n - 2a = 0 \\ a^2 - 3a + 2 = 0 \end{cases} \quad \Longleftrightarrow \begin{cases} a = 1, n \text{ odd} \\ a = 2, n \text{ even}, \end{cases}$$

which gives $P(x) = x^{2n+1} + x$ and $P(x) = x^{2n} + 2x$, both are satisfied the given equation.

Case 2. If $Q(x) = ax + 1$, then

$$[3 + (-1)^n - 2a]\, x^n + 1 - 2x^n - (a^2 - 3a + 2)x^2 - 2(a - 2)x = 0,\ \forall x,$$

which is impossible.

Thus all desired polynomials are

$$P(x) = x;\ P(x) = x^{2n} + 2x;\ P(x) = x^{2n+1} + x,\ n \geq 0.$$

4.1.62

The domain of definition is $x > 0, y > 0, y + 3x \neq 0$. The system is equivalent to

$$\begin{cases} 1 - \frac{12}{y+3x} = \frac{2}{\sqrt{x}} \\ 1 + \frac{12}{y+3x} = \frac{6}{\sqrt{y}} \end{cases} \Longleftrightarrow \begin{cases} \frac{1}{\sqrt{x}} + \frac{3}{\sqrt{y}} = 1 \\ -\frac{1}{\sqrt{x}} + \frac{3}{\sqrt{y}} = \frac{12}{y+3x}. \end{cases}$$

Multiplying the first equation by the second one yields

$$\frac{9}{y} - \frac{1}{x} = \frac{12}{y+3x} \Longleftrightarrow y^2 + 6xy - 27x^2 = 0 \Longleftrightarrow y = 3x;\ y = -9x.$$

As $x, y > 0$, we have $y = 3x$. Then both equations are equivalent to

$$\frac{1}{\sqrt{x}} + \frac{3}{\sqrt{3x}} = 1 \Longleftrightarrow \sqrt{x} = 1 + \sqrt{3},$$

which gives $x = 4 + 2\sqrt{3}$ and hence $y = 12 + 6\sqrt{3}$. This is the only solution of the given system.

4.1.63

Without loss of generality we can assume that $z > y > x \geq 0$. Put $y = x + a, z = x + a + b$ with $a, b > 0$, the right-hand side of the inequality can be written as

$$M = \left[3x^2 + 2(2a+b)x + a(a+b)\right]\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{(a+b)^2}\right).$$

Then we have

$$\begin{aligned} M &\geq a(a+b)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{(a+b)^2}\right) \\ &= \frac{a+b}{a} + \frac{a(a+b)}{b^2} + \frac{a}{a+b} \\ &= 1 + \frac{b}{a} + \frac{a(a+b)}{b^2} + 1 - \frac{b}{a+b} \\ &= 2 + \frac{a(a+b)}{b^2} + \frac{b^2}{a(a+b)} \geq 2 + 2 = 4. \end{aligned}$$

The equality occurs if and only if

$$\begin{cases} 3x^2 + 2(2a+b)x = 0, \\ a(a+b) = b^2, \end{cases}$$

which gives $x = 0$ and $a(a+b) = b^2$. For $x = 0$ we have $y = a$ and $z = a+b$, and so $b = z-y$. Then for $a(a+b) = b^2$ we have $yz = (z-y)^2$, which gives $y^2-3yz+z^2 = 0$. This equation becomes $t^2-3t+1 = 0$ with $t = \dfrac{y}{z} \in (0,1)$, and hence, $t = \dfrac{3-\sqrt{5}}{2}$. That is, $x = 0$ and $\dfrac{y}{z} = \dfrac{3-\sqrt{5}}{2}$.

4.2 Analysis

4.2.1

1) Substituting $y = a-x$ into the sum $S = x^m+y^m$ and consider a function $S(x) = x^m + (a-x)^m,\ 0 \le x \le a$. As its first order derivative has a form

$$S'(x) = mx^{m-1} - m(a-x)^{m-1},$$

we can easily check that the function $S(x)$ is decreasing on $[0, \frac{a}{2}]$, increasing on $[\frac{a}{2}, a]$, and so $S(x)$ attains its minimum at $x = \frac{a}{2}$. Thus $S = x^m + y^m$ is minimum if $x = y = \frac{a}{2}$, and the minimum value is $2\left(\frac{a}{2}\right)^m$.

2) Now consider a sum $T = x_1^m + \cdots + x_n^m$ with $x_1 + \cdots + x_n = k$ constant. If numbers $x_1, \ldots, x_n$ are not equal, there exist say x_1, x_2 such that $x_1 < \frac{k}{n} < x_2$. In this case, replacing x_1 by $x_1' = \frac{k}{n}$ and x_2 by $x_2' = x_1 + x_2 - \frac{k}{n}$, we see that the sum is unchanged, as $x_1' + x_2' = x_1 + x_2$. We prove that

$$T' = (x_1')^m + (x_2')^m + \cdots + x_n^m < T = x_1^m + \cdots + x_n^m. \qquad (1)$$

It is equivalent to $(x_1')^m + (x_2')^m < x_1^m + x_2^m$. Denote $x_1 + x_2 = x_1' + x_2' = a > 0$, the inequality becomes

$$\left(\frac{k}{n}\right)^m + \left(a - \frac{k}{n}\right)^m < x_1^m + x_2^m,\ x_1 < \frac{k}{n} < x_2.$$

Consider $S(x) = x^m + (a-x)^m$, we have $S(x_1) = S(x_2) = x_1^m + x_2^m$, and by part 1), $S(x)$ is decreasing on $[0, \frac{a}{2}]$, increasing on $[\frac{a}{2}, a]$. There are two cases:

(i) If $\frac{k}{n} \le \frac{a}{2}$: in this case since $x_1 < \frac{k}{n}$, $S(x_1) > S\left(\frac{k}{n}\right)$.

(ii) If $\frac{k}{n} > \frac{a}{2}$: similarly, since $x_2 > \frac{k}{n}$, $S(x_2) > S\left(\frac{k}{n}\right)$.

Thus (1) shows that the sum is decreasing, and moreover, among $x_1', x_2', \ldots, x_n$ there is a number with value $\frac{k}{n}$. Continuing this process, after at most n times we obtain the smallest sum, that is

$$x_1^m + \cdots + x_n^m > \left(\frac{k}{n}\right)^m + \cdots + \left(\frac{k}{n}\right)^m.$$

Thus the sum $x_1^m + \cdots + x_n^m$ is minimum if $x_1 = \cdots = x_n = \frac{k}{n}$.

4.2.2

The given function y is defined for $x \neq \frac{\pi}{6}$ and $x \neq \frac{\pi}{3}$. As

$$\cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1},$$

we can write

$$y = \frac{\cot^3 x}{\cot 3x} = \frac{\cot^2 x(3\cot^2 x - 1)}{\cot^2 x - 3}.$$

Put $t = \cot^2 x > 0$ we obtain

$$y = \frac{t(3t-1)}{t-3},$$

or a quadratic equation $3t^2 - (y+1)t + 3y = 0$. The existence of the given function y means that the last equation has roots, that is the discriminant

$$\Delta = (y+1)^2 - 36y \geq 0 \Longleftrightarrow y^2 - 34y + 1 \geq 0,$$

which gives $y \leq 17 - 12\sqrt{2}$ and $y \geq 17 + 12\sqrt{2}$. Note that $y = 17 + 12\sqrt{2}$ at

$$t = \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2} \Longleftrightarrow \tan x = \sqrt{2} - 1 \Longleftrightarrow x = \frac{\pi}{8},$$

and $y = 17 - 12\sqrt{2}$ at

$$t = \frac{1}{3 + 2\sqrt{2}} = 3 - 2\sqrt{2} \Longleftrightarrow \tan x = \sqrt{2} + 1 \Longleftrightarrow x = \frac{3\pi}{8}.$$

Thus the (local) maximum and minimum of y in the given interval are $17 \pm 12\sqrt{2}$ which has the sum 34 (integer).

4.2.3

Since $1 + \cos\alpha_i \geq 0$ $(i = 1, \ldots, n)$, by hypothesis, we can write

$$\sum_{i=1}^{n}(1 + \cos\alpha_i) = 2\sum_{i=1}^{n}\cos^2\frac{\alpha_i}{2} = 2M + 1 \text{ (M is a nonnegative integer).}$$

Note that for $x \in [0, \frac{\pi}{2}]$ there always hold

$$\sin x \cos x \begin{cases} \geq \sin^2 x, & \text{if } x \in [0, \frac{\pi}{4}], \\ \geq \cos^2 x, & \text{if } x \in [\frac{\pi}{4}, \frac{\pi}{2}]. \end{cases}$$

Without loss of generality, we can assume that $\alpha_1 \leq \cdots \leq \alpha_n$. Then for some k_0 we have

$$\begin{aligned} S &= \sum_{i=1}^{n} \sin \alpha_i = 2 \sum_{i=1}^{n} \sin \frac{\alpha_i}{2} \cos \frac{\alpha_i}{2} \\ &\geq 2 \sum_{i=1}^{k_0} \sin^2 \frac{\alpha_i}{2} + 2 \sum_{i=k_0+1}^{n} \cos^2 \frac{\alpha_i}{2} \\ &= A + B, \end{aligned}$$

where both $A, B \geq 0$.

There are two cases.

Case 1: $B \geq 1$. Then obviously $S \geq 1$.

Case 2: $B < 1$. Then we write A as follows:

$$\begin{aligned} A &= 2 \sum_{i=1}^{k_0} \sin^2 \frac{\alpha_i}{2} = 2 \sum_{i=1}^{k_0} \left(1 - \cos^2 \frac{\alpha_i}{2}\right) \\ &= 2k_0 - 2 \sum_{i=1}^{k_0} \cos^2 \frac{\alpha_i}{2} = 2k_0 - (2M + 1 - B). \end{aligned}$$

Note that $A \geq 0$, we get

$$2k_0 \geq 2M + 1 - B.$$

Also since $B < 1$,

$$2M + 1 - B > 2M.$$

Thus $2k_0 > 2M$, or $k_0 > M$, which means that $k_0 \geq M+1$. This inequality is equivalent to

$$2k_0 \geq 2M + 2.$$

Now taking into account that $B \geq 0$, we obtain

$$\begin{aligned} S \geq A + B &= 2k_0 - (2M + 1 - B) + B \\ &= 2k_0 - (2M + 1) + 2B \\ &\geq 1 + 2B \geq 1. \end{aligned}$$

The problem is solved completely.

Remark. This problem can be generalized, and proved by induction, as follows: *If* $0 \le \varphi \le \pi$, M *is integer and* $\sum_{i=1}^{n}(1+\cos\alpha_i) = 2M+1+\cos\varphi$, *then* $\sum_{i=1}^{n}\sin\alpha_i \ge \sin\varphi$.

4.2.4

1) We have

$$(1+2\sin x\cos x)^2 - 8\sin x\cos x = (1-2\sin x\cos x)^2 \ge 0,$$

which shows that

$$(1+2\sin x\cos x)^2 \ge 8\sin x\cos x.$$

From this inequality, taking into account that $(\sin x+\cos x)^2 = 1+2\sin x\cos x$, we obtain

$$(\sin x+\cos x)^4 = (1+2\sin x\cos x)^2 \ge 8\sin x\cos x = 4\sin 2x.$$

Furthermore, since $0 \le x \le \frac{\pi}{2})$, $\sin x, \cos x \ge 0$. Therefore, the last inequality is equivalent to

$$\sin x+\cos x \ge \sqrt[4]{4\sin 2x} = \sqrt{2}\sqrt[4]{\sin 2x},$$

or

$$\sqrt{2}(\sin x+\cos x) \ge 2\sqrt[4]{\sin 2x}.$$

The equality occurs if and only if $1-2\sin x\cos x = 0$, or $\sin 2x = 1$, that is $x = \frac{\pi}{4}$.

2) The inequality is defined for $y \in (0,\pi) \setminus \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$.

Replacing $\tan 2y = \dfrac{2\tan y}{1-\tan^2 y}$, $\cot 2y = \dfrac{\cot^2 y-1}{2\cot y}$ and $\cot y = \dfrac{1}{\tan y}$ in the given inequality, we have to show that

$$\frac{2+\tan^2 y-\tan^4 y}{1-\tan^2 y} \le 2. \tag{1}$$

Indeed, if $\tan^2 y < 1$, then (1) is equivalent to

$$2+\tan^2 y-\tan^4 y \le 2(1-\tan^2 y) \Longleftrightarrow 3\tan^2 y-\tan^4 y \le 0,$$

or

$$\tan^2 y(3 - \tan^2 y) \le 0.$$

But $\tan^2 y > 0$ (as $0 < y < \pi$), then the last inequality implies that $\tan^2 y \ge 3$, which contradicts the hypothesis $\tan^2 y < 1$.

Thus we should have $\tan^2 y > 1$. Then (1) is equivalent to

$$2 + \tan^2 y - \tan^4 y \ge 2(1 - \tan^2 y) \Longleftrightarrow \tan^2 y(3 - \tan^2 y) \ge 0,$$

or $\tan^2 y \le 3$.

Thus the given inequality reduces to $1 < \tan^2 y \le 3$, or $1 < |\tan y| \le \sqrt{3}$. This gives

$$\frac{\pi}{4} < y \le \frac{\pi}{3}, \text{ and } \frac{2\pi}{3} \le y < \frac{3\pi}{4}.$$

4.2.5

We observe that (u_n) consists of odd terms in the Fibonacci sequence (F_n) : $F_1 = F_2 = 1, F_{n+2} = F_{n+1} - F_n$ $(n \ge 1)$. We show the following property

$$\cot^{-1} F_1 - \cot^{-1} F_3 - \cot^{-1} F_5 - \cdots - \cot^{-1} F_{2n+1} = \cot^{-1} F_{2n+2}. \quad (1)$$

Indeed, as $\cot(a-b) = \dfrac{\cot a \cot b + 1}{\cot b - \cot a}$, we have

$$\cot^{-1} F_{2k} - \cot^{-1} F_{2k+1} = \cot^{-1} \frac{F_{2k}F_{2k+1} + 1}{F_{2k+1} - F_{2k}} = \cot^{-1} \frac{F_{2k}F_{2k+1} + 1}{F_{2k-1}}.$$

Note that $F_{i+1}F_{i+2} - F_iF_{i+3} = (-1)^i$, $\forall i$. In particular,

$$F_{2k}F_{2k+1} - F_{2k-1}F_{2k+2} = -1 \Longleftrightarrow F_{2k}F_{2k+1} + 1 = F_{2k-1}F_{2k+2}.$$

Then

$$\cot^{-1} F_{2k} - \cot^{-1} F_{2k+1} = \cot^{-1} F_{2k+2}. \quad (2)$$

Letting in (2) $k = 1, 2, \ldots, n$ and summing up the obtained equalities, we get (1).

Furthermore, from (1) it follows that

$$\cot^{-1} u_1 - \sum_{i=2}^{n} \cot^{-1} u_i = \cot^{-1} F_{2n+2}.$$

Since $\lim\limits_{n\to\infty} F_{2n+2} = +\infty$, $\lim\limits_{n\to\infty} \cot^{-1} F_{2n+2} = 0$. Therefore,

$$\lim_{n\to\infty} \left(\sum_{i=2}^{n} \cot^{-1} u_i \right) = \cot^{-1} u_1 = \frac{\pi}{4},$$

and hence

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \cot^{-1} u_i \right) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

4.2.6

If $b = 0$, then $xP(x-a) = xP(x)\ \forall x$, which implies that $P(x)$ is a constant polynomial.

For $b \neq 0$, we show that if b/a is not integer then $P(x) \equiv 0$. Indeed, there are two cases:

• If $\deg P = 0$, that is $P(x) \equiv C$, then $xC = (x-b)C \quad \forall x$, and hence $C = 0$, or $P(x) \equiv 0$.

• Consider case $\deg P = n \geq 1$ and P satisfies the equation. We prove that $b/a = n$ (a positive integer).

We write the given equation in the form

$$bP(x) = x[P(x) - P(x-a)],\ \forall x. \tag{1}$$

Suppose that $P(x) = k_0 x^n + \cdots + k_n$ $(k_0 \neq 0, n \geq 1)$. Then

$$\begin{aligned} P(x) - P(x-a) &= k_0[x^n - (x-a)^n] + \text{ polynomial of degree } (n-2) \\ &= nk_0 a x^{n-1} + \text{ polynomial of degree } (n-2). \end{aligned}$$

Substituting expressions of $P(x)$ and $P(x)-P(x-a)$ into (1), we obtain

$$k_0 b x^n + k_1 b x^{n-1} + \cdots = nk_0 a x^n + \text{ polynomial of degree } (n-1),$$

which implies that $k_0 b = nk_0 a \Longleftrightarrow b = na$, or $b/a = n$.

Thus if b/a is not a positive integer, then $P(x) \equiv 0$.

Now suppose that $b/a = n$, or $b = na$ with $n \in \mathbb{N}$. Then the given equation becomes

$$xP(x-a) = (x-na)P(x),\ \forall x.$$

From this it follows, in particular, that

$$P(0) = P(a) = P(2a) = \cdots = P[(n-1)a] = 0.$$

That is,

$$P(x) = Cx(x-a)(x-2a)\cdots[x-(n-1)a].$$

Summarizing,
* For $b=0$: $P(x) \equiv$ constant.
* For $b \neq 0$:

$$P(x) = \begin{cases} 0, & \text{if } \frac{b}{a} \notin \mathbb{N} \\ C(x-a)(x-2a)\cdots[x-(n-1)a], & \text{if } \frac{b}{a} = n \in \mathbb{N}, C = \text{const.} \end{cases}$$

4.2.7

From (1) we have $f(0)\cdot f(0) = 2f(0)$ and, due to (2), $f(0)=2$.

Also by (1)

$$f(x)\cdot f(1) = f(x+1)\cdot f(x-1) \iff f(x+1) = f(1)\cdot f(x) - f(x-1). \quad (3)$$

Since

$$f(1) = \frac{5}{2} = 2 + 2^{-1},$$

by (3)

$$f(2) = f(1)\cdot f(1) - f(0) = \frac{25}{4} - 2 = 4 + \frac{1}{4} = 2^2 + 2^{-2}.$$

Similarly,

$$f(3) = f(1)\cdot f(2) - f(1) = (2+2^{-1})(2^2+2^{-2}) - (2^2+2^{-1}) = 2^3 + 2^{-3},$$

and so on. By induction we can easily verify that

$$f(n) = 2^n + 2^{-n}.$$

Conversely, it is easy to see that $f(x) = 2^x + 2^{-x}$ satisfies the assumptions (1) and (2). So this is the unique solution of the problem.

4.2.8

Note that

$$2^m = (1+1)^m = \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} \quad (m = 1, 2, \ldots, n+1).$$

Consider a polynomial

$$P(y) = 2\left(\binom{y-1}{0} + \binom{y-1}{1} + \cdots + \binom{y-1}{n}\right),$$

whose leading term $2\binom{y-1}{n} = 2\frac{(y-1)\cdots(y-n)}{n!}$ is of degree n with respect to y. Thus $\deg P = n$.

Two polynomials $P(y)$ and $M(y)$ are of degree n and coincide at $n+1$ points $y = 1, 2, \ldots, n+1$ must be equal, and hence

$$\begin{aligned} M(n+2) &= 2\left(\binom{n+1}{0} + \binom{n+1}{1} + \cdots + \binom{n+1}{n}\right) \\ &= 2\left(2^{n+1} - \binom{n+1}{n+1}\right) \\ &= 2^{n+2} - 2. \end{aligned}$$

Remark. This problem can be solved by using the Lagrange's interpolating formula.

4.2.9

We compare the given sequence denoted by $A = \{a_n\}$ and a sequence $B = \{2n\}$ of even numbers. Let $H = \{h_n := 2n - a_n\}$, we notice that H consists of one 1, two 2, three 3, ... and therefore, $a_n = 2n - h_n$.

Group H as follows

$$(1), (2, 2), (3, 3, 3), \ldots, \underbrace{(k, k, \ldots, k)}_{\text{k times}}, \ldots$$

If h_n is in the k-th group, then $h_n = k$.

Note that there are $1 + 2 + \cdots + (k-1) = \frac{k(k-1)}{2}$ terms preceding the kth group. Therefore, if $\dfrac{(k-1)k}{2} + 1 \le n \le \dfrac{k(k+1)}{2} + 1$, then $h_n = k$.

Furthermore,

$$\frac{(k-1)k}{2} + 1 \le n \implies \frac{1 - \sqrt{8n-7}}{2} \le k \le \frac{1 + \sqrt{8n-7}}{2},$$

and

$$n \le \frac{k(k+1)}{2} + 1 \implies \frac{-1 - \sqrt{8n-7}}{2} > k, \text{ and } \frac{-1 + \sqrt{8n-7}}{2} < k.$$

Thus

$$\frac{-1 + \sqrt{8n-7}}{2} < k \le \frac{1 + \sqrt{8n-7}}{2},$$

or

$$k \le \frac{1 + \sqrt{8n-7}}{2} < k + 1,$$

which implies that $k =$ the integral part of $\frac{1+\sqrt{8n-7}}{2} = h_n$, that is

$$a_n = 2n - \left[\frac{1+\sqrt{8n-7}}{2}\right], \ (n = 1, 2, \ldots).$$

4.2.10

Denote the sum of all $\cos(\pm u_1 \pm u_2 \pm \ldots \pm u_{1987})$ by $S = \sum \cos(\pm u_1 \pm u_2 \pm \ldots \pm u_{1987})$. We first prove, by induction, that

$$\sum \cos(\pm u_1 \pm u_2 \pm \ldots \pm u_n) = 2^n \cdot \prod_{k=1}^{n} \cos u_k. \tag{1}$$

Indeed, for $n = 1$: $\cos u_1 + \cos(-u_1) = 2\cos u_1$, and (1) is true. Suppose that the formula is true for n. Then

$$\begin{aligned}
2^{n+1} \cdot \prod_{k=1}^{n+1} \cos u_k &= 2\left(2^n . \prod_{k=1}^{n} \cos u_k\right) \cdot \cos u_{n+1} \\
&= 2\left[\sum \cos(\pm u_1 \pm u_2 \pm \ldots \pm u_n)\right] \cdot \cos u_{n+1} \\
&\quad \text{(by induction hypothesis)} \\
&= \sum [2\cos(\pm u_1 \pm u_2 \pm \ldots \pm u_n) \cos u_{n+1}] \\
&= \sum \left[\cos(\pm u_1 \pm u_2 \pm \ldots \pm u_n + u_{n+1})\right. \\
&\quad \left. + \cos(\pm u_1 \pm u_2 \pm \ldots \pm u_n - u_{n+1})\right] \\
&= \sum \cos(\pm u_1 \pm u_2 \pm \ldots \pm u_n \pm u_{n+1}).
\end{aligned}$$

Now return back to our problem. We have

$$S = 2^{1987} \cdot \prod_{k=1}^{1987} \cos u_k.$$

Since

$$u_{1986} = u_1 + 1985d = \frac{\pi}{1987} + \frac{1985\pi}{2 \cdot 1987} = \frac{\pi}{2},$$

$\cos u_{1986} = 0$ and hence $S = 0$.

4.2.11

Consider $F(x) = f^2(x) + 2\cos x$ defined on $[0, +\infty)$. Then by (1)

$$|F(x)| \leq |f(x)|^2 + 2|\cos x| \leq 5^2 + 2,$$

and by (2)

$$F'(x) = 2f(x)f'(x) - 2\sin x \geq 0,$$

which shows that $F(x)$ is increasing.

Let $(x_n) := \{2\pi, 2\pi + \frac{\pi}{2}, 4\pi, 4\pi + \frac{\pi}{2}, 6\pi, 6\pi + \frac{\pi}{2}, \ldots\}$. It is clear that $x_n > 0$, (x_n) is increasing, and $x_n \to +\infty$ as $n \to \infty$.

Put $u_n = F(x_n)$, we see that (u_n) is increasing and bounded from above, and hence, by Weierstrass theorem, there does exist $\lim\limits_{n\to\infty} u_n$.

Assume that $\exists \lim\limits_{x\to+\infty} f(x)$. This implies that if $v_n = f(x_n)$, then $\exists \lim\limits_{n\to\infty} v_n$. Therefore, there does exist

$$\lim_{n\to\infty} \left[F(x_n) - f^2(x_n)\right] = \lim_{n\to\infty} u_n - \lim_{n\to\infty} (v_n)^2,$$

or equivalently, there exists $\lim\limits_{n\to\infty} \cos x_n$ which is impossible, as the sequence $\big(\cos x_n\big) = \{1, 0, 1, 0, \ldots\}$ has no limit.

4.2.12

The answer is yes.

Put $M_n = \max\{x_n, x_{n+1}\}$, we see that (M_n) is decreasing and bounded from below. So by Weierstrass theorem, $\exists \lim\limits_{n\to\infty} M_n = m$.

We prove that (x_n) has the limit M. Indeed, given $\varepsilon > 0$ we have

$$\exists N_0 \; \forall n \geq N_0 : M - \frac{\varepsilon}{3} < M_n < M + \frac{\varepsilon}{3}. \tag{1}$$

Let $p \geq N_0 + 1$, then

$$x_{p-1} \leq M_{p-1} < M + \frac{\varepsilon}{3}.$$

Consider x_p, there are two cases:

If $x_p > M - \frac{\varepsilon}{3}$, then we have

$$M - \frac{\varepsilon}{3} < x_p \leq M_n < M + \frac{\varepsilon}{3}, \tag{2}$$

If $x_p \le M - \frac{\varepsilon}{3}$, then
$$x_{p+1} > M - \frac{\varepsilon}{3}$$
(otherwise, $M_p = \max\{x_p, x_{p+1}\} \le M - \frac{\varepsilon}{3}$, which contradicts (1)). In this case, we have

$$x_p \ge 2x_{p+1} - x_{p-1} > 2\left(M - \frac{\varepsilon}{3}\right) - \left(M + \frac{\varepsilon}{3}\right) = M - \varepsilon.$$

Therefore,
$$M - \varepsilon < x_p \le M - \frac{\varepsilon}{3} < M + \varepsilon. \tag{3}$$

Combining (2) and (3) yields

$$M - \varepsilon < x_p < M + \varepsilon.$$

This is true for any $p \ge N_0 + 1$, and so there exists $\lim\limits_{p\to\infty} x_p = M$.

4.2.13

From the assumptions it follows that

$$0 \le P_n(x) \le P_{n+1}(x) \le 1,\ \forall x \in [0,1], \forall n \ge 0.$$

The first two inequalities $0 \le P_n(x)$ and $P_n(x) \le P_{n+1}(x)$ can be easily proved by induction, while the third one $P_{n+1}(x) \le 1$ is obvious, as $1 - P_{n+1}(x) = \dfrac{1-x}{2} + \dfrac{[1-P_n(x)]^2}{2} \ge 0$.

Thus for each fixed $x \in [0,1]$ a sequence $(P_n(x))$ is increasing and bounded from above, and hence, by Weierstrass theorem, there exists

$$\lim_{n\to\infty} P_n(x) := f(x).$$

Then letting $n \to \infty$ in the assumption equation for each fixed $x \in [0,1]$ gives
$$f(x) = f(x) + \frac{x - f^2(x)}{2} \Longleftrightarrow f^2(x) = x.$$
But $P_n(x) \ge 0$, then $f(x) = \lim\limits_{n\to\infty} P_n(x) \ge 0$. Therefore, we obtain $f(x) = \sqrt{x}$. Also, since $(P_n(x))$ is increasing, $f(x) \ge P_n(x)$, that is

$$0 \le \sqrt{x} - P_n(x),\ \forall x \in [0,1], \forall n \ge 0.$$

The first desired inequality is proved.

Now we prove the second inequality. Put $\alpha_n(x) = \sqrt{x} - P_n(x) \geq 0$. Then

$$\begin{aligned}\alpha_{n+1}(x) &= \alpha_n(x) \cdot \left[1 - \frac{\sqrt{x} + P_n(x)}{2}\right] \\ &\leq \alpha_n(x)\left(1 - \frac{\sqrt{x}}{2}\right), \ \forall x \in [0,1], \forall n \geq 0.\end{aligned}$$

From this it follows that

$$\begin{aligned}\alpha_n(x) &\leq \alpha_{n-1}(x)\left(1 - \frac{\sqrt{x}}{2}\right) \\ &\leq \alpha_{n-2}(x)\left(1 - \frac{\sqrt{x}}{2}\right)^2 \\ \cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \\ &\leq \alpha_1(x)\left(1 - \frac{\sqrt{x}}{2}\right)^{n-1} \\ &\leq \alpha_0(x)\left(1 - \frac{\sqrt{x}}{2}\right)^n = \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n.\end{aligned}$$

Note that a function $y = t\left(1 - \frac{t}{2}\right)^n$ has a maximum on $[0,1]$ equal to $\frac{2}{n+1}$, we arrive at

$$\alpha_n(x) \leq \frac{2}{n+1}, \ \forall x \in [0,1], \forall n \geq 0.$$

The problem is solved completely.

4.2.14

From the polynomial $f(x) = \sum_{i=0}^{m} a_i x^i$ we construct a polynomial

$$g(x) = f(x+1) - f(x) = \sum_{i=1}^{m} a_i[(x+1)^i - x^i] = \sum_{i=1}^{m} b_i x^i,$$

where $b_i = \sum_{k=0}^{m-1-i} a_{m-k}\binom{m-k}{i}, \ \forall i = 0, 1, \ldots, m-1.$

The first given sequence (u_n) is precisely the following one

$$f(1) = a \cdot 1^{1990}, f(2) = a \cdot 2^{1990}, \ldots, f(2000) = a \cdot 2000^{1990},$$

where $m = 1990, a_m = a$ and $a_i = 0, \forall i = 0, 1, \ldots, m-1$.

By the same way, we can see that the second sequence, consisting from 1999 terms, has a form

$$g(1) = f(2) - f(1), g(2) = f(3) - f(2), \ldots, g(1999) = f(2000) - f(1999),$$

where $g(x) = 1990ax^{1989} + b_{1988}x^{1988} + \cdots + b_1x + b_0$, etc.

Then the 1990th sequence has a form $h(x) = 1990!ax + c$ with $x = 1, 2, \ldots, 11$, and finally, the 1991th sequence consisting of 10 terms equal to $1990!a$.

4.2.15

1) For $x_n > 0, \forall n \in \mathbb{N}$ we must at least have $x_1 > 0$ and $x_2 > 0$. Then the inequality $x_2 > 0$ is equivalent to $\sqrt{3 - 3x_1^2} > x_1$, or $0 < x_1 < \frac{\sqrt{3}}{2}$. We show that this condition is sufficient, too.

Suppose that $0 < x_1 < \frac{\sqrt{3}}{2}$. Then there is uniquely $\alpha \in (0, 60^\circ)$ such that $\sin\alpha = x_1$. In this case

$$x_2 = \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha = \sin(60^\circ - \alpha),\ 0 < 60^\circ - \alpha < 60^\circ.$$

Also we have

$$x_3 = \frac{\sqrt{3}}{2}\cos(60^\circ - \alpha) - \frac{1}{2}\sin(60^\circ - \alpha) = \sin[60^\circ - (60^\circ - \alpha)] = \sin\alpha.$$

Continuing this process, we get

$$\begin{aligned} x_1 &= x_3 = x_5 = \cdots = \sin\alpha > 0, \\ x_2 &= x_4 = x_6 = \cdots = \sin(60^\circ - \alpha) > 0. \end{aligned}$$

Thus the necessary and sufficient condition is $0 < x_1 < \frac{\sqrt{3}}{2}$.

2) Consider two cases of x_1:

Case 1: $x_1 \geq 0$.

• If $x_2 \geq 0$, then similarly to 1), we have $x_3 \geq 0, x_4 \geq 0, \ldots$ and $x_1 = x_3 = x_5 = \cdots; x_2 = x_4 = x_6 = \cdots$.

• If $x_2 < 0$, then $x_3 > 0$ and we also have $x_3 = x_1$. Indeed, the equality

$$x_2 = \frac{-x_1 + \sqrt{3(1 - x_1^2)}}{2},$$

is equivalent to $\sqrt{3(1 - x_1^2)} = 2x_2 + x_1 > 0$ (as $|x_1| < 1$), which gives

$$3(1 - x_2^2) = (2x_1 + x_2)^2.$$

Since $x_1 \geq 0, x_2 < 0$,

$$2x_1 + x_2 = x_1 + (x_1 + x_2) = (2x_2 + x_1) + x_1 - x_2 > x_1 - x_2 > 0.$$

Then $\sqrt{3(1 - x_2^2)} = 2x_1 + x_2$, and hence

$$x_1 = \frac{-x_2 + \sqrt{3(1 - x_2^2)}}{2} = x_3,$$

and then $x_2 = x_4 = \cdots$.

Thus, if $x_1 \geq 0$ then $(x_n) = \{x_1, x_2, x_1, x_2, \ldots\}$ is periodic.

Case 2: $x_1 < 0$.

Then $x_2 > 0$ and due to the previous case, we have

$$(x_n) = \{x_1, x_2, x_3, x_2, x_3, \ldots\}$$

that is (x_n) is periodic, from the second term.

4.2.16

First we show that $x = 0$ is not a root of the polynomial, that is $a_n = f(0) \neq 0$. Indeed, let k be the greatest index such that $a_k \neq 0$. Then the left-hand side has a form

$$\begin{aligned} f(x) \cdot f(2x^2) &= \left(a_0 x^n + \cdots + a_k x^{n-k}\right) \cdot \left(a_0 2^n x^{2n} + \cdots + a_k 2^{n-k} x^{2(n-k)}\right) \\ &= a_0^2 2^n x^{3n} + \cdots + a_k^2 2^{n-k} x^{3(n-k)}, \end{aligned}$$

and the right-hand side has a form

$$\begin{aligned} f(2x^3 + x) &= a_0 (2x^3 + x)^n + \cdots + a_k (2x^3 + x)^{n-k} \\ &= a_0 2^n x^{3n} + \cdots + a_k x^{n-k}. \end{aligned}$$

So we must have

$$a_k^2 2^{n-k} x^{3(n-k)} = a_k x^{n-k}, \ \forall x \in \mathbb{R},$$

which gives $n = k$, that is $a_n = a_k \neq 0$.

Now suppose that $x_0 \neq 0$ is a root of $f(x)$. Consider a sequence

$$x_{n+1} = 2x_n^3 + x_n, \ n \geq 0.$$

It is clear that if $x_0 > 0$, then (x_n) is increasing, while if $x_1 < 0$ then (x_n) is decreasing. From the assumption of the problem

$$f(x) \cdot f(2x^2) = f(2x^3 + x), \ \forall x \in \mathbb{R},$$

it follows that if $f(x_0) = 0$ with $x_0 \neq 0$, then $f(x_k) = 0,\ \forall k$. This shows that a polynomial $f(x)$ is of degree n, non-constant, and has infinitely many roots, which is impossible.

Thus $f(x)$ has no real root at all.

Remark. We can verify that the polynomial satisfies the problem does exist, say $f(x) = x^2 + 1$.

4.2.17

Substituting $x = y = z = 0$ into the given equation, we have

$$f^2(0) - f(0) + \frac{1}{4} \leq 0 \Longleftrightarrow \left(f(0) - \frac{1}{2}\right)^2 \leq 0,$$

which gives $f(0) = \frac{1}{2}$.

Furthermore, substituting $y = z = 0$ into the equation yields

$$4f(0) - 4f(x)f(0) \geq 1,$$

which implies, due to $f(0) = 1/2$, that $f(x) \leq \frac{1}{2}$.

On the other hand, substituting $x = y = z = 1$ into the equation, we obtain

$$f(1) - f^2(1) \geq \frac{1}{4} \Longleftrightarrow \left(f(1) - \frac{1}{2}\right)^2 \leq 0,$$

which gives that $f(1) = \frac{1}{2}$.

Finally, substituting $y = z = 1$, we get

$$f(x) - f(x) \cdot f(1) \geq \frac{1}{4},$$

which, together with $f(1) = 1/2$, shows that $f(x) \geq \frac{1}{2}$.

Thus we must have $f(x) = \frac{1}{2},\ \forall x$. It is easy to check that this function satisfies the given equation.

4.2.18

We have the following equivalent transformations

$$\begin{aligned}
& \frac{x^2y}{z}+\frac{y^2z}{x}+\frac{z^2x}{y}\geq x^2+y^2+z^2 \\
\Longleftrightarrow\quad & x^3y^2+y^3z^2+z^3x^2\geq x^3yz+y^3zx+z^3xy \\
\Longleftrightarrow\quad & x^3y(y-z)+y^2z^2(y-z)+z^3(y^2-2xy+x^2)-xyz(y^2-z^2)\geq 0 \\
\Longleftrightarrow\quad & (y-z)[xy(x^2-z^2)-y^2z(x-z)]+z^3(x-y)^2\geq 0 \\
\Longleftrightarrow\quad & (y-z)(x-z)[xy(x+z)-y^2z]+z^3(x-y)^2\geq 0 \\
\Longleftrightarrow\quad & (y-z)(x-z)[x^2y+xyz-y^2z]+z^3(x-y)^2\geq 0 \\
\Longleftrightarrow\quad & (y-z)(x-z)[x^2y+yz(x-y)]+z^3(x-y)^2\geq 0.
\end{aligned}$$

The last inequality is always true for $x\geq y\geq z>0$.

Remark. The inequality is not valid without the condition $x\geq y\geq z$, as it is not symmetric with respect to x,y,z.

4.2.19

Substituting $x=0$ into the given equation we get $f(0)=f(0)+2f(0)$, or

$$f(0)=0. \tag{1}$$

Furthermore, substituting $y=-1$ yields $f(-x)=f(x)+2f(-x)$, or

$$f(-x)=-f(x). \tag{2}$$

Finally, substituting $y=-\frac{1}{2}$ we arrive to $f(0)=f(x)+2f\left(-\frac{x}{2}\right)=f(x)-2f\left(\frac{x}{2}\right)$, by (2). Combining this and (1) gives

$$f(x)=2f\left(\frac{x}{2}\right). \tag{3}$$

Now let $x\neq 0$ and t be an arbitrary real number. Substituting $y=\frac{t}{2x}$ into the given equation, by (3), we have

$$f(x+t)=f(x)+2f\left(\frac{t}{2}\right)=f(x)+f(t). \tag{4}$$

For $x=0$ we also have $f(0+t)=f(0)+f(t)$. So (4) is valid for all real x,t.

By induction, we can easily verify that

$$f(kx) = kf(x)$$

for all $x \in \mathbb{R}$ and k nonnegative integers. Therefore,

$$f(1992) = f\left(\frac{1992}{1991} \cdot 1991\right) = \frac{1992}{1991} f(1991) = \frac{1992a}{1991}.$$

4.2.20

Put $M_n = \max\{a_n, b_n, c_n\}$ and $m_n = \min\{a_n, b_n, c_n\}$, which are all positive for every $n \geq 0$. We prove that $\lim_{n\to\infty} M_n = \infty$ and $\lim_{n\to\infty} \frac{M_n}{m_n} = L \in \mathbb{R}$.

1) Consider M_n: from assumptions it follows that

$$\begin{aligned} a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 &= a_n^2 + b_n^2 + c_n^2 + 4\left(\frac{a_n}{b_n+c_n} + \frac{b_n}{c_n+a_n} + \frac{c_n}{a_n+b_n}\right) \\ &\quad + \left(\frac{2}{b_n+c_n}\right)^2 + \left(\frac{2}{c_n+a_n}\right)^2 + \left(\frac{2}{a_n+b_n}\right)^2 \\ &> a_n^2 + b_n^2 + c_n^2 + 4\left(\frac{a_n}{b_n+c_n} + \frac{b_n}{c_n+a_n} + \frac{c_n}{a_n+b_n}\right). \end{aligned}$$

Furthermore, we can easily verify that

$$\frac{a_n}{b_n+c_n} + \frac{b_n}{c_n+a_n} + \frac{c_n}{a_n+b_n} \geq \frac{3}{2}$$

(this is the well-known Nesbitt's inequality for three positive numbers).

Thus we have

$$a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 > a_n^2 + b_n^2 + c_n^2 + 6,\ \forall n \geq 0,$$

which implies that

$$a_n^2 + b_n^2 + c_n^2 > 6n,\ \forall n \geq 0. \tag{1}$$

Therefore, since $M_n = \max\{a_n, b_n, c_n\}$, $M_n^2 \geq a_n^2, b_n^2, c_n^2$, and so

$$3M_n^2 \geq a_n^2 + b_n^2 + c_n^2,\ \forall n \geq 0. \tag{2}$$

Combining (1) and (2) yields

$$M_n^2 > 2n \Longleftrightarrow M_n > \sqrt{2n},$$

which implies the first claim.

2) Consider $\dfrac{M_n}{m_n}$: we have

$$a_{n+1} = a_n + \frac{2}{b_n + c_n} \geq a_n + \frac{2}{2M_n} = a_n + \frac{1}{M_n},\ \forall n \geq 0.$$

Similarly for b_{n+1}, c_{n+1}. Hence

$$m_{n+1} = \min\{a_{n+1}, b_{n+1}, c_{n+1}\} \geq \min\left\{a_n + \frac{1}{M_n}, b_n + \frac{1}{M_n}, c_n + \frac{1}{M_n}\right\}$$

$$= \min\{a_n, b_n, c_n\} + \frac{1}{M_n} = m_n + \frac{1}{M_n},\ \forall n \geq 0. \tag{3}$$

By the analogous argument, we also have

$$M_{n+1} \leq M_n + \frac{1}{m_n},\ \forall n \geq 0. \tag{4}$$

From (3) and (4) it follows that

$$M_{n+1} \cdot m_n \leq \left(M_n + \frac{1}{m_n}\right) \cdot m_n = M_n \cdot m_n + 1 = M_n \left(m_n + \frac{1}{M_n}\right) \leq M_n \cdot m_{n+1},$$

which implies that

$$\frac{M_{n+1}}{m_{n+1}} \leq \frac{M_n}{m_n},\ \forall n \geq 0.$$

Moreover, $\dfrac{M_n}{m_n} \geq 1,\ \forall n \geq 0$. So the sequences $\left(\dfrac{M_n}{m_n}\right)$ is decreasing and bounded from below. By Weierstrass theorem, there exists

$$\lim_{n\to\infty} \frac{M_n}{m_n} = L \in \mathbb{R}.$$

Thus $\lim\limits_{n\to\infty} M_n = \infty$ and $\lim\limits_{n\to\infty} \dfrac{M_n}{m_n} = L \in \mathbb{R}$. From these facts it follows that $\lim\limits_{n\to\infty} m_n = \infty$, and hence $\lim\limits_{n\to\infty} a_n = \infty$.

4.2.21

Put $f(x) = ax^2 - x + \log(1 + x)$. The problem is equivalent to finding a so that $f(x) \geq 0,\ \forall x \in [0, +\infty)$. Note that $f(0) = 0$.

As

$$f'(x) = 2ax - 1 + \frac{1}{1+x} = 2ax - \frac{x}{1+x} = \frac{x}{1+x}[2a(1+x) - 1],$$

we see that $f'(0) = 0$, $\forall a \in \mathbb{R}$, and also for $x > 0$ the derivative $f'(x)$ has the same sign as $g(x) = 2ax + 2a - 1$.

1) If $a = 0$: in this case $g(x) = -1$, and so $f'(x) < 0\ \forall x > 0$, which means that $f(x)$ is decreasing on $[0, \infty)$. Then $f(x) \leq f(0) = 0$, the equality occurs if and only if $x = 0$.

2) If $a \neq 0$: in this case $g(x) = 0$ has a unique root $x_0 = \frac{1-2a}{2a}$. Then we have to consider three possible intervals for a, namely $(-\infty, 0), (0, \frac{1}{2})$, and $[\frac{1}{2}, +\infty)$.

• For $a < 0$: then $x_0 < 0$ and hence $g(x) < 0$, $\forall x > 0 > x_0$. In this case $f(x)$ is decreasing on $[0, \infty)$, and therefore, $f(x) \leq f(0) = 0$, the equality occurs if and only if $x = 0$.

• For $0 < a < \frac{1}{2}$: then $x_0 > 0$ and hence $g(x) < 0$, $\forall 0 < x < x_0$. In this case the same conclusion as the previous case, that is, $f(x)$ is decreasing on $[0, \infty)$, and therefore, $f(x) \leq f(0) = 0$, the equality occurs if and only if $x = 0$.

• For $a \geq \frac{1}{2}$: then $x_0 \leq 0$ and hence $g(x) > 0$, $\forall x > 0$. In this case $f(x)$ is increasing on $[0, \infty)$, which implies that $f(x) \geq f(0) = 0$, and $f(x) = 0$ if and only if $x = 0$.

Thus $f(x) \geq 0$ for all $x \geq 0$ if and only if $a \geq \frac{1}{2}$.

4.2.22

The domain of definition is $D = [-\sqrt{1995}, \sqrt{1995}]$. Note that $f(x)$ is an odd function and $f(x) \geq 0$, $\forall x \in [0, \sqrt{1995}]$. Then

$$\max_{x \in D} f(x) = \max_{x \in [0, \sqrt{1995}]} f(x); \ \min_{x \in D} f(x) = -\max_{x \in [0, \sqrt{1995}]} f(x).$$

For all $x \in [0, \sqrt{1995}]$, we have

$$\begin{aligned} f(x) &= x(1993 + \sqrt{1995 - x^2}) \\ &= x\left(\sqrt{1993} \cdot \sqrt{1993} + 1 \cdot \sqrt{1995 - x^2}\right) \\ &\leq x\left(\sqrt{1993 + 1} \cdot \sqrt{1993 + (1995 - x^2)}\right) \\ &\quad \text{(by Cauchy-Schwarz inequality)} \\ &= x\sqrt{1994} \cdot \sqrt{1993 + 1995 - x^2} \\ &\leq \sqrt{1994} \cdot \frac{x^2 + (1993 + 1995 - x^2)}{2} \\ &\quad \text{(by Cauchy inequality)} \\ &= \sqrt{1994} \cdot 1994. \end{aligned}$$

The equality occurs if and only if

$$\begin{cases} 1 = \sqrt{1995 - x^2} \\ x = \sqrt{1993 + 1995 - x^2} \end{cases} \iff x = \sqrt{1994} \in [0, \sqrt{1995}].$$

Thus

$$\max_{x \in D} f(x) = \max_{x \in [0, \sqrt{1995}]} f(x) = 1994\sqrt{1994} \text{ (attained at } x = \sqrt{1994}),$$

$$\min_{x \in D} f(x) = -\max_{x \in [0, \sqrt{1995}]} f(x) = -1994\sqrt{1994} \text{ (attained at } x = -\sqrt{1994}).$$

4.2.23

Using "trigonometrical" method, we write

$$\frac{1}{a_0} = \frac{1}{2} = \cos\frac{\pi}{3} \text{ and } b_0 = 1,$$

which imply that

$$\frac{1}{a_1} = \frac{a_0 + b_0}{2a_0 b_0} = \frac{1}{2}\left(\frac{1}{b_0} + \frac{1}{a_0}\right) = \frac{1}{2}\left(1 + \cos\frac{\pi}{3}\right) = \cos^2\frac{\pi}{6},$$

and

$$\frac{1}{b_1} = \frac{1}{\sqrt{a_1 b_0}} = \cos\frac{\pi}{6}.$$

By induction we can prove that

$$\frac{1}{a_n} = \cos\frac{\pi}{2\cdot 3}\cdot\cos\frac{\pi}{2^2\cdot 3}\cdots\cos\frac{\pi}{2^n\cdot 3}\cdot\cos\frac{\pi}{2^n\cdot 3} = \cos\frac{\pi}{2^n\cdot 3}\cdot\frac{\sin\frac{\pi}{3}}{2^n\sin\frac{\pi}{2^n\cdot 3}},$$

and

$$\frac{1}{b_n} = \cos\frac{\pi}{2\cdot 3}\cdot\cos\frac{\pi}{2^2\cdot 3}\cdots\cos\frac{\pi}{2^n\cdot 3} = \frac{\sin\frac{\pi}{3}}{2^n\sin\frac{\pi}{2^n\cdot 3}}.$$

Note that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ and $\lim\limits_{x\to 0}\cos x = 1$. In particular, $\lim\limits_{n\to\infty}\dfrac{\sin\frac{\pi}{2^n\cdot 3}}{\frac{\pi}{2^n\cdot 3}} = 1$ and $\lim\limits_{n\to\infty}\cos\dfrac{\pi}{2^n\cdot 3} = 1$. Then we finally obtain that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{2\sqrt{3}\pi}{9}.$$

4.2.24

The domain of definition is

$$\begin{cases} x > -\frac{1}{2}, \\ y > -\frac{1}{2}. \end{cases}$$

We write the given system as

$$\begin{cases} x^2 + 3x + \log(2x+1) = y, \\ x = y^2 + 3y + \log(2y+1). \end{cases}$$

Taking the sum of these equation yields

$$x^2 + 4x + \log(2x+1) = y^2 + 4y + \log(2y+1). \tag{1}$$

Consider a function

$$f(t) = t^2 + 4t + \log(2t+1),\ t \in \left(-\frac{1}{2}, +\infty\right).$$

We have

$$f'(t) = 2t + 4 + \frac{2}{2t+1} > 0,\ \forall t > -\frac{1}{2},$$

which implies that $f(t)$ is strictly increasing. Then (1) leads to $x = y$.

Substituting $y = x$ into the given system we get

$$x^2 + 2x + \log(2x+1) = 0. \tag{2}$$

Similarly, the function $g(t) = t^2 + 2t + \log(2t+1)$ is strictly increasing on $t > -\frac{1}{2}$, and hence $g(x) = 0$ if and only if $x = 0$.

Thus equation (2) has a unique solution $x = 0$, which implies that $x = y = 0$. This satisfies the given system and is its unique solution.

4.2.25

If $a = 0$: in this case $x_n = 0$, $\forall n \geq 0$, and hence $\lim\limits_{n\to\infty} x_n = 0$.
If $a > 0$: in this case from the well-known inequality $\sin x < x$, $\forall x > 0$, it follows that $x_n > 0$, $\forall n \geq 0$.

Furthermore, we also have the inequality

$$\sin x \geq x - \frac{x^3}{6}, \ \forall x \geq 0,$$

the equality occurs if and only if $x = 0$.

Then

$$\begin{aligned} &\sin x_{n-1} > x_{n-1} - \frac{x_{n-1}^3}{6} \Longleftrightarrow 6(x_{n-1} - \sin x_{n-1}) < x_{n-1}^3 \\ &\Longleftrightarrow \sqrt[3]{6(x_{n-1} - \sin x_{n-1})} < x_{n-1} \Longleftrightarrow x_n < x_{n-1}, \ \forall n \geq 1, \end{aligned}$$

which shows that (x_n) is decreasing and is bounded from below by 0. Therefore, by Weierstrass theorem, there exists $\lim\limits_{n\to\infty} x_n = L \geq 0$.

Letting n tend to infinity in the equation

$$x_n = \sqrt[3]{6(x_{n-1} - \sin x_{n-1})}$$

we obtain

$$L = \sqrt[3]{6(L - \sin L)} \Longleftrightarrow \sin L = L - \frac{L^3}{6},$$

which by the above-mentioned inequality, shows that $L = 0$. Thus $\lim\limits_{n\to\infty} x_n = 0$.

If $a < 0$: we can reduce to the case 2) by considering $(y_n = -x_n)$. Then we get the same result.

So for all $a \in \mathbb{R}$ we always have $\lim\limits_{n\to\infty} x_n = 0$.

4.2.26

Consider a function

$$f(t) = t^3 + 3t - 3 + \log(t^2 - t + 1),\ t \in \mathbb{R}.$$

The given system is written as

$$\begin{cases} f(x) = y \\ f(y) = z \\ f(z) = x \end{cases} \iff \begin{cases} f(x) = y \\ f(y) = f(f(x)) = z \\ f(z) = f(f(f(x))) = x. \end{cases}$$

We have

$$f'(t) = 3t^2 + 3 + \frac{2t-1}{t^2 - t + 1} = 3t^2 + \frac{3t^2 - t + 2}{t^2 - t + 1} > 0,\ \forall t \in \mathbb{R},$$

and so $f(t)$ is increasing on $\mathbb{R}$. Then $f(f(f(x))) = x$ is equivalent to $f(x) = x$, that is,

$$x^3 + 2x - 3 + \log(x^2 - x + 1) = 0. \tag{1}$$

The function

$$g(x) = x^3 + 2x - 3 + \log(x^2 - x + 1),$$

having

$$g'(x) = 3x^2 + 2 + \frac{2x-1}{x^2 - x + 1} = 3x^2 + \frac{2x^2 + 1}{x^2 - x + 1} > 0,\ \forall x \in \mathbb{R},$$

is increasing on $\mathbb{R}$. As $g(1) = 0$ we see that (1) gives $x = 1$ and hence $x = y = z = 1$. This is a unique solution to the given system.

4.2.27

Note that

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2},\ \forall x \in [-1, 1],$$

we then can write

$$\begin{aligned} x_n &= \frac{4}{\pi^2}\left(\cos^{-1} x_{n-1} + \frac{\pi}{2}\right)\sin^{-1} x_{n-1} \\ &= \frac{4}{\pi^2}\left(\cos^{-1} x_{n-1} + \frac{\pi}{2}\right) \cdot \left(\frac{\pi}{2} - \cos^{-1} x_{n-1}\right) \\ &= \frac{4}{\pi^2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\cos^{-1} x_{n-1}\right)^2\right] \\ &= 1 - \frac{4}{\pi^2}(\cos^{-1} x_{n-1})^2. \end{aligned}$$

Then $x_n \in (0,1)$ for all $n \geq 1$. Put $t_n = \cos^{-1} x_n$, we see that $t_n \in \left(0, \frac{\pi}{2}\right)$. Moreover, the relation

$$x_n = 1 - \frac{4}{\pi^2}(\cos^{-1} x_{n-1})^2,$$

means that

$$\cos t_n = 1 - \frac{4}{\pi^2} t_{n-1}^2. \tag{1}$$

Consider a function

$$f(t) = 1 - \frac{4}{\pi^2} t^2 - \cos t,\ t \in \left[0, \frac{\pi}{2}\right].$$

We have

$$f'(t) = \sin t - \frac{8}{\pi^2} t,$$

and

$$f''(t) = \cos t - \frac{8}{\pi^2}.$$

So

$$f''(t) = 0 \Longleftrightarrow t = t_0 := \cos^{-1} \frac{8}{\pi^2} \in \left(0, \frac{\pi}{2}\right),$$

and

$$f''(t) \begin{cases} > 0, & \text{if } t \in (0, t_0), \\ < 0, & \text{if } t \in \left(t_0, \frac{\pi}{2}\right). \end{cases}$$

This shows that $f'(t)$ is strictly increasing on $[0, t_0]$ and strictly decreasing on $[t_0, \frac{\pi}{2}]$.

Note that $f'(0) = 0$ and $f'\left(\frac{\pi}{2}\right) = 1 - \frac{\pi}{4} < 0$, there exists $t_1 \in \left(0, \frac{\pi}{2}\right)$ such that $f'(t_1) = 0$ and we also have

$$f'(t) \begin{cases} > 0, & \text{if } t \in (0, t_1), \\ < 0, & \text{if } t \in \left(t_1, \frac{\pi}{2}\right). \end{cases}$$

This in turn shows that $f(t)$ is strictly increasing on $[0, t_1]$ and strictly decreasing on $[t_1, \frac{\pi}{2}]$.

Again note that $f(0) = f\left(\frac{\pi}{2}\right) = 0$, from which it follows that $f(t) \geq 0$ on $[0, \frac{\pi}{2}]$.

Thus

$$1 - \frac{4}{\pi^2} t^2 - \cos t \leq 0 \Longleftrightarrow \cos t \geq 1 - \frac{4}{\pi^2} t^2,\ \forall t \in \left[0, \frac{\pi}{2}\right]. \tag{2}$$

Combining (1) and (2) yields

$$\cos t_n = 1 - \frac{4}{\pi^2} t_{n-1}^2 \geq \cos t_{n-1}, \ \forall n \geq 1,$$

which means that $x_n \geq x_{n-1}, \ \forall n \geq 1$.

So the sequence (x_n) with $0 < x_n < 1$ is increasing and is bounded from above. By Weierstrass theorem, there exists $\lim\limits_{n\to\infty} x_n = L$, and moreover, $L > 0$. Furthermore, from this it follows that $(t_n) \in [0, \frac{\pi}{2}]$ is decreasing and bounded from below. Again by Weierstrass theorem, there exists $\lim\limits_{n\to\infty} t_n = \alpha$, and $\frac{\pi}{2} > \alpha$.

Letting in (1) $n \to \infty$ we obtain

$$\cos \alpha = 1 - \frac{4}{\pi^2} \alpha^2,$$

which shows, by (2), that either $\alpha = 0$, or $\alpha = \frac{\pi}{2}$. The value $\frac{\pi}{2}$ is eliminated, because as already noted above, $\alpha < \frac{\pi}{2}$. Then $\alpha = 0$, which gives $L = \cos \alpha = 1$.

Thus $\lim\limits_{n\to\infty} x_n = 1$.

4.2.28

1) The relation $a_{n+1} = 5a_n + \sqrt{24a_n^2 - 96}$ is equivalent to

$$a_n^2 - 10a_{n+1} \cdot a_n + a_{n+1}^2 + 96 = 0. \tag{1}$$

This also means that

$$a_{n+2}^2 - 10a_{n+1} \cdot a_{n+2} + a_{n+1}^2 + 96 = 0. \tag{2}$$

Note that (a_n) is strictly increasing, which implies that $a_n < a_{n+2}$. Then (1) and (2) show that a_n and a_{n+2} are distinct real roots of the quadratic equation

$$t^2 - 10a_{n+1}t + a_{n+1}^2 + 96 = 0. \tag{3}$$

By Viète formula, $a_n + a_{n+2} = 10a_{n+1}$, or $a_{n+2} - 10a_{n+1} + a_n = 0, \ \forall n \geq 1$. The characteristic polynomial of this recursive equation is $x^2 - 10x + 1 = 0$, which has two roots

$$x_{1,2} = 5 \pm 2\sqrt{6}.$$

Then a general solution of the obtained recursive equation can be found in a form

$$a_n = C_1(5 - 2\sqrt{6})^n + C_2(5 + 2\sqrt{6})^n,$$

where C_1, C_2 are constants that we have to find.

Substituting $n = 0$ and $n = 1$ into the formula, we get a system

$$\begin{cases} C_1 + C_2 = a_0 = 2, \\ C_1(5 - 2\sqrt{6}) + C_2(5 + 2\sqrt{6}) = a_1 = 10, \end{cases}$$

which gives $C_1 = C_2 = 1$. Thus the general term of the sequence (a_n) is

$$a_n = (5 - 2\sqrt{6})^n + (5 + 2\sqrt{6})^n, \ \forall n \geq 0.$$

2) We prove the inequality by induction.

It is obvious for $n = 0: \ a_0 = 2 = 2 \cdot 5^0$. Assume that it is true for $n = k \geq 0$, that is $a_k \geq 2 \cdot 5^k$. Then

$$a_{k+1} = 5a_k + \sqrt{24a_k^2 - 96} \geq 5a_k \geq 2 \cdot 5^{k+1},$$

which means that the inequality is also true for $n = k + 1$.

By mathematical induction principle, we have $a_n \geq 2 \cdot 5^n$ for all integer $n \geq 0$.

4.2.29

Put $n = 1995 + k$ with $5 \leq k \leq 100$. Then

$$a = \sum_{i=1995}^{1995+k} \frac{1}{i}, \ b = 1 + \frac{k+1}{1995}.$$

Note that

$$\begin{array}{ccccc} \dfrac{1}{1995+k} & < & \dfrac{1}{1995} & = & \dfrac{1}{1995} \\ \dfrac{1}{1995+k} & < & \dfrac{1}{1995+1} & < & \dfrac{1}{1995} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{1995+k} & = & \dfrac{1}{1995+k} & < & \dfrac{1}{1995}, \end{array}$$

we get

$$\frac{k+1}{1995+k} < \sum_{i=1995}^{1995+k} \frac{1}{i} = a < \frac{k+1}{1995},$$

which implies that

$$\frac{1995}{k+1} < \frac{1}{a} < \frac{1995+k}{k+1}. \tag{1}$$

From the left inequality in (1) it follows that

$$b^{1/a} = \left(1 + \frac{k+1}{1995}\right)^{1/a} > \left(1 + \frac{k+1}{1995}\right)^{\frac{1995}{k+1}}.$$

Furthermore, by Bernoulli inequality

$$\left(1 + \frac{k+1}{1995}\right)^{\frac{1995}{k+1}} > 1 + \frac{1995}{k+1} \cdot \frac{k+1}{1995} = 2,$$

and so

$$b^{1/a} > 2. \tag{2}$$

On the other hand, from the right inequality in (1) it follows that

$$\begin{aligned} b^{1/a} = \left(1 + \frac{k+1}{1995}\right)^{1/a} &< \left(1 + \frac{k+1}{1995}\right)^{\frac{1995+k}{k+1}} \\ &< \left(1 + \frac{k+1}{1995}\right) \cdot \left(1 + \frac{k+1}{1995}\right)^{\frac{1995}{k+1}} \\ &\leq \left(1 + \frac{101}{1995}\right) \cdot \left(1 + \frac{k+1}{1995}\right)^{\frac{1995}{k+1}} \\ &< 1.06 \cdot \left(1 + \frac{k+1}{1995}\right)^{\frac{1995}{k+1}}. \end{aligned}$$

Note that $f(x) = \left(1 + \frac{1}{x}\right)^x$ is strictly increasing on $(0, +\infty)$ and $\lim\limits_{x \to +\infty} f(x) = e$, we get

$$f\left(\frac{1995}{k+1}\right) = \left(1 + \frac{k+1}{1995}\right)^{\frac{1995}{k+1}} < e,$$

and so

$$b^{1/a} < 1.06 \cdot e < 1.06 \cdot 2.8 < 3. \tag{3}$$

Combining (2) and (3) yields $[b^{1/a}] = 2$.

4.2.30

It is obvious that $\deg P := n \geq 1$. There are two cases:

Case 1: $P(x)$ is monotone on $\mathbb{R}$.

Since a graph of $P(x)$ has finitely many inflection points, for $a > 1995$ large enough $P(x) = a$ has at most one root (counted with multiplicities). Then $n = 1$ and $P(x) = px + q$ with $p > 0$.

In this case the equation $P(x) = a$ has a solution

$$x = \frac{a-q}{p}.$$

We see that $x > 1995$ for all $a > 1995$ if and only if $p > 0$ and $q \leq 1995(1-p)$.

Case 2: $P(x)$ has max and/or min points.

Then $n \geq 2$. Assume that P attains its maximum at $u_1, \ldots, u_m$ $(m \geq 1)$ and attains its minimum at $v_1, \ldots, v_k$ $(k \geq 1)$. Put

$$h = \max\{P(u_1), \ldots, P(u_m), P(v_1), \ldots, P(v_k)\}.$$

Again since a graph of $P(x)$ has finitely many inflection points, for all a large enough with $a > \max\{h, 1995\}$, the equation $P(x) = a$ has at most two solutions (counted with multiplicities), which implies that $n = 2$.

However, in this case, for the quadratic polynomial $P(x)$ with $a > 1995$ large enough, the equation $P(x) = a$ has at most one root which is larger than 1995, and so this polynomial does not satisfy our problem.

Thus all solutions of the problem are $P(x) = px + q$ with $p > 0$ and $q \leq 1995(1-p)$.

4.2.31

The given system is equivalent to

$$\begin{cases} y(x^3 - y^3) = a^2, & (1) \\ y(x+y)^2 = b^2. & (2) \end{cases}$$

There are four cases:

- If $a = b = 0$: system has infinitely many solutions $(x, 0)$ with any x.

- If $a \neq 0, b = 0$: (1) gives $y \neq 0$ and then (2) gives $x = -y$. Substituting this into (1) we get

$$-2y^4 = a^2,$$

which has no solutions, and hence the system is inconsistent.

- If $a = 0, b \neq 0$: (2) gives $y \neq 0$ and then (1) gives $x^3 = y^3$, or $x = y$. Substituting this into (2) we get

$$4y^3 = b^2,$$

which gives $y = \sqrt[3]{\dfrac{b^2}{4}}$. In this case the system has a unique solution (x, y) with $x = y = \sqrt[3]{\dfrac{b^2}{4}}$.

• If $a \neq 0, b \neq 0$: it suffices to consider $a, b > 0$ (otherwise we just take absolute values of a, b). From (2) it follows that $y > 0$, and then from (1) it follows that $x > y > 0$. Furthermore, (2) given

$$x = \frac{b}{\sqrt{y}} - y.$$

Put $t = \sqrt{y}$, we have $x = \dfrac{b}{t} - t^2$. Then (1) is equivalent to

$$t^2\left(\left(\frac{b}{t} - t^2\right)^3 - t^6\right) = a^2,$$

or

$$t^9 - (b - t^3)^3 + a^2 t = 0.$$

Consider a function $f(t) = t^9 - (b - t^3)^3 + a^2 t$ on $[0, +\infty)$, which has a derivative

$$f'(t) = 9t^8 + 9(b - t^3)^2 t^2 + a^2 \geq 0, \ \forall t \geq 0.$$

This shows that $f(t)$ is increasing on $[0, \infty)$. Note that $f(0) = -b^3 < 0$, $f(\sqrt[3]{b}) = b^3 + a^2 \sqrt[3]{b} > 0$. Then the equation $f(t) = 0$ has a unique solution $t_0 > 0$, and so the system has a unique solution $(x, y) = \left(\frac{b}{t_0} - t_0^2, t_0^2\right)$.

4.2.32

From

$$f(n) + f(n+1) = f(n+2) \cdot f(n+3) - 1996, \ \forall n \geq 1, \qquad (1)$$

we have

$$f(n+1) + f(n+2) = f(n+3) \cdot f(n+4) - 1996, \ \forall n \geq 1. \qquad (2)$$

Subtracting (1) and (2) yields

$$f(n+2) - f(n) = f(n+3)\big[f(n+4) - f(n+2)\big], \ \forall n \geq 1.$$

From this formula, by induction we can show that

$$f(3) - f(1) = f(4) \cdot f(6) \cdots f(2n) \cdot \big[f(2n+1) - f(2n-1)\big], \ \forall n \geq 2, \quad (3)$$

and

$$f(4) - f(2) = f(5) \cdot f(7) \cdots f(2n+1) \cdot \big[f(2n+2) - f(2n)\big], \ \forall n \geq 2. \quad (4)$$

• If $f(1) > f(3)$, then $f(2n-1) > f(2n+1)$, $\forall n \geq 1$. In this case (3) shows that there are infinitely many positive numbers less than $f(1)$, which is impossible. Thus we should have $f(1) \leq f(3)$.

Similarly, $f(2) \leq f(4)$.

• If $f(1) < f(3)$ and $f(2) < f(4)$, then (3) and (4) give $f(2n-1) < f(2n+1)$ and $f(2n) < f(2n+2)$, $\forall n \geq 1$. In this case $f(3) - f(1)$ has infinitely many distinct positive divisors, which is impossible. Thus we have either $f(1) = f(3)$, or $f(2) = f(4)$.

• If $f(1) = f(3)$ and $f(2) = f(4)$, then (3) and (4) give

$$f(1) = f(2n-1) \text{ and } f(2) = f(2n), \ \forall n \geq 1. \quad (5)$$

Substituting this into (1) we get

$$f(1) + f(2) = f(1) \cdot f(2) - 1996 \Longleftrightarrow \big[f(1) - 1\big] \cdot \big[f(2) - 1\big] = 1997.$$

Since 1997 is a prime number, there are two cases: either $f(1) = 2, f(2) = 1998$ or $f(1) = 1998, f(2) = 2$. Combining this and (5) we get solutions

$$f(n) = \begin{cases} 2, & \text{if } n \text{ is odd,} \\ 1998, & \text{if } n \text{ is even,} \end{cases}$$

or

$$f(n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 1998, & \text{if } n \text{ is odd.} \end{cases}$$

• If $f(1) = f(3)$ and $f(2) < f(4)$, then from (3) it follows that $f(1) = f(2n-1)$, $\forall n \geq 1$. Substituting this into (4) gives $f(4) - f(2) = [f(1)]^{n-1} \cdot \big[f(2n+2) - f(2)\big]$ and $f(2n) < f(2n+2)$, $\forall n \geq 1$.

By the same argument as above, if $f(1) > 1$, then $f(4) - f(2)$ has infinitely many distinct positive divisors, which is impossible. So we must have $f(1) = 1$ and $f(2n-1) = f(2n+1)$, $\forall n \geq 1$, y (3). Substituting this into (1) we get $f(4) - f(2) = 1997$.

Furthermore, from (4) it follows that $f(4) - f(2) = f(2n+2) - f(2n)$, $\forall n \geq 1$. So in this case the solutions are

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ k + 1997\left(\frac{n}{2} - 1\right), & \text{if } n \text{ is even,} \end{cases}$$

where k is any positive integer.

• Finally, if $f(2) = f(4)$ and $f(1) < f(3)$, then by similar argument we have solutions

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ \ell + 1997\left(\frac{n-1}{2}\right), & \text{if } n \text{ is odd,} \end{cases}$$

where ℓ is any positive integer.

4.2.33

We have

$$k^n > 2n^k \Longleftrightarrow n\log k > \log 2 + k\log n. \tag{1}$$

Consider a function $f(x) = n\log x - x\log n$ on $(1, +\infty)$. Since

$$f'(x) = \frac{n}{x} - \log n = 0 \Longleftrightarrow x = \frac{n}{\log n},$$

which shows that $f(x)$ is increasing on $\left(1, \frac{n}{\log n}\right)$ and decreasing on $\left(\frac{n}{\log n}, +\infty\right)$.

Note that $2 \le k \le n-1$, then $f(x)$ attains its minimum on $[2, n-1]$ at one of the two endpoints. So in order to prove (1) it suffices to prove that $f(2) > \log 2$ and $f(n-1) > \log 2$, for all $n \ge 7$.

For the first inequality:

$$f(2) > \log 2 \Longleftrightarrow n\log 2 - 2\log n > \log 2 \Longleftrightarrow 2^{n-1} > n^2.$$

This inequality can be proved easily by induction.

For the second inequality:

$$f(n-1) > \log 2 \Longleftrightarrow n\log(n-1) - (n-1)\log n > \log 2 \Longleftrightarrow (n-1)^n > 2n^{n-1}.$$

Put $n - 1 = t \ge 6$, the inequality is equivalent to

$$t^{t+1} > 2(t+1)^t,$$

or

$$t > 2\left(1 + \frac{1}{t}\right)^t.$$

Indeed, it is well-known that $\left(1 + \frac{1}{t}\right)^t < 3$ for $t > 0$. Then $2\left(1 + \frac{1}{t}\right)^t < 6 \le t$.

The proof is complete.

4.2.34

Since 1997 is a prime number, $(n, 1997) = 1$. Then a_i and b_j are not integers for all $i \leq 1997, j < n$.

In order to prove $c_{k+1} - c_k < 2$ we notice the following two properties:

1) $a_i \neq b_j$, $\forall i, j$.

2) For non-integers x, y with $x + y = m \in \mathbb{Z}$, it holds $[x] + [y] = m - 1$.

Let m be a fixed number $(i < m < 1997 + n)$. We count how many numbers in (a_i) which are less than or equal to m. Since

$$a_i \leq m \Longleftrightarrow \begin{cases} i < 1997, \\ \frac{i(1997+n)}{1997} \leq m, \end{cases}$$

there are $\left[\frac{1997m}{1997+n}\right]$ such numbers. Similarly, there are $\left[\frac{mn}{1997+n}\right]$ numbers in (b_j) which are less than or equal to m.

Thus the number of elements in (c_k) which less than or equal to m is

$$\left[\frac{1997m}{1997+n}\right] + \left[\frac{mn}{1997+n}\right].$$

Furthermore, since

$$\frac{1997m}{1997+n} + \frac{mn}{1997+n} = m \in \mathbb{N},$$

by the second notice said above,

$$\left[\frac{1997m}{1997+n}\right] + \left[\frac{mn}{1997+n}\right] = m - 1.$$

So in $(1, m)$ there are exactly $m - 1$ elements of (c_k). Letting $m = 2, 3, \ldots, 1996 + n$ we see that in each interval $(m - 1, m)$ there is exactly one element of (c_k). Thus $c_{k+1} - c_k < 2$, for all $k = 1, 2, \ldots, 1994 + n$.

4.2.35

Consider the following cases for a.

Case 1: $a = 0$. Then $x_n = 0$ for all n, and so $\lim\limits_{n\to\infty} x_n = 0$.

Case 2: $a = 1$. Then $x_n = 1$ for all n, and so $\lim\limits_{n\to\infty} x_n = 1$.

Case 3: $a > 0, a \neq 1$. Then as x_{n+1} has the same sign as x_n, we see that $x_n > 0$ for all n.

Note that

$$x_{n+1} - 1 = \frac{(x_n - 1)^3}{3x_n^2 + 1},$$

which shows that $x_{n+1} - 1$ and $x_n - 1$ have the same sign. Therefore,

- If $a \in (0, 1)$, then $x_n < 1$ for all n.
- If $a > 1$, then $x_n > 1$ for all n.

Now consider

$$x_{n+1} - x_n = \frac{2x_n(1 - x_n^2)}{3x_n^2 + 1},$$

which gives that (x_n) is either increasing and is bounded from above by 1, or decreasing and is bounded from below by 1. In both cases, there exists $\lim\limits_{n\to\infty} x_n = L$, and moreover $L > 0$.

Letting in the given equation $n \to \infty$, we obtain

$$L = \frac{L(L^2 + 3)}{3L^2 + 1} \Longrightarrow L = 1.$$

Case 4: $a < 0$. Consider a sequence (y_n) defined by $y_n = -x_n$, $\forall n \geq 1$. We then reduce this case to Case 3, and obtain that (y_n) converges and $\lim\limits_{n\to\infty} y_n = 1$, or equivalently, (x_n) converges and $\lim\limits_{n\to\infty} x_n = -1$.

Overall, the sequence (x_n) always converges and

$$\lim_{n\to\infty} x_n = \begin{cases} -1, & \text{if } a < 0, \\ 0, & \text{if } a = 0, \\ 1, & \text{if } a > 0. \end{cases}$$

4.2.36

From the assumptions of the problem, it follows that an is integer for all $n \geq 0$.

1) Put $b_n = a_{n+1} - a_n$, we have

$$b_{n+2} = 2b_{n+1} - b_n,$$

which means that

$$b_{n+2} - b_{n+1} = b_{n+1} - b_n = \cdots = b_1 - b_0 = a_2 - 2a_1 + a_0 = 2.$$

Thus

$$b_{n+1} = b_n + 2, \ \forall n \geq 0,$$

which implies that

$$b_n = b_0 + 2n = 2n + b - a, \ \forall n \geq 0.$$

Hence

$$a_n - a_0 = \sum_{k=0}^{n-1} b_k = 2\sum_{k=0}^{n-1} k + n(b-a) = n(n-1) + n(b-a), \ \forall n \geq 0,$$

which gives

$$a_n = n^2 + n(b-a-1) + a, \ \forall n \geq 0.$$

2) Suppose that $a_n = c^2$, where $n \geq 1998$ and c is a positive intefger. Then

$$\begin{aligned} 4c^2 &= 4a_n \\ &= 4n^2 + 4n(b-a-1) + 4a \\ &= [2n + (b-a-1)]^2 + 4a - (b-a-1)^2. \end{aligned}$$

Put $\alpha = 2n + (b-a-1), \beta = 4a - (b-a-1)^2$. Then

$$\beta = 4c^2 - \alpha^2 = (2c+\alpha)(2c-\alpha).$$

Note that α is a positive integer for n large enough.

If $\beta \neq 0$ then $2c - \alpha \neq 0$. Since $2c - \alpha$ is integer,

$$|\beta| \geq |2c + \alpha| \geq 2c + \alpha \geq \alpha = 2n + (b-a-1),$$

which is impossible for n large, as β is a constant.

So we must have $\beta = 0$, and $4a = (b-a-1)^2$. Put $b - a - 1 = 2t$, we have $a = t^2$ and $b = a + 1 + 2t = (t+1)^2$.

Conversely, if $a = t^2$ and $b = (t+1)^2$, then $a_n = n^2 + n(b-a-1) + a = (n+t)^2$.

Thus a_n is a square for all n if and only if $a = t^2, b = (t+1)^2$, where t is integer.

4.2.37

If $a = 1$: $x_n = 1$ for all n, and hence $\lim\limits_{n\to\infty} x_n = 1$.

If $a > 1$: we prove by induction that $x_n > 1$ for all n.

Indeed, $x_1 = a > 1$. Suppose that $x_n > 1$. We have

$$x_{n+1} > 1 \Longleftrightarrow \log\left(\frac{x_n^2}{1+\log x_n}\right) > 0 \Longleftrightarrow x_n^2 - 1 - \log x_n > 0.$$

Consider a function $f(x) = x^2 - 1 - \log x$ on $[1, +\infty)$. It is easy to see that $f'(x) > 0$, $\forall x \geq 1$, and hence $f(x)$ increases on $[1, +\infty)$. Moreover, $f(1) = 0$, and so $f(x) > 0$, $\forall x > 1$. In particular, $x_{n+1} > 1$. The claim is proved.

Next we prove that (x_n) is decreasing. Indeed, it is equivalent to $x_n - x_{n+1} > 0$, $\forall n \geq 1$. We have

$$x_n - x_{n+1} > 0 \Longleftrightarrow x_n - 1 - \log\left(\frac{x_n^2}{1+\log x_n}\right) > 0.$$

So consider a function

$$f(x) = x - 1 - \log\frac{x^2}{1+\log x},\ x \in [1, +\infty).$$

It has a derivative

$$f'(x) = \frac{x - 1 + x\log x - 2\log x}{x(1+\log x)},\ x \in [1, +\infty). \tag{1}$$

We in turn consider a function $g(x) = x - 1 + x\log x - 2\log x$ on $[1, +\infty)$, which has on $[1, +\infty)$ a derivative

$$g'(x) = 2\left(1 - \frac{1}{x}\right) + \log x.$$

It is clear that $g'(x) > 0$, $\forall x > 1$ and $g'(x) = 0 \Longleftrightarrow x = 1$. Then $g(x)$ increases on this interval. Also as $g(1) = 0$, we get that $g(x) > 0$, $\forall x > 1$, and $g(x) = 0$ if and only if $x = 1$.

From this fact and (1) it follows that $f'(x) > 0$, $\forall x > 1$, and $f'(x) = 0 \Longleftrightarrow x = 1$. So $f(x)$ increase on $[1, +\infty)$, and $f(x) > 0$, $\forall x > 1$, as $f(1) = 0$. In particular, since $x_n > 1$, $\forall n \geq 1$,

$$f(x_n) > 0 \Longleftrightarrow x_n > x_{n+1},\ \forall n \geq 1.$$

Thus (x_n) is decreasing, and is bounded from below by 1. By Weierstrass theorem, there exists $\lim\limits_{n\to\infty} x_n = L$, and $L \geq 1$.

Letting $n \to \infty$ in the assumption of the problem, we have

$$L = 1 + \log\left(\frac{L^2}{1+\log L}\right) \Longleftrightarrow L - 1 - \log\left(\frac{L^2}{1+\log L}\right) = 0,$$

which means that $f(L) = 0$. Thus $L = 1$.

4.2.38

Suppose that such a sequence does exist. For each $n \geq 1$ we rearrange $x_1, \ldots, x_n$ in an increasing sequence $x_{i_1}, \ldots, x_{i_n}$, where $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Note that $i_1 \neq i_n$, then either $i_1 \geq 1$ and $i_n \geq 2$, or $i_n \geq 1$ and $i_1 \geq 2$. In any cases we always have

$$\frac{1}{i_1(i_1+1)} + \frac{1}{i_n(i_n+1)} \leq \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6}.$$

Denote $M = 0.666$, we then have

$$\begin{aligned}
2M &\geq x_{i_n} - x_{i_1} \\
&= \sum_{k=2}^{n} (x_{i_k} - x_{i_{k-1}}) \\
&\geq \sum_{k=2}^{n} \left(\frac{1}{i_k(i_k+1)} + \frac{1}{i_{k-1}(i_{k-1}+1)} \right) \\
&= 2\sum_{k=1}^{n} \frac{1}{k(k+1)} - \frac{1}{i_n(i_n+1)} - \frac{1}{i_1(i_1+1)} \\
&\geq 2\sum_{k=1}^{n} \frac{1}{k(k+1)} - \frac{1}{2} - \frac{1}{6} \\
&= 2\left(1 - \frac{1}{n+1}\right) - \frac{2}{3} = \frac{4}{3} - \frac{2}{n+1}.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \left(\frac{4}{3} - \frac{2}{n+1} \right) = \frac{4}{3} > 1.332 = 2M,$$

which is impossible.

Thus there does not exist such a sequence.

4.2.39

Note that $u_3 = 5$ and (u_n) is an increasing sequence of positive numbers. Furthermore, the given equation $u_{n+2} = 3u_{n+1} - u_n$ shows that $u_{n+1} + u_{n-1} = 3u_n$. From this it follows that

$$\begin{aligned}
(u_{n+2} + u_n)u_n &= 3u_{n+1}u_n \\
&= u_{n+1}(u_{n+1} + u_{n-1}) \\
&\quad (n \geq 2),
\end{aligned}$$

or equivalently,

$$u_{n+2}u_n - u_{n+1}^2 = u_{n+1}u_{n-1} - u_n^2,\ (n \geq 2).$$

This means that

$$u_{n+2}u_n - u_{n+1}^2 = \cdots = u_3u_1 - u_2^2 = 1\ (n \geq 1),$$

which implies that

$$u_{n+2} = \frac{1 + u_{n+1}^2}{u_n},\ \forall n \geq 1.$$

Therefore,

$$u_{n+2} + u_n = \frac{1 + u_{n+1}^2}{u_n} + u_n = \frac{1}{u_n} + u_n + \frac{u_{n+1}^2}{u_n},$$

which, by Cauchy inequality, gives

$$u_{n+2} + u_n \geq 2 + \frac{u_{n+1}^2}{u_n},\ \forall n \geq 1.$$

4.2.40

Denote three positive roots of the given equation by m, n, p. By Viète formula

$$\begin{cases} m + n + p = \frac{1}{a}, \\ mn + np + pm = \frac{b}{a}, \\ mnp = \frac{1}{a}, \end{cases}$$

which implies that $a, b > 0$.

Furthermore, by the well-known inequality $(m+n+p)^2 \geq 3(mn+np+pm)$, the equality occurs if and only if $m = n = p$, we have

$$\frac{1}{a^2} \geq \frac{3b}{a} \Longleftrightarrow 0 < b \leq \frac{1}{3a}. \tag{1}$$

On the other hand, by Cauchy inequality, $m + n + p \geq 3\sqrt[3]{mnp}$, the equality occurs if and only if $m = n = p$, we also have

$$\frac{1}{a} \geq 3\sqrt[3]{\frac{1}{a}} \Longleftrightarrow 0 < a \leq \frac{1}{3\sqrt{3}}. \tag{2}$$

Consider a function

$$f(x) = \frac{5a^2 - 3ax + 2}{a^2(x - a)},\ x \in \left(0, \frac{1}{3a}\right),$$

which has derivative

$$f'(x) = -\frac{2(a^2+1)}{a^2(x-a)^2} < 0,\ \forall x \in \left(0, \frac{1}{3a}\right),$$

and hence $f(x)$ is decreasing on this interval. Then

$$f(x) \geq f\left(\frac{1}{3a}\right) = \frac{3(5a^2+1)}{a(1-3a^2)},\ \forall x \in \left(0, \frac{1}{3a}\right). \tag{3}$$

Next, note that, by (2) we have $0 < a \leq \frac{1}{3\sqrt{3}}$. So consider a function

$$g(x) = \frac{3(5x^2+1)}{x(1-3x^2)},\ x \in \left(0, \frac{1}{3\sqrt{3}}\right],$$

which has a derivative

$$g'(x) = \frac{15x^4+14x^2-1}{x^2(3x^2-1)^2} < 0,\ \forall x \in \left(0, \frac{1}{3\sqrt{3}}\right].$$

We get that $g(x)$ is decreasing and hence

$$g(x) \geq g\left(\frac{1}{3\sqrt{3}}\right) = 4\sqrt{3}. \tag{4}$$

Combining (3) and (4) yields

$$P = \frac{5a^2-3ab+2}{a^2(b-a)} \geq 12\sqrt{3}.$$

The equality occurs if and only if $a = \frac{1}{\sqrt{3}}, b = \sqrt{3}$, or $m = n = p = \sqrt{3}$.

Thus the minimum value of P is $12\sqrt{3}$.

4.2.41

Substituting $y = 0$ and $y = 1$ into the given equation we have

$$2f(x) = 3f\left(\frac{2x}{3}\right),\ \forall x \in [0,1], \tag{1}$$

and

$$2f(x) = 3f\left(\frac{2x+1}{3}\right),\ \forall x \in [0,1]. \tag{2}$$

Since $f(x)$ is defined and continuous on $[0,1]$, there exists

$$M := \max_{x\in[0,1]} f(x) = f(x_0),$$

for some $x_0 \in [0,1]$.

Note that $f(0) = f(1) = 0$, we have $M \geq 0$. Consider two cases:

Case 1: $x_0 \in \left[0, \frac{2}{3}\right]$. Then $0 \leq \frac{3x_0}{2} \leq 1$. From (1) it follows that

$$2f\left(\frac{3x_0}{2}\right) = 3f\left(\frac{2}{3}\cdot\frac{3x_0}{2}\right) = 3f(x_0),$$

or

$$M = f(x_0) = \frac{2}{3}f\left(\frac{3x_0}{2}\right) \leq \frac{2}{3}M,$$

which gives $M = 0$.

Case 2: $x_0 \in [2/3, 1]$. Then $0 < \frac{3x_0 - 1}{2} \leq 1$. Similarly, from (2) we have

$$2f\left(\frac{3x_0-1}{2}\right) = 3f\left(\frac{2\frac{3x_0-1}{2}+1}{3}\right) = 3f(x_0),$$

or

$$M = f(x_0) = \frac{2}{3}f\left(\frac{3x_0-1}{2}\right) \leq \frac{2}{3}M,$$

which also gives $M = 0$.

Thus in both cases $f(x) \leq 0$ for all $x \in [0,1]$.

Now consider a function $g(x) = -f(x)$ on $[0,1]$. We can easily verify that this function $g(x)$ satisfies all requirements of the problem, as $f(x)$ does. By the proof above, we have $g(x) \leq 0$ for all $x \in [0,1]$, or $f(x) \geq 0$ for all $x \in [0,1]$.

Thus $f(x) = 0$ for all $x \in [0,1]$.

4.2.42

Replacing x by $1 - x$ we have

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4,\ \forall x. \tag{1}$$

On the other hand, from the assumption we also have

$$f(1-x) = 2x - x^4 - x^2 f(x),\ \forall x. \tag{2}$$

Substituting (2) into (1) yields

$$f(x)(x^2 - x - 1)(x^2 - x + 1) = (1 - x)(1 + x^3)(x^2 - x - 1),\ \forall x,$$

or, due to $x^2 - x + 1 \neq 0,\ \forall x$, equivalently

$$(x^2 - x - 1)f(x) = (1 - x^2)(x^2 - x - 1),\ \forall x,$$

which gives $f(x) = 1 - x^2$ for all $x \neq \alpha, \beta$, where α, β are two roots of the equation $x^2 - x - 1 = 0$.

Furthermore, by Viète formula $\alpha + \beta = 1, \alpha\beta = -1$. Substituting these values into the given equation we obtain

$$\begin{cases} \alpha^2 f(\alpha) + f(\beta) = 2\alpha - \alpha^4, \\ \beta^2 f(\beta) + f(\alpha) = 2\beta - \beta^4. \end{cases}$$

Then $f(\alpha) = k$ and $f(\beta) = 2\alpha - \alpha^4 - \alpha^2 k$, where k ia an arbitrary real number.

Conversely, it is easy to verify that this function satisfies the given equation.

Thus the answer is

$$f(x) = \begin{cases} k, & \text{if } x = \alpha, \\ 2\alpha - \alpha^4 - \alpha^2 k, & \text{if } x = \beta, \\ 1 - x^2, & \text{if } x \neq \alpha, \beta \end{cases}$$

where α, β are two roots of the equation $x^2 - x - 1 = 0$ and k is an arbitrary real number.

4.2.43

We have x_1 is defined for any $x_0 \in (0, c)$ if and only if

$$c \geq \sqrt{c + x_0} \Longrightarrow c(c-1) \geq x_0 \Longrightarrow c(c-1) \geq c \Longrightarrow c \geq 2.$$

Conversely, we can prove, by induction, that if $c > 2$ then x_n is well defined for any $n \geq 1$. Indeed, from $x_0 < c$ and $c > 2$ it follows that

$$c + x_0 < 2c \Longrightarrow \sqrt{c + x_0} < \sqrt{2c} < c \Longrightarrow c - \sqrt{c + x_0} > 0,$$

which shows that x_1 is well defined.

Suppose that x_k $(k \geq 1)$ is well defined. Then $0 < x_k < \sqrt{c}$, and so

$$0 < c + x_k < 2c \Longrightarrow \sqrt{c + x_k} < \sqrt{2c} < c \Longrightarrow c - \sqrt{c + x_k} > 0,$$

which implies that x_{k+1} is well defined. By mathematical principle, we conclude that with $c > 2$ all x_n is well defined.

Suppose, at the moment, that (x_n) has a limit L, of course, $0 < L < \sqrt{c}$. Then letting $n \to \infty$ in the given equation we obtain

$$\begin{aligned}
&\sqrt{c-\sqrt{c+L}}=L \\
\iff\ & c+L=(c-L^2)^2 \\
\iff\ & L^4-2cL^2-L+c^2-c=0 \\
\iff\ & (L^4+L^3+L^2-cL^2)-(L^3+L^2+L-cL)-(cL^2+cL+c-c^2)=0 \\
\iff\ & (L^2+L+1-c)(L^2-L-c)=0.
\end{aligned}$$

Note that $L^2 - L - c < 0$, as $0 < L < \sqrt{c}$. Then we get $L^2 + L + 1 - c = 0$, which has two roots of opposite signs. Therefore, L must be a positive root of this equation.

Now we show that (x_n) converges to this limit. Indeed, let L be a positive root of the equation $L^2+L+1-c = 0$, that is, $L = \dfrac{-1 + \sqrt{4c-3}}{2}$. In this case we have

$$\begin{aligned}
|x_{n+1} - L| &= \left|\sqrt{c - \sqrt{c + x_n}} - L\right| = \frac{\left|c - \sqrt{c + x_n} - L^2\right|}{\sqrt{c - \sqrt{c + x_n}} + L} \\
&= \frac{|(L + 1) - \sqrt{c + x_n}|}{\sqrt{c - \sqrt{c + x_n}} + L} \le \frac{|(L + 1) - \sqrt{c + x_n}|}{L} \\
&= \frac{|(L + 1)^2 - c - x_n|}{L\left(L + 1 + \sqrt{c + x_n}\right)} = \frac{|x_n - L|}{L\left(L + 1 + \sqrt{c + x_n}\right)} \\
&\le \frac{|x_n - L|}{L^2 + L + L\sqrt{c}} = \frac{|x_n - L|}{c - 1 + L\sqrt{c}}.
\end{aligned}$$

Note that $c - 1 + L\sqrt{c} \ge 1 + L\sqrt{c} > 1$. Then from the inequality

$$|x_{n+1} - L| \le \frac{|x_n - L|}{c - 1 + L\sqrt{c}},$$

it follows that there exists

$$\lim_{n\to\infty} x_n = L = \frac{-1 + \sqrt{4c-3}}{2}.$$

4.2.44

1) We have

$$\begin{aligned} P_3(x) &= x^3\sin\alpha - x\sin 3\alpha + \sin 2\alpha \\ &= x^3\sin\alpha - x(3\sin\alpha - 4\sin^3\alpha) + 2\sin\alpha\cos\alpha \\ &= \sin\alpha(x+2\cos\alpha)(x^2-2x\cos\alpha+1). \end{aligned}$$

Note that $f(x) = x^2 - 2x\cos\alpha + 1$ has no real roots, as $\alpha \in (0,\pi)$. So $f(x)$ is only quadratic polynomial of the form $x^2 + ax + b$ that $P_3(x)$ is divisible by.

Moreover, for any $n \geq 3$

$$\begin{aligned} P_{n+1}(x) &= x^{n+1}\sin\alpha - x\sin(n+1)\alpha + \sin(n\alpha) \\ &= x^{n+1}\sin\alpha - x[\sin(n-1)\alpha - 2\sin(n\alpha)\cos\alpha] + \sin(n\alpha) \\ &= x[x^n\sin\alpha - x\sin(n\alpha) + \sin(n-1)\alpha] + (x^2-2x\cos\alpha+1)\sin(n\alpha) \\ &= xP_n(x) + (x^2-2x\cos\alpha+1)\sin(n\alpha). \end{aligned}$$

So, $f(x) = x^2 - 2x\cos\alpha + 1$ is the desired polynomial.

2) Assume, in contrary, that there is a linear binomial $g(x) = x + c$ with $c \in \mathbb{R}$ such that $P_n(x)$ is divisible by $g(x)$ for all $n \geq 3$. Then there exists x_0 (namely, $x_0 = -c$) for which $P_n(x_0) = 0$ for all $n \geq 3$.

By 1) we have $P_{n+1}(x) - xP_n(x) = (x^2 - 2x\cos\alpha + 1)\sin(n\alpha) = f(x)\sin(n\alpha)$. Substituting $x = x_0$ into this equation we obtain

$$0 = P_{n+1}(x_0) - x_0P_n(x_0) = f(x_0)\sin(n\alpha),\ \forall n \geq 3,$$

which implies that $\sin(n\alpha) = 0$ for all $n \geq 3$. In particular,

$$\sin(4\alpha) = \sin(3\alpha) = 0.$$

However, $\sin(4\alpha) = \sin(3\alpha+\alpha) = \sin(3\alpha)\cos\alpha + \cos(3\alpha)\sin\alpha$, and so we arrive to $\sin\alpha = 0$, which is impossible, as $0 < \alpha < \pi$.

Thus such a linear function does not exists.

4.2.45

Note that $x_n > 0$ for all n. Put $y_n = \dfrac{2}{x_n}$, we have

$$y_1 = 3,\ y_{n+1} = 4(2n+1) + y_n,\ \forall n \geq 1. \tag{1}$$

From (1), by induction, it follows that

$$y_n = (2n-1)(2n+1), \ \forall n \geq 1.$$

Therefore,

$$x_n = \frac{2}{y_n} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1},$$

which gives

$$\sum_{i=1}^{2001} x_i = 1 - \frac{1}{4003} = \frac{4002}{4003}.$$

4.2.46

1) Let $b=1$. Consider two cases.

Case 1: $a = k\pi$ ($k \in \mathbb{Z}$). Then $x_n = a$ for all n, and so (x_n) converges and

$$\lim_{n\to\infty} x_n = a$$

Case 2: $a \neq k\pi$ ($k \in \mathbb{Z}$). Then denote $f(x) = x + \sin x, \ x \in \mathbb{R}$, we can rewrite (x_n) in the form

$$x_0 = a, \ x_{n+1} = f(x_n), \ \forall n \geq 0.$$

Note that $f'(x) = 1+\cos x \geq 0, \ \forall x$, and so $f(x)$ is increasing, which implies that the sequence (x_n) is monotone.

• For $a \in (2k\pi, (2k+1)\pi), \ k \in \mathbb{Z}$: $\sin a > 0$, and so $x_0 < x_1$, which gives that (x_n) is increasing.

By induction we can prove that $x_n \in (2k\pi, (2k+1)\pi)$ for all n. Indeed, it is obvious for $n=0$. Suppose that the statement is true for $n = m \geq 0$, that is $x_m \in (2k\pi, (2k+1)\pi)$. In this case, since $f(x)$ is increasing,

$$2k\pi = f(2k\pi) < f(x_m) = x_{m+1} < f\big((2k+1)\pi\big) = (2k+1)\pi,$$

which means that the statement is also true for $n = m+1$.

Thus (x_n) is increasing and is bounded from above by $(2k+1)\pi$. By Weierstrass theorem, there exists $\lim\limits_{n\to\infty} x_n = L$, and moreover, $2k\pi < a \leq L \leq (2k+1)\pi$ and $\sin L = 0$. From this it follows that $L = (2k+1)\pi$.

• For $a \in ((2k-1)\pi, 2k\pi), k \in \mathbb{Z}$: it is similar. We get that (x_n) is decreasing and is bounded from below by $(2k-1)\pi$, and so there exists $\lim\limits_{n\to\infty} x_n = (2k-1)\pi$.

Thus for any given a, the sequence (x_n) always converges and its limit, as we can easily see, be written in the following general formula:

$$\lim_{n\to\infty} x_n = \left(2\left[\frac{a}{2\pi}\right] + \operatorname{sign}\left(\left\{\frac{a}{2\pi}\right\}\right)\right),$$

where $\{x\} = x - [x]$, and $\operatorname{sign}(x)$ is the sign function of x.

2) Let $b > 2$ be given. Consider a function

$$g(x) = \frac{\sin x}{x},$$

which is continuous on $(0, \pi]$, and moreover, $g(\pi) = 0$, $\lim\limits_{x\to 0} g(x) = 1$.

Then from $0 < \frac{2}{b} < 1$ it follows that there exists $a_0 \in (0, \pi)$ such that

$$\frac{\sin a_0}{a_0} = \frac{2}{b},$$

or

$$2a_0 = b \sin a_0.$$

Take $a = \pi - a_0$, we have

$$\begin{aligned}
x_0 &= a = \pi - a_0,\\
x_1 &= x_0 + b\sin x_0 = \pi - a_0 + b\sin(\pi - a_0)\\
&= \pi - a_0 + b\sin a_0 = \pi - a_0 + 2a_0 = \pi + a_0,\\
x_2 &= x_1 + b\sin x_1 = \pi + a_0 + b\sin(\pi + a_0)\\
&= \pi + a_0 - b\sin a_0 = \pi + a_0 - 2a_0 = \pi - a_0,
\end{aligned}$$

from which it follows that $x_0 = x_2 = \cdots = x_{2n} = \pi - a_0,\ x_1 = x_3 = \cdots = x_{2n+1} = \pi + a_0,\ \forall n \geq 0$. That is, the sequence (x_n) is periodic with the period 2, so it diverges as $n \to \infty$.

4.2.47

We have

$$\frac{(1-x^2)^2}{(1+x^2)^2} f\big(g(x)\big) = (1-x^2) f(x),\ \forall x \in (-1,1). \tag{1}$$

Put $h(x) = (1-x^2)f(x),\ \forall x \in (-1,1)$. Then we can verify that $f(x)$ is continuous on $(-1,1)$ and satisfies (1) if and only if $h(x)$ is continuous on $(-1,1)$ and satisfies

$$h\big(g(x)\big) = h(x),\ \forall x \in (-1,1). \tag{2}$$

Note that $\varphi(x) = \frac{1-x}{1+x},\ x > 0$, is a bijection from $(0, +\infty)$ onto $(-1, 1)$. So we can write (2) as follows

$$h\left(g\left(\frac{1-x}{1+x}\right)\right) = h\left(\frac{1-x}{1+x}\right),\ \forall x > 0,$$

or

$$h\left(\frac{1-x^2}{1+x^2}\right) = h\left(\frac{1-x}{1+x}\right),\ \forall x > 0. \tag{3}$$

Consider a function $k(x) = h\left(\frac{1-x}{1+x}\right), x > 0$. We can verify that $h(x)$ is continuous on $(-1, 1)$ and satisfies (3) if and only if $k(x)$ is continuous on $(0, +\infty)$ and satisfies

$$k(x^2) = k(x),\ \forall x > 0.$$

By induction we can prove that

$$k(x) = k\left(\sqrt[2^n]{x}\right),\ \forall x > 0,\ \forall n \geq 1,$$

which gives $\lim\limits_{n\to\infty} \sqrt[2^n]{x} = 1$, and since $k(x)$ is continuous on $(0, +\infty)$, $h(x) = h(1),\ \forall x > 0$.

Thus $h(x) = C$ (const) for all $x \in (-1, 1)$, and hence

$$f(x) = \frac{C}{1-x^2},\ \forall x \in (-1, 1), \tag{4}$$

where C is an arbitrary real number.

Conversely, all functions defined by (4) satisfy requirements of the problem, so they are exactly what we are searching for.

4.2.48

Substituting $y = f(x)$ and $y = x^{2002}$ into the given equation, we have

$$f(0) = f\big(x^{2002} - f(x)\big) - 2001[f(x)]^2,$$

and

$$f\big(x^{2002} - f(x)\big) = f(0) - 2001x^{2002}f(x).$$

Adding these two equations gives

$$f(x)\left(f(x) + x^{2002}\right) = 0.$$

Then either $f(x) = 0$ for $x = 0$, or

$$f(x) = -x^{2002},\ \text{for all } x \text{ such that } f(x) \neq 0. \tag{1}$$

We can verify that two functions $f_1(x) \equiv 0$ and $f_2(x) = -x^{2002}$, satisfy $f(0) = 0$ and (1).

Now we prove that there does not exists a function $f(x)$ satisfying the given equation which is different from both f_1 and f_2. Indeed, by the arguments above, such a function f should satisfy $f(0) = 0$ and (1). Moreover, since $f \neq f_2$, there is $x_0 \neq 0$ such that $f(x_0) = 0$. Also since $f \neq f_1$, there is $y_0 \neq 0$ such that $f(y_0) \neq 0$.

Substituting $x = 0$ into the given equation and taking into account that $f(0) = 0$, we have $f(y) = f(-y)$ for all y. So we can assume that $y_0 > 0$.

Since $f(y_0) \neq 0$, by (1),

$$f(y_0) = -y_0^{2002}. \tag{2}$$

On the other hand, substituting $x = x_0$ and $y = -y_0$ into the given equation, we get

$$f(-y_0) = f\left(x_0^{2002} + y_0\right). \tag{3}$$

From (1), (2), and (3) we obtain

$$0 \neq -y_0^{2002} = f(y_0) = f(-y_0) = f\left(x_0^{2002} + y_0\right) = -\left(x_0^{2002} + y_0\right)^{2002} < -y_0^{2002},$$

because $y_0 > 0$, which is impossible.

By verifying directly the functions f_1 and f_2, we see that the only solution of the problem is $f(x) \equiv 0$.

4.2.49

Put

$$f_n(x) = \frac{1}{2x} + \frac{1}{x-1^2} + \frac{1}{x-2^2} + \cdots + \frac{1}{x-n^2}.$$

1) We note that $f_n(x)$ is continuous and decreasing on $(0, 1)$. Also

$$\lim_{x \to 0^+} f_n(x) = +\infty, \quad \lim_{x \to 1^-} f_n(x) = -\infty.$$

Then for each $n \geq 1$ the equation $f_n(x) = 0$ has a unique solution $x_n \in (0, 1)$.

2) We have

$$\begin{aligned} f_{n+1}(x_n) &= \frac{1}{2x_n} + \frac{1}{x_n - 1^2} + \frac{1}{x_n - 2^2} + \cdots + \frac{1}{x_n - n^2} + \frac{1}{x_n - (n+1)^2} \\ &= \frac{1}{x_n - (n+1)^2} < 0, \end{aligned}$$

as a solution $x_n \in (0, 1)$.

Since $\lim_{x\to 0^+} f_{n+1}(x) = +\infty$, the last inequality shows that for each $n \geq 1$ the equation $f_{n+1}(x) = 0$ has a unique solution in the interval $(0, x_n)$. Note that $(0, x_n) \subset (0, 1)$ for all n, and then by the result of 1), $x_{n+1} \in (0, x_n)$ for all n, or $x_{n+1} < x_n$ for all n.

Thus the sequence (x_n) is decreasing and is bounded from below by 0, by Weierstrass theorem, it converges.

4.2.50

Denote

$$f_n(x) = \frac{1}{x-1} + \frac{1}{2^2x-1} + \cdots + \frac{1}{n^2x-1} - \frac{1}{2}.$$

1) For each n the function $f_n(x)$ is continuous, decreasing on $(1, +\infty)$. Moreover, $\lim_{x\to 1^+} f_n(x) = +\infty$, $\lim_{x\to+\infty} f_n(x) = -\frac{1}{2}$. Hence the equation $f_n(x) = 0$ has a unique solution $x_n > 1$.

2) For each n we have

$$f_n(4) = -\frac{1}{2} + \frac{1}{2^2-1} + \cdots + \frac{1}{(2k)^2-1} + \cdots + \frac{1}{(2n)^2-1}$$

$$= \frac{1}{2}\left(-1 + 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2k+1} + \cdots + \frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= -\frac{1}{2(2n+1)} < 0 = f_n(x_n).$$

So the function $f_n(x)$ is decreasing on $(1, +\infty)$, and hence

$$x_n < 4, \text{ for all } n. \tag{1}$$

On the other hand, $f_n(x)$ is differentiable on $[x_n, 4]$, and so, by Lagrange theorem, there exists $t \in (x_n, 4)$ such that

$$\frac{f_n(4) - f_n(x_n)}{4 - x_n} = f_n'(t) = -\frac{1}{(t-1)^2} - \cdots - \frac{1}{(n^2t-1)^2} < -\frac{1}{9},$$

or

$$-\frac{1}{2(2n+1)(4-x_n)} < -\frac{1}{9},$$

which implies that

$$x_n > 4 - \frac{9}{2(2n+1)}, \text{ for all } n. \tag{2}$$

From (1) and (2) it follows that

$$4 - \frac{9}{2(2n+1)} < x_n < 4, \text{ for all } n,$$

which gives the desired limit.

4.2.51

The given equation is equivalent to

$$(x+2)(x^2+x+1)P(x-1) = (x-2)(x^2-x+1)P(x), \ \forall x. \tag{1}$$

Substituting $x=-2$ and $x=2$ into (1) we get

$$P(-2) = P(1) = 0.$$

Also substituting $x=-1$ and $x=1$ into the given equation we obtain

$$P(-1) = P(0) = 0.$$

From these facts it follows that

$$P(x) = (x-1)x(x+1)(x+2)Q(x), \ \forall x, \tag{2}$$

where $Q(x)$ is a polynomial with real coefficients. Then

$$P(x-1) = (x-2)(x-1)x(x+1)Q(x-1), \ \forall x. \tag{3}$$

Combining (1), (2) and (3) yields

$$\begin{aligned} & (x-2)(x-1)x(x+1)(x+2)(x^2+x+1)Q(x-1) \\ = \ & (x-2)(x-1)x(x+1)(x+2)(x^2-x+1)Q(x), \ \forall x, \end{aligned}$$

which in turn implies that

$$(x^2+x+1)Q(x-1) = (x^2-x+1)Q(x), \ \forall x \neq 0, \pm 1, \pm 2.$$

As both sides are polynomials of a variable x, the last equality is valid for all $x \in \mathbb{R}$, that is,

$$(x^2+x+1)Q(x-1) = (x^2-x+1)Q(x), \ \forall x. \tag{4}$$

Taking into account that $(x^2+x+1, x^2-x+1) = 1$, we get

$$Q(x) = (x^2+x+1)R(x), \ \forall x, \tag{5}$$

where $R(x)$ is a polynomial with real coefficients, and so

$$Q(x-1) = (x^2 - x + 1)R(x-1),\ \forall x. \tag{6}$$

Combining (4), (5) and (6) yields

$$\begin{aligned} & (x^2 + x + 1)(x^2 - x + 1)R(x-1) \\ = \ & (x^2 - x + 1)(x^2 + x + 1)R(x),\ \forall x, \end{aligned}$$

or equivalently,

$$R(x-1) = R(x),\ \forall x,$$

as $(x^2 + x + 1)(x^2 - x + 1) \neq 0,\ \forall x.$

The last equation shows that $R(x) = \text{const}$. Thus

$$P(x) = C(x-1)x(x+1)(x+2)(x^2 + x + 1),\ \forall x,$$

where C is an arbitrary constant.

Conversely, by direct verification we wee that the above-mentioned polynomials satisfy the requirement of the problem, and so they are all the required polynomials.

4.2.52

1) Consider two cases.

Case 1: $\alpha = -1$. Then $x_n = 0$ for all n.

Case 2: $\alpha \neq -1$. Then $x_n \neq -\alpha$ for all n. So we can write

$$x_{n+1} = \frac{\alpha + 1}{x_n + \alpha},\ \forall n \geq 1. \tag{1}$$

Since $x_1 = 0$, $x_2 = \dfrac{\alpha+1}{\alpha}$. Putting $u_1 = 0,\ u_2 = \alpha + 1$ and $v_1 = 1,\ v_2 = \alpha$, we can see that

$$x_1 = 0 = \frac{u_1}{v_1},\ x_2 = \frac{\alpha+1}{\alpha} = \frac{u_2}{v_2}.$$

Suppose that

$$x_k = \frac{u_k}{v_k},$$

for $k \geq 1$. Then, by (1), we get x_{k+1} has a form

$$x_{k+1} = \frac{u_{k+1}}{v_{k+1}},$$

where $(u_k), (v_k)$ defined by

$$\begin{cases} u_{k+1} = (\alpha+1)v_k, \\ v_{k+1} = \alpha v_k + u_k. \end{cases}$$

So by the principle of mathematical induction, we can conclude that

$$x_n = \frac{u_n}{v_n},$$

where (u_n) and (v_n) are defined by

$$\begin{aligned} &u_1 = 0,\ u_2 = \alpha+1,\ v_1 = 1,\ v_2 = \alpha, \\ &u_{n+1} = (\alpha+1)v_n,\ v_{n+1} = \alpha v_n + u_n,\ \forall n \geq 1. \end{aligned}$$

From the equations for $(u_n), (v_n)$ it follows that (v_n) is defined by

$$v_1 = 1,\ v_2 = \alpha, v_{n+1} = \alpha v_n + (\alpha+1)v_{n-1}.$$

Note that the characteristic equation of (v_n) is $x^2 - \alpha x - (\alpha+1) = 0$, which gives two roots $x_1 = -1, x_2 = \alpha + 1$. In this case we have:

- If $\alpha = -2$, then $v_n = (-1)^{n-1} + (-1)^{n-1}(n-1),\ \forall n \geq 1$.
- If $\alpha \neq -2$, then $v_n = \frac{(-1)^{n-1} + (\alpha+1)^n}{\alpha+2},\ \forall n \geq 1$.

From this it follows that

- If $\alpha = -2$, then $u_1 = 0,\ u_n = -\left[(-1)^{n-2} + (-1)^{n-2}(n-2)\right],\ \forall n \geq 2$.
- If $\alpha \neq -2$: $u_1 = 0,\ u_n = \frac{(-1)^{n-2} + (\alpha+1)^{n-1}}{\alpha+2}(\alpha+1),\ \forall n \geq 2$.

Finally, we get

$$x_n = \begin{cases} \frac{n-1}{n},\ \forall n \geq 1, & \text{if } \alpha = -2, \\ \frac{\left[(-1)^{n-2} + (\alpha+1)^{n-1}\right](\alpha+1)}{(-1)^{n-1} + (\alpha+1)^n},\ \forall n \geq 1, & \text{if } \alpha \neq -2. \end{cases}$$

2) We can easily verify that

- If $\alpha = -1$, then $\lim\limits_{n\to\infty} x_n = 0$.
- If $\alpha = -2$: $\lim\limits_{n\to\infty} x_n = 1$.
- For the case $\alpha \neq -2$, we have

$$\begin{cases} x_{2k-1} = \frac{(\alpha+1)^{2k-1} - (\alpha+1)}{1 + (\alpha+1)^{2k-1}},\ \forall k \geq 1, \\ x_{2k} = \frac{(\alpha+1) + (\alpha+1)^{2k}}{(\alpha+1)^{2k} - 1},\ \forall k \geq 1. \end{cases}$$

Therefore,

• If $|\alpha+1| > 1$, then $\lim\limits_{k\to\infty} x_{2k-1} = 1$, $\lim\limits_{k\to\infty} x_{2k} = 1$, and hence $\lim\limits_{n\to\infty} x_n = 1$.

• If $|\alpha+1| < 1$, then $\lim\limits_{k\to\infty} x_{2k-1} = -(\alpha+1)$, $\lim\limits_{k\to\infty} x_{2k} = -(\alpha+1)$, and hence $\lim\limits_{n\to\infty} x_n = -(\alpha+1)$.

4.2.53

We have

$$\begin{aligned} f(\cot x) &= \sin 2x + \cos 2x \\ &= \frac{2\cot x}{\cot^2 x + 1} + \frac{\cot^2 x - 1}{\cot^2 x + 1} \\ &= \frac{\cot^2 x + 2\cot x - 1}{\cot^2 x + 1}. \end{aligned}$$

Note that there is a one-to-one correspondence between $t \in \mathbb{R}$ and $x \in (0, \pi)$ by $t = \cot x$, we have

$$f(t) = \frac{t^2 + 2t - 1}{t^2 + 1}, t \in \mathbb{R}.$$

Then

$$g(x) = f(\sin^2 x) \cdot f(\cos^2 x) = \frac{\sin^4 2x + 32\sin^2 2x - 32}{\sin^4 2x - 8\sin^2 2x + 32}, \quad x \in \mathbb{R}. \qquad (1)$$

Put $u = \frac{1}{4}\sin^2 2x$. In this case u has the range $[0, \frac{1}{4}]$, and so $g(x)$ becomes

$$h(u) = \frac{u^2 + 8u - 2}{u^2 - 2u + 2}.$$

From (1) one gets

$$\min_{x\in\mathbb{R}} g(x) = \min_{u\in[0,\frac{1}{4}]} h(u), \ \max_{x\in\mathbb{R}} g(x) = \max_{u\in[0,\frac{1}{4}]} h(u).$$

Since

$$h'(u) = \frac{2(-5u^2 + 4u + 6)}{(u^2 - 2u + 2)^2} > 0, \ \forall u \in [0, \frac{1}{4}],$$

$h(u)$ is increasing on this interval. So $\min h(u) = h(0) = -1, \max h(u) = h\left(\frac{1}{4}\right) = \frac{1}{25}$.

Thus $\min g(x) = -1$ occurs at $x = 0$, and $\max g(x) = \frac{1}{25}$ occurs at $x = \frac{\pi}{4}$.

4.2.54

Note that $f(x) = \dfrac{1}{2}x \in \mathcal{F}$, and so $a \leq \frac{1}{2}$.

Furthermore, from assumptions of the problem, namely,

$$\begin{cases} f(3x) \geq f\big(f(2x)\big) + x, \ \forall x > 0, \\ f(x) > 0, \forall x > 0, \end{cases}$$

it follows that

$$f(x) \geq \frac{1}{3}x, \ \forall x > 0. \tag{1}$$

Consider (a_n) defined by

$$a_1 = \frac{1}{3}, \ a_{n+1} = \frac{2a_n^2 + 1}{3}, \ n \geq 1.$$

It is clear that $a_n > 0$ for all n. Then we can prove by induction that

$$f(x) \geq a_n x, \ \forall x > 0. \tag{2}$$

Indeed, for $n = 1$ it is true, by (1). Suppose that (2) is true for $n = k \geq 1$. Then for all $x > 0$ we have

$$\begin{aligned} f(x) &\geq f\left(f\left(\frac{2x}{3}\right)\right) + \frac{x}{3} \\ &\geq a_k \cdot f\left(\frac{2x}{3}\right) + \frac{x}{3} \\ &\geq a_k \cdot a_k \frac{2x}{3} + \frac{x}{3} \\ &= \frac{2a_k^2 + 1}{3}x = a_{k+1}. \end{aligned}$$

By the principle of mathematical induction, the inequality (2) is true.

Furthermore, it is easily verified by induction that (a_n) is bounded from above by $\frac{1}{2}$. Then

$$a_{n+1} - a_n = \frac{1}{3}(a_n - 1)(2a_n - 1) > 0,$$

which shows that (a_n) is increasing. By Weierstrass theorem, there exists $\lim\limits_{n\to\infty} a_n = L$, which satisfies the equation $(L-1)(2L-1) = 0$, and moreover, $L \leq \frac{1}{2}$. Hence $L = \frac{1}{2}$.

Thus $f(x) \geq \frac{1}{2}x, \ \forall x > 0, \ \forall f \in \mathcal{F}$, and so $a = \frac{1}{2}$.

4.2.55

Note that $x_n > 0$ for all n. We have

$$\begin{aligned} 2x_{n+1}+1 &= \frac{(4+2\cos 2\alpha+2-2\cos 2\alpha)x_n+2\cos^2\alpha+2-\cos 2\alpha}{(2-2\cos 2\alpha)x_n+2-\cos 2\alpha} \\ &= \frac{3(2x_n+1)}{(2-2\cos 2\alpha)x_n+2-\cos 2\alpha}, \end{aligned}$$

which leads to

$$\frac{1}{2x_{n+1}+1} = \frac{(1-\cos 2\alpha)(2x_n+1)+1}{3(2x_n+1)} = \frac{1}{3}\left(2\sin^2\alpha+\frac{1}{2x_n+1}\right).$$

This shows that

$$\frac{1}{2x_{n+1}+1}-\sin^2\alpha = \frac{1}{3}\left(\frac{1}{2x_n+1}-\sin^2\alpha\right),$$

or

$$z_{n+1} = \frac{1}{3}z_n,$$

where $z_n = \dfrac{1}{2x_n+1} - \sin^2\alpha$.

Thus (y_n) is a geometric progression with $z_1 = \frac{1}{3} - \sin^2\alpha$ and ratio $q = \frac{1}{3}$. Then

$$\sum_{k=1}^{n} z_k = \left(\frac{1}{3}-\sin^2\alpha\right)\cdot\frac{1-\frac{1}{3^n}}{1-\frac{1}{3}} = \frac{3}{2}\left(\frac{1}{3}-\sin^2\alpha\right)\left(1-\frac{1}{3^n}\right).$$

Then

$$\begin{aligned} y_n &= \sum_{k=1}^{n}\frac{1}{2x_k+1} = \sum_{k=1}^{n}(z_k+\sin^2\alpha) \\ &= \frac{3}{2}\left(\frac{1}{3}-\sin^2\alpha\right)\left(1-\frac{1}{3^n}\right)+n\sin^2\alpha \\ &= \frac{1}{2}(1-3\sin^2\alpha)\left(1-\frac{1}{3^n}\right)+n\sin^2\alpha. \end{aligned}$$

Since the sequence $\left(\frac{1}{3^n}\right)$ converges, the sequence (y_n) converges if and only if the sequence $(n\sin^2\alpha)$ converges, or $\sin^2\alpha = 0$, that is $\alpha = k\pi$ (k is integer). Then

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{3^n}\right) = \frac{1}{2}.$$

4.2.56

Denote $f(0) = k$. Substituting $x = y = 0$ into the given equation, we get

$$f(k) = k^2. \tag{1}$$

Then substituting $x = y$ and taking into account (1), we have $f(k) = [f(x)]^2 - x^2$, or

$$[f(x)]^2 = x^2 + k^2. \tag{2}$$

This shows that $[f(-x)]^2 = [f(x)]^2$, or

$$[f(x) + f(-x)] \cdot [f(x) - f(-x)] = 0, \ \forall x \in \mathbb{R}. \tag{3}$$

Assume that there is $x_0 \neq 0$ such that $f(x_0) = f(-x_0)$. Then substituting $y = 0$ into the given equation one gets

$$f(f(x)) = kf(x) - f(x) - k, \ \forall x \in \mathbb{R},$$

and substituting $x = 0, y = -x$ one obtains

$$f(f(x)) = kf(-x) + f(-x) - k,$$

which together yields

$$k[f(-x) - f(x)] + f(-x) + f(x) = 2k, \tag{4}$$

from which, by substituting $x = x_0$, we have

$$f(x_0) = k. \tag{5}$$

On the other hand, from (2) it follows that if $f(x) = f(y)$ then $x^2 = y^2$. Then by (5), the equality $f(x_0) = k = f(0)$ shows that $x_0 = 0$, which contradicts to $x_0 \neq 0$.

Thus $f(-x) \neq f(x)$ for all $x \neq 0$. Then (3) gives $k[f(x)-1] = 0, \ \forall x \neq 0$, which implies that $k = 0$ (otherwise $f(x) = 1, \ \forall x \neq 0$ contradicts to $f(-1) \neq f(1)$).

So we arrive to $[f(x)]^2 = x^2$.

Now assume that there is $x_0 \neq 0$ such that $f(x_0) = x_0$. Then $x_0 = f(x_0) = -f(f(x_0)) = -f(x_0) = -x_0$, which implies that $x_0 = 0$. This is impossible, as $x_0 \neq 0$. Hence, $f(x) \neq x, \ \forall x \neq 0$.

Then from $[f(x)]^2 = x^2$ it follows that $f(x) = -x, \ \forall x \in \mathbb{R}$.

Conversely, by direct verification we see that this solution satisfies the requirement of the problem. Thus the answer is $f(x) = -x$.

4.2.57

Substituting $x = \frac{t}{2},\ y = -\frac{t}{2},\ z = 0\ (t \in \mathbb{R})$ into the given equation we have

$$f(t) \cdot f\left(-\frac{t}{2}\right)^2 + 8 = 0$$

which shows that $f(t) < 0$ for all t. Then we can write $f(x) = -2^{g(x)}$, where $g(x)$ is a function we have to find.

Now the given equation becomes

$$g(x-y) + g(y-z) + g(z-x) = 3. \tag{1}$$

Put $u = x - y,\ v = y - z$, then $z - x = -(u + v)$. Also be denoting $h(x) = g(x) - 1$ we get

$$h(u) + h(v) = -h(-u-v). \tag{2}$$

Note that for $u = v = 0$ and $u = x, v = 0$ we have $h(0) = 0$ and $h(-x) = -h(x)$, respectively, and hence (2) can be written as

$$h(u) + h(v) = h(u+v).$$

The last functional equation is the Cauchy equation, which has all solutions $h(t) = Ct$ with $C \in \mathbb{R}$. Then $g(x) = Ct + 1$ and $f(x) = -2^{Cx+1}$.

Conversely, by direct verification we see that the obtained functions satisfy the requirement of the problem. Thus the solutions are

$$f(x) = -2^{Cx+1},$$

where C is an arbitrary real constant.

4.2.58

The domain of definition of the system is $x, y, z < 6$. Then the system is equivalent to

$$\begin{cases} \frac{x}{\sqrt{x^2-2x+6}} = \log_3(6-y), \\ \frac{y}{\sqrt{y^2-2y+6}} = \log_3(6-z), \\ \frac{z}{\sqrt{z^2-2z+6}} = \log_3(6-x). \end{cases}$$

Consider a function

$$f(x) = \frac{x}{\sqrt{x^2-2x+6}},\ x < 6,$$

which has a derivative

$$f'(x) = \frac{6-x}{(x^2-2x+6)\sqrt{x^2-2x+6}} > 0,\ \forall x < 6,$$

and so $f(x)$ is increasing, while a function $g(x) = \log_3(6-x),\ \forall x < 6$ is obviously decreasing.

Let (x, y, z) is a solution of the system. We prove that $x = y = z$.

Without loss of generality, we can assume that $x = \max\{x, y, z\}$. There are two cases:

Case 1: $x \geq y \geq z$. In this case, since $f(x)$ increases,

$$\log_3(6-y) \geq \log_3(6-z) \geq \log_3(6-x),$$

and hence, since $g(x)$ decreases, $x \geq z \geq y$. Then $y \geq z$ and $z \geq y$ give $y = z$, and therefore, $x = y = z$.

Case 2: $x \geq z \geq y$. Similarly, we get $x \geq z$ and $z \geq x$ which give $x = z$ and therefore $x = y = z$.

Thus the system becomes $f(x) = g(x) = 6,\ x < 6$. Note that $f(x)$ increases, and $g(x)$ decreases, then the equation $f(x) = g(x)$ has at most one solution. Since $x = 3$, as can be easily seen, is a solution, the unique solution of the equation, and therefore, of the system, is $(3, 3, 3)$.

4.2.59

We have

$$f(x+y) + b^{x+y} = (f(x) + b^x) \cdot 3^{b^y + f(y) - 1}, \forall x, y \in \mathbb{R}. \qquad (1)$$

Put $g(x) = f(x) + b^x$. Then (1) becomes

$$g(x+y) = g(x) \cdot 3^{g(y)-1}, \forall x, y \in \mathbb{R}. \qquad (2)$$

Substituting $y = 0$ into (2) we get

$$g(x) = g(x) \cdot 3^{g(0)-1},\ \forall x \in \mathbb{R},$$

which gives either $g(x) = 0, \forall x \in \mathbb{R}$, or $g(0) = 1$.

If $g(x) = 0,\ \forall x$, then $f(x) = -b^x$.

If $g(0) = 1$, then substituting $x = 0$ into (2) gives

$$g(y) = g(0) \cdot 3^{g(y)-1} = 3^{g(y)-1},$$

or

$$3^{g(y)-1} - g(y) = 0, \forall y \in \mathbb{R}.$$

Consider a function $h(t) = 3^{t-1} - t$ which has derivative $h'(t) = 3^{t-1}\log 3 - 1$. Note that $h'(t) = 0 \iff t = \log_3(\log_3 e + 1) < 1$. Form this it follows that $h(t) = 0$ has two solutions $t_1 = 1$ and $t_2 = c$ with $0 < c < 1$ (as $h(0) = \frac{1}{3}$). Thus $g(y) = 3^{g(y)-1}$ gives either $g(y) = 1, \forall y \in \mathbb{R}$, or $g(y) = c \in (0,1), \forall y \in \mathbb{R}$.

We show that the second case is impossible. Indeed, if there is $y_0 \in \mathbb{R}$ such that $g(y_0) = c$, then

$$1 = g(0) = g(y_0 - y_0) = g(-y_0) \cdot 3^{g(y_0)-1} = c \cdot g(-y_0),$$

which gives $g(-y_0) = \frac{1}{c} \neq c$: a contradiction.

Thus $g(y) = 1, \forall y \in \mathbb{R}$, and hence $f(x) = 1 - b^x, \forall x \in \mathbb{R}$.

By direct verification shows that these two functions satisfy all requirements of the problem. Thus we have two solutions: $f(x) = -b^x$ and $f(x) = 1 - b^x$.

4.2.60

For each n put $g_n(x) = f_n(x) - a$, it is continuous, increasing on $[0, +\infty)$. Note that $g_n(0) = 1 - a < 0, g_n(1) = a^{10} + n + 1 - a > 0$, and therefore the equation $g_n(x) = 0$ has a unique solution $x_n \in (0, +\infty)$.

Furthermore,

$$g_n\left(1 - \frac{1}{a}\right) = a^{10}\left(1 - \frac{1}{a}\right)^{n+10} + \frac{1 - \left(1 - \frac{1}{a}\right)^{n+1}}{\frac{1}{a}} - a$$

$$= a\left(1 - \frac{1}{a}\right)^{n+1}\left(a^9\left(1 - \frac{1}{a}\right)^9 - 1\right) = a\left(1 - \frac{1}{a}\right)^{n+1}\left[(a-1)^9 - 1\right] > 0,$$

which gives

$$x_n < 1 - \frac{1}{a}, \ \forall n \geq 1.$$

Also we have

$$g_n(x_n) = a^{10}x_n^{n+10} + x_n^n + \cdots + 1 - a = 0,$$

which gives

$$x_n g_n(x_n) = a^{10}x_n^{n+11} + x_n^{n+1} + \cdots + x_n - x_n a = 0,$$

or

$$g_{n+1}(x_n) = x_n g_n(x_n) + 1 + a x_n - a = a x_n + 1 - a < 0$$

as $x_n < 1 - \frac{1}{a}$.

Since $g_{n+1}(x)$ increases, and $0 = g_{n+1}(x_{n+1}) > g_{n+1}(x_n)$, it follows that $x_n < x_{n+1}$. Thus (x_n) is increasing and bounded from above, hence it converges.

Remark. We can prove that the limit is $1 - \dfrac{1}{a}$.

4.2.61

Note that if (x, y) is a solution, then $x, y > 1$. Put $t = \log_3 x > 0$, that is $x = 3^t$, the second equation becomes $y = 2^{\frac{1}{t}}$. Then the first equation has the form

$$9^t + 8^{\frac{1}{t}} = a. \tag{1}$$

The number of solutions of the given system is the same as the number of solutions of equation (1).

Consider a function $f(t) = 9^t + 8^{\frac{1}{t}} - a$ on $(0, +\infty)$. We have

$$f'(t) = 9^t \cdot \log 9 - \frac{8^{\frac{1}{t}} \cdot \log 8}{t^2}.$$

Note that on the interval $(0, +\infty)$ both functions $8^{\frac{1}{t}} \cdot \log 8$ and $\dfrac{1}{t^2}$ are decreasing and positive. So

$$-\frac{8^{\frac{1}{t}} \cdot \log 8}{t^2}$$

is increasing and hence $f'(t)$ is increasing too. Also since

$$f'(\frac{1}{2}) \cdot f'(1) = 18(\log 9 - \log 2^{256})(\log 27 - \log 16) < 0,$$

there exists $t_0 \in \left(\frac{1}{2}, 1\right)$ such that $f'(t_0) = 0$.

From all said above it follows that $f(t)$ is decreasing on $(0, t_0)$, increasing on $(t_0, +\infty)$, $\lim\limits_{t \to 0} f(t) = \lim\limits_{t \to +\infty} f(t) = +\infty$. Moreover, $f(1) = 17 - a \leq 0$. So equation (1) has exactly 2 positive solutions.

4.2.62

Consider a function

$$f(x) = 2^{-x} + \frac{1}{2},\ x \in \mathbb{R}.$$

For each n we have

$$x_{n+4} = f(x_{n+2}) = f(f(x_n)),$$

or $x_{n+4} = g(x_n)$, where $g(x) = f(f(x))$ on $\mathbb{R}$.

Note that $f(x)$ is decreasing, and hence $g(x)$ is increasing. So for each $k = 1, 2, 3, 4$ a sequence (x_{4n+k}) is monotone. Moreover, from definition of (x_n) it follows that $0 \le x_n \le 2$ for all n. Thus, for each $k = 1, 2, 3, 4$ the sequence (x_{4n+k}) converges.

Put $\lim_{n\to\infty} x_{4n+k} = L_k,\ k = 1, 2, 3, 4$, then $0 \le L_k \le 2$. As $g(x)$ is continuous on $\mathbb{R}$, we have $g(L_k) = L_k$.

Consider s function $h(x) = g(x) - x$ on $[0, 2]$. We see that

$$h'(x) = 2^{-(f(x)+x)} \cdot (\log 2)^2 - 1 < 0$$

on $[0, 2]$ (as $f(x) + x > 0$ on this interval). Thus $h(x)$ is decreasing on $[0, 2]$, and hence the equation $h(x) = 0$ has at most one solution, that is $g(x) = x$. Note that $g(1) = 1$, and therefore we obtain $L_k = 1$, for each $k = 1, 2, 3, 4$.

Finally, since (x_n) is the union of four subsequences (x_{4n+k}), it converges and the limit is 1.

4.3 Number Theory

4.3.1

We have to find a number

$$\overline{xxyy} = 10^3x + 10^2x + 10y + y = 11(100x + y). \tag{1}$$

This number is divisible by 11, and by 11^2 (as it is a square). So $100x + y$ is divisible by 11, which mean that $x + y$ is divisible by 11. Here $0 < x \le 9, 0 \le y \le 9$.

As the number is a square, the last digit y can be in within the set $\{0, 1, 4, 5, 6, 9\}$. Then $x = 11 - y$ can be in within the set $\{11, 10, 7, 6, 5, 2\} \setminus \{11, 10\}$.

Thus we have the following pairs $(x, y) = (7, 4), (6, 5), (5, 6), (2, 9)$ which mean that the number is in within $7744, 6655, 5566, 2299$. Among these numbers there is only one which is a square, 7744. This is the answer.

4.3.2

Factorizing into prime numbers, we have $1890 = 2 \cdot 3^3 \cdot 5 \cdot 7, 1930 = 2 \cdot 5 \cdot 193, 1970 = 2 \cdot 5 \cdot 197$. Then

$$N = 1890 \cdot 1930 \cdot 1970 = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 193 \cdot 197,$$

which shows that each of its divisors has a form $2^{k_1} \cdot 3^{k_2} \cdot 5^{k_3} \cdot 7^{k_4} \cdot 193^{k_5} \cdot 197^{k_6}$, where $k_1, k_2, k_3 \in \{0, 1, 2, 3\}$ and $k_4, k_5, k_6 \in \{0, 1\}$.

The unknown number is not divisible by $45 = 3^2 \cdot 5$, which means that it cannot be both true that $k_2 \geq 2$ and $k_3 \geq 1$, or equivalently, either $k_2 \leq 1$ or $k_3 = 0$. From this it can be inferred that the set of divisors of N is such that k_1, k_4, k_5, k_6 can have any value said above, and either $k_2 \in \{0, 1\}$, $k_3 \in \{0, 1, 2, 3\}$, or $k_3 = 0$, $k_2 \in \{0, 1, 2, 3\}$.

So k_1 can be four possible values, a pair (k_2, k_3) can be ten possible values, and k_4, k_5, k_6 can be two possible values. In total there are $4 \cdot 10 \cdot 2 \cdot 2 \cdot 2 = 320$ divisors of N that are not divisible by 45.

4.3.3

1) We have $\tan(\alpha + \beta) = \tan 45^\circ = 1$, or $\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = 1$. From this it follows that

$$\tan\beta = \frac{1 - \tan\alpha}{1 + \tan\alpha}, \text{ or } \frac{p}{q} = \frac{1 - \frac{m}{n}}{1 + \frac{m}{n}} = \frac{n - m}{n + m}.$$

Consider two cases:

a) Two numbers m, n are of different parity. Then $n - m$ and $n + m$ are both odd, and so they only have odd common divisors. These divisors then divide $(n - m) + (n + m) = 2n$ and $(n + m) - (n - m) = 2m$, which shows that these divisors divide both m and n. This is impossible, as $(m, n) = 1$, or the fraction $\frac{n-m}{n+m}$ is irreducible. So we must have $p = n - m, q = n + m$. In order to have $p > 0$ we must have $n > m$.

b) Both m, n are odd. Then $n - m$ and $n + m$ are even, and so $(n - m, n + m) = (2n, 2m) = 2$, as $(n, m) = 1$. From this we infer that two numbers $\frac{n-m}{2}$ and $\frac{n+m}{2}$ are co-prime, and hence $p = \frac{n-m}{2}, q = \frac{n+m}{2}$ with $n > m$.

Thus, if $n > m$ then there is always a unique solution, namely

$$\text{if } n - m \text{ is odd, then } p = n - m, q = n + m, \tag{1}$$

while

$$\text{if } n - m \text{ is even, then } p = \tfrac{n-m}{2}, q = \tfrac{n+m}{2}. \tag{2}$$

2) We follow the two cases said above.

a) We find a solution for the case $n-m$ odd, that is when (1) is satisfied. Then

$$m = q - n, \;\; p = 2n - q. \tag{3}$$

As $m, \; p > 0$, we must have $n < q < 2n$. Furthermore, since $p = 2n - q$ and q are co-prime, q must be odd and $(q, n) = 1$. In this case, by (1), $(m, n) = 1$. So in order to have (3) we must have $n < q < 2n$, q is odd, and $(q, n) = 1$.

b) We find a solution for the case $n-m$ even, that is when (2) is satisfied. Then

$$m = 2q - n, \;\; p = n - q. \tag{4}$$

As $m, p > 0$, we must have $q < n < 2q$. Furthermore, since $m = 2q - n$ and n are co-prime, n must be odd and $(q, n) = 1$. In this case, by (2), $(p, q) = 1$. So in order to have (4) we must have $q < n < 2q$, n odd, and $(q, n) = 1$.

Thus if $(n, q) = 1$, the greater number is odd and less than twice the other number, then there is a unique solution, namely

$$\text{if } n < q, \text{ then } m = q - n, \;\; p = 2n - q,$$

while

$$\text{if } n > q, \text{ then } m = 2q - n, \;\; p = n - q.$$

3) Similarly, in case a) we have $n = q - m > 0, \; p = q - 2m > 0$ which gives $q > 2m$. Then there is one solution if q is odd and $(q, m) = 1$.

In case b) we have $n = 2q - m > 0, \; p = q - m > 0$ that gives $q > m$, and hence there is one solution if m is odd and $(m, q) = 1$.

Thus,

- If $m < q < 2m$, m is odd, then there is one solution $n = 2q - m, \; p = q - m$.
- If $2m < q$, m is even, q is odd, then there is one solution $n = q - m, \; p = q - 2m$.
- If $2m < q$, m is odd, q is even, then there is one solution $n = 2q - m, \; p = q - m$.
- If $2m < q$, m is odd, q is odd, then there are two solutions $n = q - m, \; p = q - 2m$ and $n = 2q - m, \; p = q - m$.

4.3.4

1) We have $f(2) = (-1)^0 = 1$, and $f(2) = f(2^r) = 1 \cdot f(p) = 1 + (-1)^{\frac{p-1}{2}}$, as 2 and 2^r have 1 as the only odd divisor.

2) If a prime number p has a form $p = 4k+1$, then it has only two divisors 1 and $4k+1$. Thus $f(p) = 1 + (-1)^{2k} = 2$. On the other hand, if a prime number has a form $p = 4k-1$, then $f(p) = 1 + (-1)^{2k-1} = 0$.

Furthermore, as p^r has $r+1$ odd divisors $1, p, p^2, \ldots, p^r$, we have

$$f(p^r) = 1 + (-1)^{\frac{p-1}{2}} + (-1)^{\frac{p^2-1}{2}} + \cdots + (-1)^{\frac{p^r-1}{2}}.$$

So if $p = 4k+1$, then $f(p^r) = r+1$; and if $p = 4k-1$, then $f(p^r) = \begin{cases} 1, & \text{if } r \text{ is even,} \\ 0, & \text{if } r \text{ is odd.} \end{cases}$

3) A product $f(N) \cdot f(M)$ consists of such numbers that have the form

$$(-1)^{\frac{n-1}{2}} \cdot (-1)^{\frac{m-1}{2}} = (-1)^{\frac{n+m-2}{2}},$$

where n and m are odd divisors of N and M respectively.

On the other hand, $f(N \cdot M)$ consists of numbers of the form $(-1)^{\frac{mn-1}{2}}$. Note that

$$(-1)^{\frac{mn-1}{2}} : (-1)^{\frac{n+m-2}{2}} = (-1)^{\frac{(n-1)(m-1)}{2}} = 1,$$

(as m, n are odd, which imply that $\frac{(n-1)(m-1)}{2}$ is even). This means that the summands of two sums are the same and hence

$$f(N \cdot M) = f(N) \cdot f(M).$$

From this it follows that

$$f(5^4 \cdot 11^{28} \cdot 19^{19}) = f(5^4) \cdot f(11^{28}) \cdot f(17^{19}) = (1+4) \cdot 1 \cdot (1+19) = 5 \cdot 20 = 100,$$

and

$$f(1980) = f(2^2 \cdot 3^2 \cdot 5 \cdot 11) = f(2^2) \cdot f(3^2) \cdot f(5) \cdot f(11) = 1 \cdot 1 \cdot 1 \cdot 0 = 0,$$

as 11 is a prime number of the form $4k-1$.

Finally, we can see the following rule for computing $f(N)$: factorizing N into prime numbers that consist of three types, $2, p_i = 4k+1, q_j = 4k-1$, we can write N as

$$N = 2^r p_1^{\alpha_1} \cdots p_r^{\alpha_r} q_1^{\beta_1} \cdots q_s^{\beta_s},$$

which gives that

$$f(N) = \begin{cases} (1+\alpha_1) \cdots (1+\alpha_r), & \text{if all } \beta_j \text{ are even,} \\ 0, & \text{if there is some } \beta_j \text{ odd.} \end{cases}$$

4.3.5

1) We have

$$A = \underbrace{11\ldots1}_{2n\ \text{times}} - \underbrace{77\ldots7}_{n\ \text{times}} = \frac{10^{2n}-1}{10-1} - 7\cdot\frac{10^n-1}{10-1} = \frac{10^{2n}-7\cdot10^n+6}{9}.$$

* For $n=1$: $A=4=2^2$.

* For $n\geq 2$: $(10^n-4)^2 < 10^{2n}-7\cdot10^n+6 < (10^n-3)^2$ which shows that the numerator of A cannot be a square, and hence neither can A.

2) Similarly,

$$B = \frac{10^{2n}-b\cdot10^n+(b-1)}{9}.$$

Denote by C the numerator of B. Since $C=9B$, it follows that if B is a square, then so is C. As $n>0$, $(b-1)$ should be the last digit of C, and so $b\in\{1,2,5,6,7\}$.

The case $b=7$ was already considered. By substituting $b=1,2,5,6$ into the expression of B we see that the only possible case is $b=2$, for which $C=(10^n-1)^2$. In this case $B=\left(\frac{10^n-1}{3}\right)^2$ is a square, as $10^n-1=\underbrace{99\ldots9}_{n\ \text{times}}$ is divisible by 3.

4.3.6

1) From $n=9u, n+1=25v$ it follows that $25v=9u+1$ which gives $u=11+25t, v=4+9t$, where t is any integer. So there are infinitely many pairs $(99+225t, 100+225t)$, $t\in\mathbb{Z}$, satisfying the requirements in the problem.

2) Since both 21 and 165 are divisible by 3 and $(n,n+1)=1$, there is no solution.

3) We have $n=9u, n+1=25v, n+2=4w$. From the first two equations, by 1) we get the form of n. So in order to satisfy the third equation we must have

$$(99+225t)+2=4w \implies 4w=225t+101,$$

which gives $t=3+4s, w=194+225s$, where s is any integer. Thus there are infinitely many triples $(774+900s, 775+900s, 776+900s)$ with $s\in\mathbb{Z}$, satisfying the requirements of the problem.

4.3.7

We have the first term $a_1 = -1$ and the difference $d = 19$, which gives the general form $a_n = a_1 + (n-1)d = 19n - 20,\ n \geq 1$. We need to find all n for which

$$19n - 20 = \underbrace{55\ldots5}_{k \text{ times}} = 5 \cdot \frac{10^k - 1}{9}, \text{ for some } k \geq 1.$$

This is equivalent to $5 \cdot 10^k \equiv -4 \pmod{19}$, or $5 \cdot 10^k \equiv 15 \pmod{19}$, which reduces to $10^k \equiv 3 \pmod{19}$.

Now consider a sequence of congruences for $10^k \pmod{19}$. We have consecutively

$$10^0 \equiv 1, 10^1 \equiv 10, 10^2 \equiv 5, 10^3 \equiv 12, 10^4 \equiv 6, 10^5 \equiv 3.$$

Furthermore,

$$10^6 \equiv 11, 10^7 \equiv 15, \ldots, 10^{18} \equiv 1.$$

So we obtain $10^{18\ell+5} \equiv 3 \pmod{19}$ which implies that $k = 18\ell + 5,\ \ell \geq 0$.

Conversely, if $k = 18\ell + 5,\ \ell \geq 0$, then $10^k \equiv 3 \pmod{19}$. This implies $5 \cdot 10^k \equiv -4 \pmod{19}$; that is, $5 \cdot 10^k = 19s - 4$ for some s, which is equivalent to $5(10^k - 1) = 19s - 9$. The left-hand side is divisible by 9 and hence s is also divisible by 9; that is, $s = 9r$. Then we have

$$19r - 1 = 5 \cdot \frac{10^k - 1}{9} = \underbrace{55\ldots5}_{k \text{ times}}.$$

Thus the answer is all terms of the form $\underbrace{55\ldots5}_{18\ell+5 \text{ times}},\ \ell \geq 0$.

4.3.8

Note that n must be divisible by 3, so the smallest and largest are 102 and 999. Note that

$$102 \leq n \leq 999 \Longleftrightarrow 68 \leq \frac{2}{3}\overline{abc} \leq 666 \Longleftrightarrow 68 \leq a!b!c! \leq 666. \qquad (1)$$

From $a!b!c! \leq 666$ it follows that no digit is greater than 5, as $6! = 720 > 666$. Also, since $a!b!c! \geq 68$, we see that either each digit is ≥ 2, or one among them is ≥ 4 (if two digits are 1 and 2, or both are different from 1), or two digits are ≥ 3 (then the third digit must be 2). All cases give that $a!b!c!$ is divisible by 8, that is, n is divisible by 4.

Thus $n = \overline{abc}$ is divisible by 3, by 4, and its digits must satisfy the conditions $0 < a,\ b,\ c \leq 5$. As n is even, so is c, that is $c \in \{2, 4\}$.

• If $c = 2$: as $\overline{abc}$ is divisible by 4, so is $\overline{bc}$. Then $b \in \{1, 3, 5\}$, and $\overline{bc} \in \{12, 32, 52\}$.

• If $c = 4$: similarly, $b \in \{2, 4\}$, and $\overline{bc} \in \{24, 44\}$.

However, n is divisible by 3, then $a + b + c$ is divisible by 3. Combining this fact and the fact that $a!b!c!$ is divisible by 8 yields the following numbers:

$$312\ (a = 3), 432\ (a = 4), 252\ (a = 2), 324\ (a = 3), 144\ (a = 1).$$

Among these candidates the only number that satisfies the requirement of the problem is 432.

4.3.9

Consider $P(x) = ax^3 + bx^2 + cx + d$. For $x = 0, 1, -1, 2$ we have $d, a + b + c + d, -a + b - c + d, 8a + 4b + 2c + d$ are integers. So we get the following integers

$$\begin{aligned}
&a + b + c = (a + b + c + d) - d,\\
&2b = (a + b + c + d) + (-a + b - c + d) - 2d,\\
&6a = (8a + 4b + 2c + d) + 2(-a + b - c + d) - 6b - 3d.
\end{aligned}$$

Thus $P(x)$ is integer implies that $6a, 2b, a + b + c, d$ are integers.

Conversely, we write

$$P(x) = 6a\frac{(x-1)x(x+1)}{6} + 2b\frac{x(x-1)}{2} + (a + b + c)x + d$$

and note that for any integer x, we always have that $(x - 1)x(x + 1)$ is divisible by 6, and since $x(x - 1)$ is divisible by 2, $P(x)$ is an integer.

Thus the answer is that $6a, 2b, a + b + c, d$ are integers.

4.3.10

We have

$$2(100a+10b+c) = (100b+10c+a)+(100c+10a+b) \Longleftrightarrow 7a = 3b+4c. \quad (1)$$

There are the obvious nine solutions $a = b = c \in \{1, 2, \ldots, 9\}$. If any two of a, b, c are equal, then (1) shows that they are all equal.

Now consider the case where all are distinct. Writing equation (1) as

$$\frac{a-b}{c-a} = \frac{4}{3},$$

we get $a - b = 4k, c - a = 3k$ $k \in \mathbb{Z}$, and hence $c - b = 7k$. Note that $c - b < 10$, so that $k = \pm 1$.

• For $k = 1$: $a - b = 4, c - a = 3 \implies b = a - 4 \geq 0, a = c - 3 \leq 6 \implies 4 \leq a \leq 6$. Thus $a = 4, 5, 6$ which gives respectively $b = 0, 1, 2$ and $c = 7, 8, 9$. So we have $407, 518, 629$.

• Similarly, for $k = -1$ we obtain $370, 481, 592$.

Thus there are 15 solutions.

4.3.11

By long division we have

$$f(x) = g(x) \cdot h(x) + r(x),$$

where the remainder $r(x) = (m^3 + 6m^2 - 32m + 15)x^2 + (5m^3 - 24m^2 + 16m + 33)x + (m^4 - 6m^3 + 4m^2 + 5m + 30) = Ax^2 + Bx + C$. In order to have $r(x) = 0$ for all x we should have $A = B = C = 0$, which gives $m = \pm 1, \pm 3$. Among them only $m = 3$ is suitable.

4.3.12

We have

$$2^x(1 + 2^{y-x} + 2^{z-x}) = 2^5 \cdot 73.$$

Put $M = 1+2^{y-x}+2^{z-x}$, we see that M is odd, and hence $2^x = 2^5, M = 73$. So $x = 5$ and $2^{y-x}+2^{z-x} = 72 \iff 2^{y-x}(1+2^{z-y}) = 2^3 \cdot 9$. By the same argument, we have $2^{y-x} = 2^3, 1 + 2^{z-y} = 9$, which give $y - x = z - y = 3$, and so $y = 8, z = 11$.

Thus the answer is $x = 5, y = 8, z = 11$.

4.3.13

Note that if $b > 2$ then $2^b - 2^{b-1} = 2^{b-1} > 2$; that is $2^{b-1} + 1 < 2^b - 1$. So if $a < b$, that is $a \leq b - 1$, then $2^a \leq 2^{b-1}$, which implies that

$$2^a + 1 < 2^b - 1. \tag{1}$$

In this case $2^a + 1$ cannot be divisible by $2^b - 1$.

Now suppose that $a = b$. Then we have

$$\frac{2^a + 1}{2^b - 1} = 1 + \frac{2}{2^b - 1},$$

which also shows that $2^a + 1$ cannot be divisible by $2^b - 1$.

Finally, in the case $a > b$ we write $a = bq + r$, where q is a positive number, r is either 0, or a positive number less than b. Then

$$\frac{2^a + 1}{2^b - 1} = \frac{2^a - 2^r}{2^b - 1} + \frac{2^r + 1}{2^b - 1} = \frac{2^a - 2^{a-qb}}{2^b - 1} + \frac{2^r + 1}{2^b - 1}.$$

For the first fraction, since $2^a - 2^{a-qb} = 2^{a-qb}(2^{qb} - 1)$, it is divisible by $2^{qb} - 1 = (2^b)^q - 1$, and also by $2^b - 1$. For the second fraction, by (1), it is always less than 1.

Summarizing all the arguments above, we conclude that such a and b do not exist.

4.3.14

1) We can see that 1 cannot be represented in such a form. Indeed, for the first six largest fractions

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} < 0.34 + 0.2 + 0.15 + 0.12 + 0.1 + 0.08 = 0.99 < 1.$$

2) For this case it is possible:

$$1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{15} + \frac{1}{35} + \frac{1}{45} + \frac{1}{231}.$$

Generalization. For any odd number $k \geq 9$ we can have

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}.$$

Indeed, note that

$$\frac{1}{3} = \frac{1}{5} + \frac{1}{9} + \frac{1}{45}.$$

Then from 2) we replace

$$\frac{1}{231} = \frac{1}{3 \cdot 77} = \frac{1}{5 \cdot 77} + \frac{1}{9 \cdot 77} + \frac{1}{45 \cdot 77}$$

to get $k = 11$. Then replace the smallest fraction $\frac{1}{3m}$ (here $m = 15 \cdot 77$) by the sum

$$\frac{1}{3m} = \frac{1}{5m} + \frac{1}{9m} + \frac{1}{45m},$$

to get $k = 13$, and so on.

4.3.15

Put $\alpha = \min |f(x, y)|$, where $f(x, y) = 5x^2 + 11xy - 5y^2$. Since the equation $f(x, y) = 0$ has no real root, α is a positive integer. On the other hand, $\alpha \leq |f(1, 0)| = 5$, and therefore $\alpha = 1, 2, 3, 4, 5$.

Note that if $x = 2k$ and $y = 2m$ then $f(2k, 2m) = 4f(k, m)$. In this case pairs $(2k, 2m)$ cannot make the given quantity minimal. So it suffices to consider the case when x, y are not simultaneously even. If so $f(x, y)$ is odd integer and $\alpha = 1, 3, 5$.

We prove that $\alpha \neq 1, \neq 3$.

Suppose $\alpha = 1$, that is there is a pair (x_0, y_0) such that $|f(x_0, y_0)| = 1$. Consider case $f(x_0, y_0) = 1$ (the case $f(x_0, y_0) = -1$ is similar). We then have

$$(10x_0 + 11y_0)^2 - 221y_0^2 = 20.$$

Let $t = 10x_0 + 11y_0$, then the last equation is rewritten as $(t^2 - 7) = 13 + 13 \cdot 17y_0^2$. This is impossible, as $t^2 - 7$ is never divisible by 13.

Similarly, the case $\alpha = 3$ cannot occur, as $t^2 - 8 = 52 + 13 \cdot 17y_0^2$ and $t^2 - 8$ can never be divisible by 13.

So $\alpha = 5$ is the desired value, as $f(1, 0) = 5$.

4.3.16

If $x = 0$ then $y = -2$; if $y = 0$ then $x = 2$.

Consider $x, y \neq 0$. There are two cases.

1) **Case 1**: $xy < 0$.

a. If $x > 0, y < 0$ then $x^3 = y^3 + 2xy + 8 < 8$, which implies that $x = 1$ and the equation becomes $y^3 + 2y + 7 = 0$. There is no integer solution.

b. If $x < 0, y > 0$ then $y^3 - x^3 = -2xy - 8 < -2xy$. Also $y^3 - x^3 = y^3 + (-x)^3 \geq y^2 + (-x)^2 \geq -2xy$. This is impossible.

2) **Case 2**: $xy > 0$.

We note that $2xy + 8 > 0$, and so $x^3 - y^3 = (x - y)[(x - y)^2 + 3xy] > 0$, which implies that $x - y > 0$.

a. If $x - y = 1$, then we have $y^2 + y - 7 = 0$. There is no integer solution.

b. If $x - y \geq 2$, then we get $2xy + 8 = (x-y)[(x-y)^2 + 3xy] \geq 2(4 + 3xy) = 8 + 6xy$, which implies that $xy \leq 0$, a contradiction.

Thus there are two pairs of solutions $(0, -2)$ and $(2, 0)$.

4.3.17

Put $(b, m) = d$ we have $b = b_1 d, m = m_1 d$ with $(b_1, m_1) = 1$. So $(a^n - 1)b$ is divisible by m if and only if $a^n - 1$ is divisible by m_1. We study this condition.

First we note that the necessity is $(a, m_1) = 1$, say if a and m_1 have a common prime divisor p then $a^n - 1$ is not divisible by p, while m_1 is divisible by this number.

We prove that this condition is also sufficient. Indeed, consider a sequence $a, a^2, \ldots, a^{m_1+1}, a^{m_1+2}$ consisting of $m_1 + 2$ terms. By the Pigeonhole principle, there exit two terms a^k, a^ℓ such that $a^k \equiv a^\ell \pmod{m_1}$, or $a^k - a^\ell = a^\ell(a^{k-\ell} - 1)$ is divisible by m_1. This shows, due to $(a, m_1) = 1$, that $a^{k-\ell} - 1$ is divisible by m_1. So taking $n = k - \ell$ we get that $a^n - 1$ is divisible by m_1.

Thus $(a^n - 1)b$ is divisible by m if and only if $(a, m_1) = 1$. But $(b_1, m_1) = 1$, and so

$$(a, m_1) = 1 \Longleftrightarrow (ab_1, m_1) = 1 \Longleftrightarrow (ab_1 d, m_1 d) = d \Longleftrightarrow (ab, m) = d.$$

This means that $(ab, m) = (b, m) = d$.

4.3.18

First note that (x_n) and (y_n) are sequences of positive integers.

We have

$$y_1 - y_0 = y_0^4 - 1952 = 63584 = 32 \cdot 1987,$$

and so $y_1 \equiv y_0 \pmod{1987}$.

Next consider

$$y_2 - y_1 = y_1^4 - 1952 \equiv y_0^4 - 1952 \pmod{1987} \equiv 0 \pmod{1987}.$$

Thus $y_2 \equiv y_1 \pmod{1987}$, which implies that $y_2 \equiv y_0 \pmod{1987}$.

Similarly, we obtain

$$y_k \equiv y_0 \pmod{1987} \equiv 16 \pmod{1987}, \ \forall k \geq 1.$$

On the other hand,

$$x_1 - x_0 = x_0^{1987} + 1622 = (365^{1987} - 365) + 1987.$$

But $365^{1987} - 365 \equiv 0 \pmod{1987}$, by Fermat's theorem, and so $x_1 \equiv x_0 \pmod{1987}$.

Also we have

$$x_2 - x_1 = x_1^{1987} + 1622 = x_0^{1987} + 1622 \equiv 0 \pmod{1987},$$

that is $x_2 \equiv x_1 \pmod{1987}$, which implies that $x_2 \equiv x_0 \pmod{1987}$.

Similarly, we obtain

$$x_n \equiv x_0 \pmod{1987} \equiv 365 \pmod{1987},\ \forall n \geq 1.$$

Thus, for all $k, n \geq 1$ we always have $|y_k - x_n| > 0$, as 365 and 16 are not congruent by $\pmod{1987}$.

4.3.19

1) Put $g(n) = 4n^2 + 33n + 29$. Then $g(n) = 1989(n^2 + n + 1) - f(n)$, and hence $f(n)$ is divisible by 1989 if and only if $g(n)$ is so. Consider the following sequence $-1, 1, 0, 1, 1, 2, \ldots$, denoted by (F_n), $n \geq 0$ (that is, added three numbers $-1, 1, 0$ to the given sequence). For the new sequence we also have $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$.

Let r_i be the remainder of division of F_i by 1989, so $0 \leq r_i \leq 1988$. By the Pigeonhole principle, among the first $1989^2 + 1$ pairs $(r_0, r_1), (r_1, r_2), \ldots$ there are at least two pairs that coincide, say $(r_p, r_{p+1}) = (r_{p+\ell}, r_{p+\ell+1})$, that is $r_p = r_{p+\ell}, r_{p+1} = r_{p+\ell+1}$. Note that $F_{n-1} = F_{n+1} - F_n$, we get

$$r_{p-1} = r_{p+\ell-1}, r_{p-2} = r_{p+\ell-2}, \ldots, r_2 = r_{\ell+2}, r_1 = r_{\ell+1}, r_0 = r_\ell,$$

from which it follows that $r_i = r_{i+\ell}$ for all $i \geq 0$.

Thus $r_0 = r_\ell = r_{2\ell} = \cdots = r_{k\ell}$ for all $k \geq 1$. Therefore, we have

$$F_{k\ell} = 1989t + r_0 = 1989t - 1,\ t \in \mathbb{Z},$$

which gives

$$g(F_{k\ell}) = g\left(1989t - 1\right) = 4(1989t-1)^2 + 33(1989t-1) + 29 = 1989A,\ A \in \mathbb{Z}.$$

On the other hand, $F_{k\ell}$, $k \geq 1$ all are Fibonacci numbers, so there are infinitely many such numbers F that $f(F)$ is divisible by 1989.

2) We prove that there is no Fibonacci number G for which $f(G)+2$ is divisible by 1989.

Note that

$$f(n)+2 = 1989(n^2+n+1) - 26(n+1) - (4n^2+7n+1),$$

and both 1989 and 26 are divisible by 13. So we suffice to show that for all $n \in \mathbb{N}$ a number $4n^2+7n+1$ is not divisible by 13.

Indeed, $16(4n^2+7n+1) = (8n+7)^2 - 7 - 13 \cdot 2$. Put $8n+7 = 13t \pm r$ $(0 \leq r \leq 6)$, t, r are integers. We have $(8n+7)^2 = (13t \pm r)^2 = (13t)^2 \pm 2 \cdot 13tr + r^2 = 13(13t^2 \pm 2tr) + r^2$, and so $16(4n^2+7n+1) = r^2 - 7 + 13m$ for some integer m. Direct verification shows that $r^2 - 7$ is not divisible by 13 for any $r \in \{0, 1, \ldots, 6\}$.

4.3.20

Note the following equalities

$$100 = 9^2 + 19 \cdot 1^2, \tag{1}$$

$$1980 = 21^2 + 19 \cdot 9^2, \tag{2}$$

and

$$(x^2+19y^2)(a^2+19b^2) = (xa - 19yb)^2 + 19(xb+ya)^2, \tag{3}$$

for all real numbers x, y, a, b.

We prove by induction that for each $m \in \mathbb{N}$ the number 100^m has the property that there exist two integers x, y such that $x-y$ is not divisible by 5 and $100^m = x^2 + 19y^2$.

Indeed, for $m = 1$ it is true by (1). Suppose that it is true for $m = k \geq 1$, that is

$$100^k = x^2 + 19y^2,$$

for some integers x, y such that $x-y$ is not divisible by 5. Then for $m = k+1$ we have

$$10^{k+1} = 100 \cdot 100^k = (9^2 + 19 \cdot 1^2)(x^2+19y^2) = (9x-19y)^2 + 19(x+9y)^2,$$

by (3). Also since $x-y$ is not divisible by 5, neither is $(9x-19y)-(x+9y) = 8(x-y) - 20y$. The claim is proved.

Now $10^{1988} = 100^{994}$ has such a property. Suppose that $100^{994} = A^2 + 19B^2$, where $A - B$ is not divisible by 5. From (2) and (3) it follows that

$$198 \cdot 10^{1989} = 1980 \cdot 100^{994} = (21^2 + 19 \cdot 9^2)(A^2 + 19B^2)$$

$$= (21A - 171B)^2 + 19(21B + 9A)^2 = x^2 + 19y^2, \qquad (4)$$

where $x = 21A - 171B, y = 21B + 9A$. Moreover, $x - y = (21A - 171B) - (21B + 9A) = 12(A - B) - 180B$ is not divisible by 5.

Also note, by (4), that if either of x, y is divisible by 5 then so is the other. Since $x - y$ is not divisible by 5, neither are x and y.

4.3.21

Let S be the sum of the removed numbers. The problem is equivalent to finding the minimum value of S.

Let $a_1 < \cdots < a_p \in [1, 2n - 1]$ be the removed numbers. Note that $p \geq n - 1$, by the assumption.

• If $a_1 = 1$ then 2 is removed, then $1 + 2 = 3, 1 + 3 = 4, \ldots$ are all must be removed. In this case S is maximum, or the sum of the remaining numbers is 0.

• If $a_1 > 1$, then $a_1 + a_p \geq 2n$, otherwise $a_1 + a_p \leq 2n - 1$ should be removed, while $a_p < a_1 + a_p$. This contradicts the fact that a_p is the greatest removed number.

Next we also have $a_{p-1} + a_2 \geq 2n$. Indeed, note that $a_{p-1} + a_1 \geq a_p$, otherwise this number $a_{p-1} + a_1$ should be removed, while it is in between two consecutive removed numbers a_{p-1} and a_p, which is impossible. So $a_{p-1} + a_1 \geq a_p$ and therefore, $a_{p-1} + a_2 > a_{p-1} + a_1 \geq a_p$. This implies that $a_{p-1} + a_2 \geq 2n$ (otherwise, this number should be removed, but it is greater than a_p, again contradiction).

Continuing this argument, we have

$$a_{p+1-i} + a_i \geq 2n, \text{ for all } 1 \leq i \leq \frac{p+1}{2}. \qquad (1)$$

From this it follows that

$$2S = (a_1 + a_p) + (a_2 + a_{p-1}) + \cdots + (a_p + a_1) \geq 2n.p \geq 2n(n - 1),$$

or

$$S = \sum_{i=1}^{p} a_i \geq n(n - 1).$$

The equality occurs if and only if (1) is satisfied, that is $a_{p+1-i} + a_i = 2n$ for all $1 \leq i \leq \frac{p+1}{2}$. In this case $a_p = a_1 + a_{p-1} = 2a_1 + a_{p-2} = \cdots = pa_1$, which implies that $2n = a_1 + a_p = (p + 1)a_1 \geq na_1$ (as $p \geq n - 1$ already noted above), or $a_1 \leq 2$.

Combining $a_1 > 1$ and $a_1 \leq 2$ yields $a_1 = 2$ as well as $p = n - 1$, and so $a_i = 2i$ $(1 \leq i \leq n - 1)$.

Summarizing: the desired removed numbers must be $2, 4, \ldots, 2n-2$, and the maximum value of the sum of remaining numbers is $1+3+\cdots+(2n-1) = n^2$.

4.3.22

Note that each positive number A can be written in the form $A = 2^r B$ with B odd. By the requirement of the problem, we have to find a formula for $f(n)$ in the representation $k^n - 1 = 2^{f(n)} B$, where B is odd.

Write $n = 2^t m$ with m odd, also $k - 1 = 2^r u,\ k + 1 = 2^s v$ with u, v odd, and $r, s \geq 1$ (as $k > 1$). Then

$$k^n - 1 = (2^r u + 1)^n - 1 = \sum_{i=0}^{n} \binom{n}{i} (2^r u)^i - 1 = \sum_{i=1}^{n} \binom{n}{i} (2^r u)^i$$

$$= \binom{n}{1} 2^r u + \sum_{i=2}^{n} \binom{n}{i} (2^r u)^i = 2^r nu + 2^{2r} M. \tag{1}$$

1) If $n = m$ odd, then (1) gives $k^m - 1 = 2^r(mu + 2^r M)$, and so $f(m) = r$.

2) If $n = 2p$ even, then $k^{2p} - 1 = (k^p - 1)(k^p + 1)$. Note that

$$k^p + 1 = (2^s v - 1)^p + 1 = \sum_{i=0}^{p} \binom{p}{i} (2^s v)^i (-1)^{p-i} + 1$$

$$= \binom{p}{0}(-1)^p + \binom{p}{1}(2^s v)(-1)^{p-1} + \sum_{i=2}^{p} \binom{p}{i} (2^s v)^i (-1)^{p-i} + 1$$

$$= 2^{2s} N + (-1)^{p-1} 2^s pv + (-1)^p + 1. \tag{2}$$

(i) If p is odd, by (2), $k^p + 1 = 2^{2s} N + 2^s pv = 2^s(2^s N + pv)$. Then by (1) we have

$$k^{2p} - 1 = (k^p - 1)(k^p + 1) = 2^r(2^r M + pu) \cdot 2^s(2^s N + pv),$$

which implies that $f(2p) = r + s$, as $r, s \geq 1$ and p, u, v are all odd.

(ii) If p is even, by (2), $k^p + 1 = 2^{2s} N - 2^s pv + 2 = 4P + 2 = 2(2P + 1)$. Then

$$k^{2p} - 1 = (k^p - 1)(k^p + 1) = 2^{f(p)} Q \cdot 2(2P + 1) = 2^{f(p)+1} Q(2P + 1),$$

which implies that $f(2p) = 1 + f(p)$, as Q is odd.

So, for $n = 2^t m$, with $t \geq 1$, m odd, we have

$$f(n) = f(2^t m) = (t-1)f(2m) = r + s + t - 1.$$

Thus,

$$\begin{cases} f(m) = r, \text{ if } m \text{ is odd,} \\ f(2^t m) = r + s + t - 1, \text{ if } t \geq 1 \text{ and } m \text{ odd.} \end{cases}$$

Moreover, $k-1$ and $k+1$ are two consecutive even numbers, then $k \equiv 1 \pmod 4 \iff r > 1, s = 1$, while $k \equiv 3 \pmod 4 \iff r = 1, s > 1$. So the final answer is

$$\begin{cases} f(m) = r, \text{ if } m \text{ is odd,} \\ f(2^t m) = r + t, \text{ if } t \geq 1, r \geq 2 \text{ and } k \equiv 1 \pmod 4, \\ f(2^t m) = s + t, \text{ if } t \geq 1, s \geq 2 \text{ and } k \equiv 3 \pmod 4. \end{cases}$$

4.3.23

Let A and B be sets of numbers which end with digits 1 or 9, and 3 or 7, respectively. We note that for each $n \in \mathbb{N}$, if $a \in A$ then $a^n \in A$; if $b \in B$ then $b^{2n} \in A$, while $b^{2n+1} \in B$.

Now put $n = 2^\alpha 5^\beta k$ with k odd, not divisible by 5. We can verify that $f(n) = f(k)$ and $g(n) = g(k)$. It suffices to prove that $f(k) \geq g(k)$.

1) For $k = 1$ it is obvious.

2) For $k > 1$: we prove by induction on s, where s is the number of prime divisors of k. If $s = 1$ then $k = p^\ell$ ($p \notin \{2, 5\}, \ell \in \mathbb{N}$). By the note above, we have

(i) If $p \in A$ then $f(k) = \ell + 1 \ > \ g(k) = 0$.

(ii) If $p \in B$ then $f(k) = \left[\frac{\ell}{2}\right] + 1$ and

$$g(k) = \begin{cases} \left[\frac{\ell}{2}\right] & \text{(if } \ell \text{ even),} \\ \left[\frac{\ell}{2}\right] + 1 & \text{(if } \ell \text{ odd)} \end{cases},$$

where $[x]$ is the integral function of x. In this case $f(k) \geq g(k)$.

Suppose that the claim is true for $s \geq 1$. In the case $s+1$, $k = p_1^{\ell_1} \cdots p_s^{\ell_s} \cdot p_{s+1}^{\ell_{s+1}}$, where $p_i \notin \{2, 5\}, \ell_i \in \mathbb{N}$. Denote $k' = p_1^{\ell_1} \cdots p_s^{\ell_s}$, we see that k' is odd, not divisible by 5, and $k = k' p_{s+1}^{\ell_{s+1}}$.

Note that d is a divisor of k if and only if $d = d'p_{s+1}^{\ell}$, where d' is a divisor of k' and $0 \le \ell \le \ell_{s+1}$. By the note above, we have

$$f(k) = f(k')f(p_{s+1}^{\ell_{s+1}}) + g(k')g(p_{s+1}^{\ell_{s+1}}),$$

$$g(k) = f(k')g(p_{s+1}^{\ell_{s+1}}) + g(k')f(p_{s+1}^{\ell_{s+1}}).$$

So $f(k)-g(k) = [f(k')-g(k')]\cdot[f(p_{s+1}^{\ell_{s+1}})-g(p_{s+1}^{\ell_{s+1}})] \ge 0$, that is, $f(k) \ge g(k)$. Everything is proved.

4.3.24

1) For each n denote $b_n \in [0,3],\ c_n \in [0,4]$ the remainder of division of a_n by 4 and 5, respectively. Then

$$b_0 = 1, b_1 = 3, b_{n+2} \equiv b_{n+1} + b_n \pmod 4,$$

and

$$c_0 = 1, c_1 = 3, c_{n+2} \equiv \begin{cases} c_{n+1} - c_n \pmod 5 \ (n \text{ even}), \\ -c_{n+1} \pmod 5 \ (n \text{ odd}). \end{cases}$$

Direct computations give

$$b_0 = 1,\ b_1 = 3,\ b_2 = 0,\ b_3 = 3,\ b_4 = 3,\ b_5 = 2,\ b_6 = 1, \dots$$

and

$$c_0 = 1,\ c_1 = 3,\ c_2 = 2,\ c_3 = 3,\ c_4 = 1,\ c_5 = 4,\ c_6 = 3, \dots,$$

which lead to $b_k = b_{k+6\ell},\ c_k = c_{k+8\ell}$ for all $k \ge 2, l \ge 1$.

Note that $1995 = 3+6\cdot332 = 3+8\cdot249$, and $1996 = 4+6\cdot332 = 4+8\cdot249$, we then have

$$b_{1995} = b_3 = 3,\ b_{1996} = b_4 = 3,$$

and

$$c_{1995} = c_3 = 3,\ c_{1996} = c_4 = 1.$$

Therefore, $b_{1997} = 2,\ b_{1998} = 1,\ b_{1999} = 3,\ b_{2000} = 0$ and $c_{1997} = 4,\ c_{1998} = 3,\ c_{1999} = 2,\ c_{2000} = 4$.

Thus

$$\sum_{k=1995}^{2000} a_k^2 \equiv \sum_{k=1995}^{2000} b_k^2 \equiv 0 \pmod 4,$$

and

$$\sum_{k=1995}^{2000} a_k^2 \equiv \sum_{k=1995}^{2000} c_k^2 \equiv 0 \pmod 5.$$

Since $(4,5)=1$, we get $\sum\limits_{k=1995}^{2000} a_k^2 \equiv 0 \pmod{20}$.

2) Note that $2n+1$ has a from either $6k+1$ or $6k+3$ or $6k+5$. By 1) either $a_{2n+1} \equiv 3 \pmod 4$, or $a_{2n+1} \equiv 2 \pmod 4$.

On the other hand, a is a square if and only if $a \equiv 0 \pmod 4$, or $a \equiv 1 \pmod 4$. From this it follows that a_{2n+1} is never a square for any n.

4.3.25

From the second assumption it follows that for all $n \in \mathbb{Z}$

$$f(f(n)) = n, \tag{1}$$

$$f\big(f(n)+3\big) = n-3. \tag{2}$$

Then by (2)

$$f(n-3) = f\big(f(f(n)+3)\big)$$

and by (1)

$$f\big(f(f(n)+3)\big) = f(n)+3,$$

which give $f(n) = f(n-3)-3$.

By induction we can show that

$$f(3k+r) = f(r) - 3k,\ 0 \le r \le 2,\ k \in \mathbb{Z}. \tag{3}$$

The first assumption and (3) give

$$1996 = f(1995) = f(3\cdot 665+0) = f(0) - 1995 \implies f(0) = 3991. \tag{4}$$

From (1) and (3) it follows that

$$0 = f\big(f(0)\big) = f(3991) = f(3\cdot 1330+1) = f(1) - 3990 \implies f(1) = 3990. \tag{5}$$

Put $f(2) = 3s+r$ with $0 \le r \le 2,\ s \in \mathbb{Z}$, by (1) and (3)

$$2 = f\big(f(2)\big) = f(3s+r) = f(r) - 3s \implies f(r) = 3s+2,$$

which implies that $r=2$ and s is an arbitrary integer, because by (4) and (5), $f(0) = 3\cdot 1330+1, f(1) = 3\cdot 1330$, both are not of the form $3s+2$.

Substituting (4), (5) and (6) into (3) we obtain

$$f(n) = \begin{cases} 3991 - n, & \text{if } n \neq 3k+2,\ k \in \mathbb{Z}, \\ 3s + 4 - n, & \text{if } n = 3k+2,\ k \in \mathbb{Z}. \end{cases}$$

We can verify that this function satisfies the requirements of the problem.

4.3.26

1) By direct testing we can guess the following relation

$$a_{n+1}^2 - a_n a_{n+2} = 7^{n+1}, \tag{1}$$

which can be proved by induction.

From (1) it follows that the number of positive divisors of $a_{n+1}^2 - a_n a_{n+2}$ is $n+2$.

2) We can get, by (1), that

$$a_{n+1}^2 - a_n(45a_{n+1} - 7a_n) - 7^{n+1} = 0$$

$$\iff a_{n+1}^2 - 45a_n a_{n+1} + 7a_n^2 - 7^{n+1} = 0.$$

This shows that the equation $x^2 - 45a_n x + 7a_n^2 - 7^{n+1} = 0$ has an integer solution. So $\Delta = (45a_n)^2 - 4(7a_n^2 - 7^{n+1}) = 1997a_n^2 + 4 \cdot 7^{n+1}$ must be a square.

4.3.27

For $n = 1, 2$ we choose $k = 2$. Consider $n \geq 3$. We note that

$$19^{2^{n-2}} - 1 = 2^n t_n,\ t_n \text{ is odd.} \tag{1}$$

Indeed, this can be proved by induction. For $n = 3$ it is obvious. If it is true for $n \geq 3$, then

$$19^{2^{n-1}} - 1 = (19^{2^{n-2}} + 1) \cdot (19^{2^{n-2}} - 1) = (2^n t_n + 2) \cdot 2^n t_n = 2s_n \cdot 2^n t_n = 2^{n+1}(s_n t_n),$$

where s_n and t_n are odd. The note is proved.

We will now solve the problem again, by induction. For $n = 3$ it is true. Suppose that there exists $k_n \in \mathbb{N}$ such that $19^{k_n} - 97 = 2^n A$.

a) If A is even, then $19^{k_n} - 97$ is divisible by 2^{n+1}.

b) If A is odd, then putting $k_{n+1} = k_n + 2^{n-2}$, by the note above, we have

$$19^{k_{n+1}} - 97 = 19^{2^{n-2}}(19^{k_n} - 97) + 97(19^{2^{n-2}} - 1) = 2^n(19^{2^{n-2}}A + 97t_n)$$

is divisible by 2^{n+1}. The problem is solved completely.

4.3.28

We have

$$\begin{aligned} x_{n+2} &= 22y_{n+1} - 15x_{n+1} = 22(17y_n - 12x_n) - 15x_{n+1} \\ &= 17(x_{n+1} + 15x_n) - 22 \cdot 12x_n - 15x_{n+1}, \end{aligned}$$

which gives

$$x_{n+2} = 2x_{n+1} - 9x_n, \ \forall n. \tag{1}$$

Similarly,

$$y_{n+2} = 2y_{n+1} - 9y_n, \ \forall n. \tag{2}$$

1) From (1) it follows that $x_{n+2} \equiv 2x_{n+1} \pmod 3$. This together with $x_1 = 1, \ x_2 = 29$ implies that x_n is not divisible by 3, and so

$$x_n \neq 0, \ \forall n. \tag{3}$$

Furthermore,

$$x_{n+3} = 2x_{n+2} - 9_{n+1} = -5x_{n+1} - 18x_n,$$

or equivalently,

$$x_{n+3} + 5x_{n+1} + 18x_n = 0, \ \forall n. \tag{4}$$

Assume that there is in (x_n) a finite number of positive (or negative) terms. Then all terms (x_n), for n large enough, are positive (or negative), this contradicts (3) and (4).

The argument is similar for (y_n).

2) From (1) we have

$$x_{n+4} = -28x_{n+1} = 45x_n,$$

which gives

$$x_n \equiv 0 \pmod 7 \Longleftrightarrow x_{n+4} \equiv 0 \pmod 7 \Longleftrightarrow x_{4k+n} \equiv 0 \pmod 7.$$

Since $1999^{1945} \equiv (-1)^{1945} \pmod 4 \equiv 3 \pmod 4$ and $x_3 = 49 = 7^2$, $x_{1999^{1945}}$ is divisible by 7.

Similarly, y_n is not divisible by 7 if and only if y_{4k+n} is not divisible by 7. Since $y_3 = 26$ is not divisible by 7, neither is $y_{1999^{1945}}$.

4.3.29

Put $f(2000) = a$ with $a \in T$, and $b = 2000 - a$. Then $1 \le b \le 2000$. We have the following notes.

Note 1. For any $0 \le r < b$ there holds $f(2000 + r) = a + r$.
Indeed, if $0 \le r < b$ then $a + r < a + b = 2000$, and hence

$$f(2000 + r) = f\big(f(2000) + f(r)\big) = f(a + r) = a + r.$$

Note 2. For any $k \ge 0,\ 0 \le r < b$ there holds $f(2000 + kb + r) = a + r$.
This can easily be proved by induction.

From these two notes it follows that if f is a function satisfying the requirements of the problem, then

$$\begin{cases} f(n) = n,\ \forall n \in T, \\ f(2000) = a, \\ f(2000 + m) = a + r, \end{cases} \tag{1}$$

for $r \equiv m \pmod{(2000 - a)}$ and $0 \le r < 2000 - a$, where a is an arbitrary element from T.

Conversely, given $a \in T$. Consider a function f defined on nonnegative integers and satisfies relations (1). We show that f satisfies the requirements of the problem.

First, it is clear that $f(n) \in T,\ \forall n \ge 0$ and $f(t) = t,\ \forall t \in T$.

Next, we can easily verify the following relations:

$$f(n + b) = f(n),\ \forall n \ge a,\ \text{with } b = 2000 - a. \tag{2}$$

and

$$n \equiv f(n) \pmod b,\ \forall n \ge 0. \tag{3}$$

From these it follows that $f(n) \in T,\ \forall n \ge 0$ and $f(n) = n,\ \forall n \in T$. We next verify that

$$f(m + n) = f\big(f(m) + f(n)\big),\ \forall m, n \ge 0.$$

It suffices to check this for the case when at least one of two numbers m, n is not in T. Suppose that $m \geq 2000$, Then $m+n \geq 2000 > a$ and $f(m)+f(n) \geq a$ (as $f(m) \geq a$).

On the other hand, by (3),

$$m+n \equiv f(n)+f(m) \pmod{b},$$

which implies, by (2), that $f(m+n) = f\big(f(m)+f(n)\big)$.

Thus all functions are defined by (1), and so there are 2000 such functions.

4.3.30

We make the following notes.

Note 1. If d is an odd prime divisor of $a^{6^n}+b^{6^n}$ with $(a,b)=1$, then $d \equiv 1 \pmod{2^{n+1}}$.

Indeed, let $a^{6^n}+b^{6^n} = kd$, $k \in \mathbb{N}$. We write $d = 2^m t + 1$, where $m \in \mathbb{N}$, t is an odd positive integer. Assume that $m \leq n$. Since $(a,b)=1$, $(a,d)=(b,d)=1$. By Fermat's Little Theorem,

$$\left(a^{3^n \cdot 2^{n-m}}\right)^{d-1} \equiv \left(b^{3^n \cdot 2^{n-m}}\right)^{d-1} \equiv 1 \pmod{d},$$

or equivalently,

$$\left(a^{6^n}\right)^t \equiv \left(b^{6^n}\right)^t \equiv 1 \pmod{d}. \tag{1}$$

On the other hand,

$$\left(a^{6^n}\right)^t = \left(kd - b^{6^n}\right)^t = rd - (b^{6^n})^t \equiv -(b^{6^n})^t \pmod{d}.$$

Combining this with (1) yields $(b^{6^n})^t \equiv -(b^{6^n})^t \pmod{d}$, and we get a contradiction. Thus it must be $m \geq n+1$, which implies that $d \equiv 1 \pmod{2^{2n+1}}$.

Note 2. If $\ell \equiv 1 \pmod{r^k}$ then $\ell^{r^m} \equiv 1 \pmod{r^{m+k}}$.

Indeed, $\ell = tr^k + 1$ with $t \in \mathbb{Z}$. Then

$$\ell^{r^m} = (tr^k+1)^{r^m} = sr^{m+k} + 1 \equiv 1 \pmod{r^{m+k}},\ s \in \mathbb{Z}.$$

We return back to the problem. Since p,q are odd divisors of $a^{6^n}+b^{6^n}$ and $(a,b)=1$, by Note 1, we have $p^{3^n} \equiv q^{3^n} \equiv 1 \pmod{2^{n+1}}$. Then by Note 2, we obtain

$$p^{6^n} \equiv q^{6^n} \equiv 1 \pmod{2^{2n+1}}. \tag{2}$$

Also since $(a,b) = 1$, $p^{6^n} \equiv q^{6^n} \not\equiv 0 \pmod 3$, we see that $(p,3) = (q,3) = 1$. In this case $p^{2^n} \equiv q^{2^n} \equiv 1 \pmod 3$. By Note 2, we get

$$p^{6^n} \equiv q^{6^n} \equiv 1 \pmod{3^{n+1}}. \tag{3}$$

From (2) and (3), as $(2,3) = 1$, it follows that $p^{6^n} \equiv q^{6^n} \equiv 1 \pmod{6 \cdot (12)^n}$, which gives

$$p^{6^n} + q^{6^n} \equiv 2 \pmod{6 \cdot (12)^n}.$$

4.3.31

It is clear that

$$|\mathcal{T}| = 2^{2002} - 1.$$

For each $k \in \{1, 2, \ldots, 2002\}$, put $m_k = \sum m(X)$, where the sum is taken over all sets $X \in \mathcal{T}$ with $|X| = k$. Then we have to compute

$$\sum m(X) = \sum_{k=1}^{2002} m_k.$$

Let a be an arbitrary number in $\mathcal{S}$. It is easy to see that a appears in $\binom{2001}{k-1}$ sets X in $\mathcal{T}$ with $|X| = k$. Then

$$k \cdot m_k = (1 + 2 + \cdots + 2002) \cdot \binom{2001}{k-1} = 1001 \cdot 2003 \cdot \binom{2001}{k-1},$$

which gives

$$\sum m(X) = \sum_{k=1}^{2002} m_k = 1001 \cdot 2003 \cdot \sum_{k=1}^{2002} \frac{\binom{2001}{k-1}}{k}$$

$$= \frac{2003}{2} \cdot \sum_{k=1}^{2002} \binom{2002}{k} = \frac{2003(2^{2002} - 1)}{2}.$$

Thus

$$m = \frac{\sum m(X)}{|\mathcal{T}|} = \frac{2003}{2}.$$

4.3.32

Suppose that p is a prime divisor of $\binom{2n}{n}$, with the multiplicity m. We prove that $p^m \leq 2n$.

Assume that it is not true. That is $p^m > 2n$. In this case the integral part $\left[\frac{2n}{p^m}\right] = 0$. Therefore,

$$m = \left(\left[\frac{2n}{p}\right] - 2\left[\frac{n}{p}\right]\right) + \left(\left[\frac{2n}{p^2}\right] - 2\left[\frac{n}{p^2}\right]\right) + \cdots + \left(\left[\frac{2n}{p^{m-1}}\right] - 2\left[\frac{n}{p^{m-1}}\right]\right). \tag{1}$$

Note that for $x \in \mathbb{R}$, it always holds that

$$2[x] + 2 > 2x \geq [2x] \implies [2x] - 2[x] \leq 1.$$

Then from (1) it follows that $m \leq m - 1$, a contradiction.

Thus, $p^m \leq 2n$ and hence

$$\binom{2n}{n} = (2n)^k \Longleftrightarrow k = 1,$$

and

$$\binom{2n}{n} = 2n \Longleftrightarrow n = 1.$$

4.3.33

The given equation is equivalent to

$$(x + y + u + v)^2 = n^2 xyuv.$$

That is

$$x^2 + 2(y + u + v)x + (y + u + v)^2 = n^2 xyuv. \tag{1}$$

Let n be such a required number. Denote by (x_0, y_0, u_0, v_0) a solution of (1) with the minimum sum, and without loss of generality we can assume that $x_0 \geq y_0 \geq u_0 \geq v_0$. It is easy to verify the following notes.

Note 1. $(y_0 + u_0 + v_0)^2$ is divisible by x_0.

Note 2. x_0 is a positive integer solution of the quadratic function

$$f(x) = x^2 + [2(y_0 + u_0 + v_0) - n^2 y_0 u_0 v_0]x + (y_0 + u_0 + v_0)^2.$$

From these notes, by Viète formula, we see that beside x_0 the function $f(x)$ has a positive integer solution

$$x_1 = \frac{(y_0 + u_0 + v_0)^2}{x_0}.$$

This shows that (x_1, y_0, u_0, v_0) is also a solution of (1). So by the assumption for (x_0, y_0, u_0, v_0), we have

$$x_1 \geq x_0 \geq y_0 \geq u_0 \geq v_0. \tag{2}$$

Since y_0 is outside of the root interval $[x_0, x_1]$ of the quadratic equation $f(x) = 0$ said above, it follows that $f(y_0) \geq 0$.

On the other hand, by (2)

$$\begin{aligned} f(y_0) &= y_0^2 + 2(y_0 + u_0 + v_0)y_0 + (y_0 + u_0 + v_0)^2 - n^2 y_0^2 u_0 v_0 \\ &\leq y_0^2 + 2(y_0 + y_0 + y_0)y_0 + (y_0 + y_0 + y_0)^2 - n^2 y_0^2 u_0 v_0 \\ &= 16y_0^2 - n^2 y_0^2 u_0 v_0. \end{aligned}$$

So we arrive at

$$16y_0^2 - n^2 y_0^2 u_0 v_0 \geq 0 \implies n^2 u_0 v_0 \leq 16.$$

However, $n^2 \leq n^2 u_0 v_0$, and so $n^2 \leq 16 \implies n \in \{1, 2, 3, 4\}$.

We then can easily verify that for each value $n = 1,\ 2,\ 3$ and 4 the equation (1) has a unique solution, which is $(4, 4, 4, 4), (2, 2, 2, 2), (1, 1, 2, 2)$ and $(1, 1, 1, 1)$ respectively.

4.3.34

We prove that the given system has no integer solutions if $n = 4$, and therefore, it has no integer solutions for all $n \geq 4$.

Note that if $k \in \mathbb{Z}$ then

$$k^2 \equiv \begin{cases} 1 \pmod 8, & \text{if } k \equiv \pm 1 \pmod 4, \\ 0 \pmod 8, & \text{if } k \equiv 0 \pmod 4, \\ 4 \pmod 8, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

This implies that for any two integers k, ℓ we have

$$k^2 + \ell^2 \equiv \begin{cases} 2, 1, 5 \pmod 8, & \text{if } k \equiv \pm 1 \pmod 4, \\ 1, 0, 4 \pmod 8, & \text{if } k \equiv 0 \pmod 4, \\ 5, 4, 0 \pmod 8, & \text{if } k \equiv 2 \pmod 4. \end{cases}$$

Assume that there are integers x, y_1, y_2, y_3, y_4 satisfying

$$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = (x+3)^2 + y_3^2 = (x+4)^2 + y_4^2.$$

Then, due to the fact that $x+1, x+2, x+3, x+4$ form a complete congruent system modulo 4, there exists an integer m such that

$$m \in \{2; 1; 5\} \cap \{1; 0; 4\} \cap \{5; 4; 0\} = \emptyset,$$

which is impossible. Thus there is no integer solution for $n \geq 4$.

When $n = 3$ we can see that $(-2, 0, 1, 0)$ is a solution. Thus $n_{\max} = 3$.

4.3.35

Note that if $(x, y, z) = (a, b, c)$ is a solution, then (b, a, c) is also a solution. So we first find solutions (x, y, z) with $x \leq y$.

In this case x, y are odd, $z \geq 2$ and there is a positive integer $m < z$ such that

$$x + y = 2^m, \tag{1}$$

and

$$1 + xy = 2^{z-m}. \tag{2}$$

We note that

$$(1 + xy) - (x + y) = (x - 1)(y - 1) \geq 0,$$

which gives $2^{z-m} \geq 2^m$, that is, $m \leq \frac{z}{2}$.

Consider two cases:

Case 1: If $x = 1$, then from (1) and (2) it follows that

$$\begin{cases} y = 2^m - 1, \\ z = 2m. \end{cases}$$

By direct testing we obtain that $(1, 2^m - 1, 2m)$ with $m \in \mathbb{N}$ and $x \leq y$ satisfies the given equation.

Case 2: If $x > 1$, then since x is odd, $x \geq 3$. This implies that

$$2^m = x + y \geq 6 \implies m \geq 3.$$

Also note that

$$x^2 - 1 = x(x + y) - (1 + xy) = 2^m x - 2^{z-m} = 2^m(x - 2^{z-2m}),$$

and so (x^2-1) is divisible by 2^m. Since $\gcd(x-1,x+1)=2$, one of the two numbers $x\pm 1$ must divisible by 2^{m-1}.

Moreover, since $x \le y$, we have

$$0 < x-1 \le \frac{x+y}{2} - 1 = 2^{m-1}-1 < 2^{m-1}.$$

Therefore, $x-1$ is not divisible by 2^{m-1}, and so $x+1$ is divisible by 2^{m-1}.

Note that $1 < x \le y$ and hence $x+1 < x+y = 2^m$. Then $x+1 = 2^{m-1}$, as it must be divisible by 2^{m-1}. So $x = 2^{m-1}-1$, which implies that $y = 2^m - x = 2^{m-1}+1$. Combining these values of x, y with (2) yields $2^{2(m-1)} = 2^{z-m}$, which gives $z = 3m-2$.

By direct testing we get that $(2^{m-1}-1, 2^{m-1}+1, 3m-2)$, where $m \in \mathbb{N}, m \ge 3$ and $x \le y$, is a solution.

Summarizing, we have

$$(x,y,z) = \begin{cases} (1, 2^m-1, 2m), \ m \in \mathbb{N}, \\ (2^m-1, 1, 2m), \ m \in \mathbb{N}, \\ (2^{m-1}-1, 2^{m-1}+1, 3m-2), \ m \in \mathbb{N}, m \ge 3, \\ (2^{m-1}+1, 2^{m-1}-1, 3m-2), \ m \in \mathbb{N}, m \ge 3. \end{cases}$$

4.3.36

Let $M = \{1, 2, \ldots, 16\}$. We can easily verify that a subset $S = \{2, 4, 6, \ldots, 14, 16\}$ which consists of all 8 even numbers, cannot be a solution, as if $a, b \in S$, then a^2+b^2 is always a composite number. Thus k must be greater than 8.

By direct computations of all sums a^2+b^2 with $a, b \in M$, we can obtain a partition of M consisting of 8 subsets, each of which has two elements with sum of square being a prime number:

$$M = \{1,4\} \cup \{2,3\} \cup \{5,8\} \cup \{6,11\} \cup \{7,10\} \cup \{9,16\} \cup \{12,13\} \cup \{14,15\}.$$

By the Pigeonhole principle, among any 9 elements of M there exist two that belong to the same subset of the partition. In other words, in any subset with 9 elements of M there always exist two distinct numbers a, b such that a^2+b^2 is prime. So, $k_{\min} = 9$.

Remark: The partition said above is not unique.

4.3.37

We have the following notes.

Note 1. The smallest positive integer m that satisfies $10^m \equiv 1 \pmod{2003}$ is 1001.

Indeed, since $1001 = 7\cdot 11\cdot 13$, the positive divisors of 1001 are $1, 7, 11, 13, 77, 91, 143$ and 1001. Note that $10^{1001} \equiv 1 \pmod{2003}$ and therefore, if k is the smallest positive integer such that $10^k \equiv 1 \pmod{2003}$, then k must be a positive divisor of 1001. By direct computation, we have $10^1, 10^7, \dots, 10^{143}$ all are not congruent to 1 by $\pmod{2003}$. The claim follows.

Note 2. There does not exist a positive multiple of 2003 that is of the form $10^k + 1$, where $k \in \mathbb{N}$.

Indeed, if not so, that is, there is $k \in \mathbb{N}$ such that $10^k + 1 \equiv 0 \pmod{2003}$, then $10^{2k} \equiv 1 \pmod{2003}$, and by Note 1, $2k$ is a multiple of 1001. From this it follows that k is divisible by 1001 (as $(2, 1001) = 1$), which in turn, implies that $10^k \equiv 1 \pmod{2003}$, a contradiction.

Note 3. There exists a positive multiple of 2003 that is of the form $10^k + 10^h + 1$, where $k, h \in \mathbb{N}$.

Indeed, consider the 2002 positive integers $a_1, \dots, a_{1001}, b_1, \dots, b_{1001}$, where a_k, b_k are remainders of the division of 10^k and $-10^k - 1$ by 2003, respectively.

We have $a_k \neq 0, b_k \neq 2002$ and by Note 2, also $a_k \neq 2002, b_k \neq 0$, for all $k = 1, \dots, 1001$. So all of these numbers a_k, b_k are in the set $\{1, 2, \dots, 2001\}$. By the Pigeonhole principle, there are two equal numbers.

Furthermore, $a_i \neq a_j, b_i \neq b_j$, otherwise, if for $1 \leq i < j \leq 1001$ either $a_i = a_j$ or $b_i = b_j$, then $1 \leq j - i \leq 1000$ and $10^j - 10^i \equiv 10^i(10^{j-i} - 1) \equiv 0 \pmod{2003}$, or $10^{j-i} \equiv 1 \pmod{2003}$. This contradicts Note 1.

So there are $k, h \in \{1, \dots, 1001\}$ such that $a_k = b_h$, or $10^k + 10^h + 1 \equiv 0 \pmod{2003}$.

Now we return to the problem. Let m be a positive multiple of 2003. It is easy to see that m is not in the form 10^k and $2 \cdot 10^k$ with $k \in \mathbb{N}$. By Note 2, we have $S(m) \geq 3$.

On the other hand, if $h, k \in \mathbb{N}$ then $S(10^k + 10^h + 1) = 3$, and by Note 3, there exists a positive multiple m_0 of 2003 such that $S(m_0) = 3$.

Thus $\min S(m) = 3$.

4.3.38

We write

$$\frac{x!}{n!} + \frac{y!}{n!} = 3. \tag{1}$$

Let (x, y, n) be a solution. Without loss of generality we can assume that $x \leq y$. We have the following notes, the first two of which are obvious.

Note 1. If $x < n$ and $y < n$, then

$$\frac{x!}{n!} + \frac{y!}{n!} < 2.$$

Note 2. If $x < n$ and $y > n$, then

$$\frac{x!}{n!} + \frac{y!}{n!} \notin \mathbb{N}.$$

Note 3. If $x > n$ then

$$\frac{x!}{n!} + \frac{y!}{n!} \geq 4.$$

Indeed, since $x > n$, we have $x \geq n+1$, and hence $y \geq n+1$ (as $y \geq x$). So

$$\frac{x!}{n!} \geq n + 1 \geq 2$$

and

$$\frac{y!}{n!} \geq n + 1 \geq 2,$$

which gives the desired inequality.

By Notes 1–3, from (1) it follows that $x = n$. In this case (1) gives

$$\frac{y!}{n!} = 2. \tag{2}$$

Then $y \geq n + 1$, and $\dfrac{y!}{n!} \geq n + 1 \geq 2$. Therefore, (2) is equivalent to

$$\begin{cases} \frac{y!}{n!} = n + 1 \\ n + 1 = 2 \end{cases} \iff y = 2, n = 1,$$

and hence $x = 1$. So we obtain $(1, 2, 1)$. Furthermore, by interchanging x and y, we get also $(2, 1, 1)$.

Conversely, it is easy to verify that these two triples satisfy the problem.

4.3.39

We write

$$x! + y! = 3^n \cdot n! \tag{1}$$

Let (x, y, n) be a solution. We should have $n \geq 1$, and without loss of generality we can assume that $x \leq y$. There are two possibilities:

1) **Case 1**: $x \leq n$. Equation (1) is equivalent to

$$1 + \frac{y!}{x!} = 3^n \frac{n!}{x!}, \tag{2}$$

which implies that $1 + \frac{y!}{x!} \equiv 0 \pmod 3$. This first shows that $x < y$. Furthermore, $\frac{y!}{x!}$ is not divisible by 3, and as a product of three consecutive integers is divisible by 3 and $n \geq 1$, we must have $y \leq x + 2$. Thus either $y = x + 1$, or $y = x + 2$.

(i) If $y = x + 2$, then from (2) it follows that

$$1 + (x + 1)(x + 2) = 3^n \frac{n!}{x!}. \tag{3}$$

Note that a product of two consecutive integers is divisible by 2. Thus the left-hand side of (3) is an odd number, which shows that the right-hand side of (3) cannot divisible by 2. Therefore, we must have $n \leq x + 1$, that is, either $n = x$ or $n = x + 1$.

If $n = x$, then from (3) it follows that

$$1 + (x + 1)(x + 2) = 3^x \Longleftrightarrow x^2 + 3x + 3 = 3^x.$$

This implies that $x \equiv 0 \pmod 3$, as $x \geq 1$. So $x \geq 3$, and we get

$$-3 = x^2 + 3x - 3^x \equiv 0 \pmod 9,$$

which is impossible.

If $n = x + 1$, then from (3) it follows that

$$1 + (x + 1)(x + 2) = 3^n (x + 1).$$

This implies that $x + 1$ must be a positive divisor of 1, that is, $x = 0$, and hence $y = 2, n = 1$. So we get a triple $(0, 2, 1)$.

(ii) If $y = x + 1$, then (2) gives

$$x + 2 = 3^n \frac{n!}{x!}, \tag{4}$$

which implies that $x \geq 1$ as $n \geq 1$. In this case we can write $x + 2 \equiv 1 \pmod{(x+1)}$. Then from (4) it follows that $n = x$ (otherwise, the right-hand side of (4) would be divisible by $x + 1$), and we have

$$x + 2 = 3^x.$$

It is easy to verify that for $x \geq 2$ there always holds $3^x > x + 2$, and so $x = 1$ is a unique solution of the last equation, and we get a triple $(1, 2, 1)$.

Thus for the first case we obtain two solutions $(0, 2, 1)$ and $(1, 2, 1)$.

2) **Case 2**: $x > n$. We have

$$\frac{x!}{n!} + \frac{y!}{n!} = 3^n. \tag{5}$$

Note that $n + 1$ and $n + 2$ cannot both be powers of 3 simultaneously, and so from (5) we must have $x = n + 1$. Then

$$n + 1 + \frac{y!}{n!} = 3^n. \tag{6}$$

Since $y \geq x$, $y \geq n + 1$. Putting $M = \frac{y!}{(n+1)!}$, we can write (6) as

$$n + 1 + M(n + 1) = 3^n \Longleftrightarrow (n + 1)(M + 1) = 3^n.$$

Since a product of three consecutive integers, as noted above, is divisible by 3, it is clear that if $y \geq n+4$ then $M \equiv 0 \pmod 3$, and so $M+1$ cannot be a power of 3. Thus we must have $y \leq n+3$, and hence $y \in \{n+1, n+2, n+3\}$.

(i) If $y = n + 3$, then $M = (n + 2)(n + 3)$, and hence (6) gives

$$(n + 1)[(n + 2)(n + 3) + 1] = 3^n \Longleftrightarrow (n + 2)^3 - 1 = 3^n.$$

This implies that $n > 2$ and $n + 2 \equiv 1 \pmod 3$. Since $n + 2 > 4$, $n + 2 = 3k + 1$, $k \geq 2$ and we arrive at

$$9k(3k^2 + 3k + 1) = 3^{3k-1},$$

which shows that $3k^2 + 3k + 1$ is a power of 3. This is impossible.

(ii) If $y = n + 2$, then $M = n + 2$. In this case we have

$$n + 1 + (n + 2)(n + 1) = 3^n \Longleftrightarrow (n + 1)(n + 3) = 3^n.$$

However, $n + 1$ and $n + 3$ cannot be at the same time powers of 3, and so this case is also impossible.

(iii) If $y = n+1$, then $M = 1$. This gives $2(n+1) = 3^n$, which cannot happen.

Thus if $x \leq y$ then the case $x > n$ is impossible.

Combing two cases, taking into account a possible interchanging of x and y, we have four pairs (x, y, n), that satisfy the problem by direct test. They are $(0, 2, 1)$, $(2, 0, 1)$, $(1, 2, 1)$ and $(2, 1, 1)$.

4.3.40

1) Let $T = \{a_1, \ldots, a_n\}$ with $a_1 < \cdots < a_n$. Then $M = \{a_2 - a_1, a_3 - a_1, \ldots, a_n - a_1\}$ is a subset of S with $n-1$ elements. From the property of T, it follows that $T \cap M = \emptyset$. Indeed, if not so, then there exists $a_p - a_1 \in T$ for some $1 \leq p \leq n$. In this case, a_1 and $a_p - a_1$ both are in T, but their sum $a_1 + a = a_1 + (a_p - a_1) = a_p$ is also in T, which is impossible.

Thus $T \cap M = \emptyset$, and so $|T| + |M| = n + (n-1) \leq 2006$, or $n \leq 1003$.

2) Let $S = \{a_1, \ldots, a_{2006}\}$. Denote by P the product of all odd divisors of $\prod_{k=1}^{2006} a_k$. It is clear that there exists a prime number of the form $p = 3q + 2$ with $q \in \mathbb{N}$, which is a divisor of $3P + 2$. Note that $(p, a_k) = 1$ for all k.

We see that for each $a_k \in S$ a sequence $(a_k, 2a_k, \ldots, (p-1)a_k)$ is a permutation of $(1, 2, \ldots, p-1)$ by $\mod p$. Therefore, there exists a set A_k consisting of $q+1$ integers $x \in \{1, 2, \ldots, p-1\}$ such that xa_k by $\pmod p$ belongs to $A = \{q+1, \ldots, 2q+1\}$.

For each $x \in \{1, 2, \ldots, p-1\}$ denote $S_x = \{a_k \in S : xa_k \in A\}$. We then have

$$|S_1| + |S_2| + \cdots + |S_{p-1}| = \sum_{a_k \in S} |A_k| = 2006(q+1).$$

So there exists x_0 such that

$$|S_{x_0}| \geq \frac{2006.(q+1)}{3q+1} > 668.$$

Now we can choose T to be a subset of S_{x_0} that consists from 669 elements. This is a desired subset. Indeed, if $u, v, w \in T$ (u, v can be equal), then $x_0u, x_0v, x_0w \in A$. Also we can verify that $x_0u + x_0v \neq x_0w \pmod p$, and hence $u + v \neq w$.

4.3.41

First we note that

$$x^4y^{44} - 1 = x^4(y^{44} - 1) + x^4 - 1,$$

where $x^4 - 1$ is divisible by $x + 1$ and $y^{44} - 1$ is divisible by $y^4 - 1$. From this it follows that the problem will be proved if we can show that $y^4 - 1$ is divisible by $x + 1$.

Put

$$\frac{x^4 - 1}{y + 1} = \frac{a}{b}, \ \frac{y^4 - 1}{x + 1} = \frac{c}{d},$$

where $a, b, c, d \in \mathbb{Z}, (a, b) = 1, (c, d) = 1, b > 0, d > 0$.

From the assumption it follows that

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} = k \Longleftrightarrow ad + bc = kbd,$$

for some integer k. This relation shows that d is divisible by b as well as b is divisible by d, which imply that $b = d$.

On the other hand, since

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{x^4 - 1}{x + 1} \cdot \frac{y^4 - 1}{y + 1} = (x^2 + 1)(x - 1)(y^2 + 1)(y - 1)$$

is an integer and $(a, b) = (c, d) = 1$, we get $b = d = 1$. Thus $y^4 - 1$ is divisible by $x + 1$. The desired result follows.

4.3.42

Note that $(m, 10) = 1$, and so

$$n(2n + 1)(5n + 2) \equiv 0 \pmod{m}$$

is equivalent to

$$100n(2n + 1)(5n + 2) \equiv 0 \pmod{m}.$$

That is

$$10n(10n + 4)(10n + 5) \equiv 0 \pmod{m}. \tag{1}$$

As $m = 3^{4016} \cdot 223^{2008}$, put $10n = x$, $3^{4016} = \ell_1$, $223^{2008} = \ell_2$ and note that $(\ell_1, \ell_2) = 1$, we therefore have

$$(1) \Longleftrightarrow \begin{cases} x(x + 4)(x + 5) \equiv 0 \pmod{\ell_1}, \\ x(x + 4)(x + 5) \equiv 0 \pmod{\ell_2}. \end{cases}$$

We can see that

(i) $x(x+4)(x+5) \equiv 0 \pmod{\ell_1}$ if and only if $x \equiv 0 \pmod{\ell_1}$, or $x \equiv -5 \pmod{\ell_1}$, or $x \equiv -4 \pmod{\ell_1}$.

(ii) $x(x+4)(x+5) \equiv 0 \pmod{\ell_2}$ if and only if $x \equiv 0 \pmod{\ell_2}$, or $x \equiv -5 \pmod{\ell_2}$, or $x \equiv -4 \pmod{\ell_2}$.

From these observations, together with $x \equiv 0 \pmod{10}$, it follows that n is a solution of the problem if and only if $n = \frac{x}{10}$, where x satisfies the following system

$$\begin{cases} x \equiv 0 \pmod{10} \\ x \equiv r_1 \pmod{\ell_1} \\ x \equiv r_2 \pmod{\ell_2} \end{cases} \quad (0 \leq x \leq 10\ell_1\ell_2), \tag{2}$$

where $r_1, r_2 \in \{0, -4, -5\}$.

Next, we prove that for each pair (r_1, r_2) said above the system has a unique solution.

Consider any such a pair (r_1, r_2). Put $m_1 = 10\ell_1, m_2 = 10\ell_2$, we have $(m_1, \ell_1) = (m_2, \ell_2) = 1$. So there exist integers s_1, s_2 such that

$$s_1 m_1 \equiv 1 \pmod{\ell_1}, \; s_2 m_2 \equiv 1 \pmod{\ell_2}.$$

Then $M = r_1 s_1 m_1 + r_2 s_2 m_2$ satisfies conditions

$$M \equiv 0 \pmod{10},$$

$$M \equiv r_1 \pmod{\ell_1},$$

and

$$M \equiv r_2 \pmod{\ell_2}.$$

In this case we can choose an integer $x \in [0, 10\ell_1\ell_2)$ such that $x \equiv M \pmod{10\ell_1\ell_2}$, which is a solution.

Now assume that system (2) has two solution $x' > x''$. As $x', x'' \in [0, 10\ell_1\ell_2)$ and $x' \equiv x'' \pmod{10\ell_1\ell_2}$, we see that $0 < x' - x'' < 10\ell_1\ell_2$, while $x' - x'' \equiv 0 \pmod{10\ell_1\ell_2}$, a contradiction.

Thus, the system has a unique solution. It is easy to see that different pairs of (r_1, r_2) give different solutions. So there are $3^2 = 9$ distinct pairs (r_1, r_2), which implies that there are 9 values of x that satisfy 9 corresponding systems. As there is a one-to-one correspondence between x and n, we obtain 9 numbers satisfying the requirements of the problem.

Remark: System (2) is in fact the Chinese Remainder Theorem.

4.4 Combinatorics

4.4.1

Denote the number of edges between vertices of A and B by s. Assume that there is no vertex of B that can be joined to all vertices of A. Then $s \le k(n-1)$.

On the other hand, since each vertex of A can be joined to at least $k-p$ vertices of B, $s \ge n(k-p)$. Moreover, by the hypothesis, $np < k$. This inequality gives $n(k-p) = nk - np > nk - k = k(n-1)$.

Thus, $s \le k(n-1) < n(k-p) \le s$, which is impossible.

4.4.2

Denote by $P(n)$, the numbers of regions divided by n circles. We have $P(1) = 2, P(2) = 4, P(3) = 8, P(4) = 14, \ldots$ and from this we notice that

$$\begin{aligned}
&P(1) = 2,\\
&P(2) = P(1) + 2,\\
&P(3) = P(2) + 4,\\
&P(4) = P(3) + 6,\\
&\cdots\ \cdots\ \cdots\\
&P(n) = P(n-1) + 2(n-1).
\end{aligned}$$

Summing up these equations we obtain

$$\begin{aligned}
P(n) &= 2 + 2 + 4 + \cdots + 2(n-1)\\
&= 2 + 2\big(1 + 2 + \cdots + (n-1)\big)\\
&= 2 + 2 \cdot \frac{n(n-1)}{2}\\
&= 2 + n(n-1).
\end{aligned}$$

We now prove this formula by induction.

For $n = 1$ it is obviously true.

Suppose the formula is true for $n = k \ge 1$, that is $P(k) = 2 + k(k-1)$. Consider $k+1$ circles, the $(k+1)$-th circle intersects k other circles at $2k$ points, which means that this circle is divided into $2k$ arcs, each of which divides the region it passes into two sub-regions. Therefore, we have in addition $2k$ regions, and so

$$\begin{aligned} P(k+1) &= P(k) + 2k \\ &= 2 + k(k-1) + 2k \\ &= 2 + k(k+1). \end{aligned}$$

4.4.3

For each ray Ox_i let π_i be the half-space divided by a plane perpendicular to Ox_i at O, that does not contains Ox_i $(i = 1, \ldots, 5)$.

Assume that all angles between any two rays are greater than 90°. Then the rays Ox_2, Ox_3, Ox_4, Ox_5 are in the half-space π_1. Similarly, all rays Ox_3, Ox_4, Ox_5 are in the half-space π_2. This means that Ox_3, Ox_4, Ox_5 are in the dihedral angle $\pi_1 \cap \pi_2$, whose linear angle is less than 90°.

By the same reasoning, rays Ox_4, Ox_5 must be in the intersection $\pi_1 \cap \pi_2 \cap \pi_3$. This intersection is either empty, or a trihedral angle whose face angles have the sum less than 90°. From this it follows that the angle between Ox_4 and Ox_5 is less than 90°. We arrive at a contradiction.

4.4.4

Let X and Y be two children one next to other in the direction of giving sweets. At the moment n when X is to give a_n sweets to Y, keeping x_n sweets (that is, Y has not received sweets, and therefore a_n sweets are not counted for neither X nor Y), suppose that X, Y have x_n, y_n sweets, respectively. Denote by M_n, m_n the max and min of sweets of all children at the moment n, not counting a_n, of course.

At the moment $n+1$ when Y is to give a_{n+1} sweets to the next child, Y has, by the assumptions of the problem

$$y_{n+1} = a_{n+1} = \begin{cases} \frac{a_n + y_n}{2}, & \text{if } a_n + y_n \text{ is even,} \\ \frac{a_n + y_n + 1}{2}, & \text{if } a_n + y_n \text{ is odd.} \end{cases}$$

Note that at that moment the child next to Y has not yet received any sweets and the number of sweets for every one, except Y, is a constant in comparison with the moment n.

1) If $x_n = a_n = y_n$: in this case $y_{n+1} = y_n = x_n$, and by the note, $M_{n+1} = M_n, m_{n+1} = m_n$.

2) If $x_n = a_n \neq y_n$: consider separately M_{n+1} and m_{n+1}.
For M_{n+1}, we have

$$y_{n+1} \leq \frac{a_n + y_n + 1}{2} \leq \frac{M_n + M_n + 1}{2} = M_n + \frac{1}{2},$$

which means that $y_{n+1} \leq M_n$, as both M_n, y_{n+1} are integers. Together with the note, we obtain $M_{n+1} \leq M_n$. Thus (M_n) is a non-increasing sequence of natural numbers.

For m_{n+1}: if $a_n < y_n$ then $y_n \geq a_n + 1 = x_n + 1 \geq m_n + 1$, while if $a_n > y_n$ then $a_n \geq y_n + 1 \geq m_n + 1$. In both cases we always have

$$y_{n+1} \geq \frac{a_n + y_n}{2} \geq \frac{m_n + m_n + 1}{2} = m_n + \frac{1}{2},$$

which implies that $y_{n+1} \geq m_n + 1$, as both m_n, y_{n+1} are integers. From this and the note, it follows that either $m_{n+1} > m_n$ if at the n-th moment there is only one number $y_n = m_n$, or $m_{n+1} = m_n$ if there is someone else, beside Y, who has m_n sweets.

Overall, (m_n) is a non-decreasing sequence of natural numbers; moreover when $y_n = m_n < x_n$ then up to the $(n+1)$-th moment we have $y_{n+1} \geq m_n + 1$, which means that m_n loses one time, and if the process of sweets' transferring is continuing, then after a finite number of steps, the number m_n is gone, that is there is a case when (m_n) increases strictly.

Thus the sequence of natural numbers (M_n) is non-increasing, while a sequence of natural numbers (m_n) is non-decreasing and there is a moment it increases strictly. This shows that at some moment i it must be true that $M_i = m_i$, and then the number of sweets of all students (not counting sweets being on the way of transferring) are equal.

4.4.5

Consider two cases:

Case 1: There are 3^n students sitting in a circle, then after the first counting there remain 3^{n-1} students and student B who counts 1 first at the first round will count the same 1 first at the second round. So B will remain the last one.

Case 2: There are 1991 students. Since $3^6 = 729 < 1991 < 3^7 = 2187$, we'll reduce this to Case 1 till there remains 3^6 students, then student A should be the one who count 1 first among 3^6 students.

If so we need remove $1991 - 729 = 1262$ students, that corresponds to 631 groups (each group of three students two are left). So we need

$631 \cdot 3 = 1893$ students sitting before A, that is the winner should choose the 1894-th place counting clockwise from A (who counts the first 1 among 1991 students).

4.4.6

We write on each square a natural number by the following rule: in each row, from left to right, write down numbers from 1 to 1992. Then three numbers written on consecutive squares in a row are consecutive numbers, while three numbers written on consecutive squares in a column are equal. When we color a square, the number written on that square will be erased. Therefore, from the second step, we will always erase three numbers whose sum is divisible by 3. Moreover, three numbers written on the squares $(r, s), (r+1, s+1), (r+2, s+1)$ are $s, s+1, s+1$ whose sum gives the remainder 2 when divided by 3.

If we can color all squares in the rectangle, then the sum S of all numbers written on squares must be a number of the form $3a+2$. However, $S = 1991 \cdot (1+2+\cdots+1992) = 1991 \cdot 1993 \cdot 996$ is divisible by 3. This contradiction gives the negative answer to the problem.

4.4.7

We replace the sign $(+)$ by 1 and $(-)$ by -1, then the change of signs becomes the number-changes: the number assigned to the vertex A_i is changed into the product of the numbers which were assigned to A_i and A_{i+1}. Let a_i be the number assigned to A_i at the beginning and let $f_j(a_i)$ be the number assigned to A_i after j consecutive number-changes. The problem now is to prove that there exists an integer $k \geq 2$ such that $f_k(a_i) = f_1(a_i)$ for all $i = 1, \ldots, 1993$.

Note that the number-change is a mapping from the set of 1993 vertices into the set $\{\pm 1\}$, there is only a finite number of distinct number-changes. Then, by the Pigeonhole principle, there exist two integers $m > n \geq 1$ such that $f_m(a_i) = f_n(a_i)$, for all i.

(i) If $n = 1$: it is done.

(ii) If $n \geq 2$: then from $f_m(a_i) = f_n(a_i)$ it follows that

$$f_{m-1}(a_i) \cdot f_{m-1}(a_{i+1}) = f_{n-1}(a_i) \cdot f_{n-1}(a_{i+1}), \ (i = 1, \ldots, 1993),$$

and so either

$$\frac{f_{m-1}(a_i)}{f_{n-1}(a_i)} = 1, \ \forall i \leq 1993,$$

or

$$\frac{f_{m-1}(a_i)}{f_{n-1}(a_i)} = -1, \ \forall i \leq 1993.$$

Furthermore, it is easy to verify that $f_j(a_1) \cdot f_j(a_2) \cdots f_j(a_{1993}) = 1, \ \forall j = 1, \ldots, m$. Then $\frac{f_{m-1}(a_i)}{f_{n-1}(a_i)} = 1$, that is $f_{m-1}(a_i) = f_{n-1}(a_i)$ for all $i = 1, \ldots, 1993$. Continuing this process we obtain $f_{m-n+1}(a_i) = f_1(a_i)$ for all $i = 1, \ldots, 1993$ and as $m - n + 1 \geq 2$ we finish the proof.

4.4.8

Let S be a set of ordered k-tuples $(a_1, \ldots, a_k)$, where $k \leq n$, $a_i \in \{1, \ldots, n\}$, $i = 1, \ldots, k$. Denote by S_1 the set of all ordered k-tuples satisfying either (1) or (2) of the problem. Consider the following set of ordered k-tuples

$$\begin{aligned} S_2 \quad = \quad & \big\{(a_1, \ldots, a_k) : a_i < a_{i+1} \ (i = 1, \ldots, k-1) \\ & a_i \equiv i \pmod 2 \ (i = 1, \ldots, k)\big\}. \end{aligned}$$

It is clear that $S_2 \subset S$ and $S_1 = S \setminus S_2$. Then

$$|S_1| = |S| - |S_2| = \frac{n!}{(n-k)!} - |S_2|.$$

For each $(a_1, \ldots, a_k) \in S_2$ we have $a_i + i \neq a_j + j$ for all $i \neq j \in \{2, \ldots, k\}$, $a_i + i$ is even, and $a_i + i \in \{2, \ldots, n+k\}$ for all $i = 1, \ldots, k$. Denote

$$\begin{aligned} T \quad = \quad & \big\{(b_1, \ldots, b_k) : b_i \in \{2, \ldots, n+k\}, \ b_i \text{ is even } (i = 1, \ldots, k), \\ & 1 + b_i < b_{i+1} \ (i = 1, \ldots, k-1)\big\}. \end{aligned}$$

Consider a mapping $f \colon S_2 \to T$ by the rule

$$(a_1, \ldots, a_k) \in S_2 \mapsto (b_1, \ldots, b_k) = (a_1 + 1, \ldots, a_k + k) \in T.$$

It is clear that if $a, a' \in S_2$ and $a \neq a'$, then $f(a) \neq f(a')$, that is, f is injective.

We prove that f is surjective. Let $(b_1, \ldots, b_k) \in T$, consider a tuple $(b_1 - 1, \ldots, b_k - k)$. It suffices to prove that this tuple belongs to S_2, because it is clear in this case that $f(b_1 - 1, \ldots, b_k - k) = (b_1, \ldots, b_k)$.

From the assumption that b_i is even, it follows that

$$b_i - i \equiv i \pmod 2.$$

Furthermore, the assumption $1 + b_i < b_{i+1}$ $(i = 1, \ldots, k-1)$ implies that

$$b_i - i < b_{i+1} - (i+1) \ (i = 1, \ldots, k-1).$$

Finally, it is clear that $b_1 < b_2 < \cdots < b_k$. Then from $2 \le b_1$ and $b_k \le n+k$ it follows, respectively, that $i+1 \le b_i$, and $b_i \le n+i$ for all $i = 1, \ldots, k$, or equivalently

$$1 \le b_i - i \le n \ (i = 1, \ldots, k).$$

All these show that $(b_1 - 1, \ldots, b_k - k) \in S_2$.

Thus f is a bijection from S_2 onto T. Then

$$|S_2| = |T| = \binom{[\frac{n+k}{2}]}{k},$$

and so

$$|S_1| = |S| - |S_2| = \frac{n!}{(n-k)!} - \binom{[\frac{n+k}{2}]}{k}.$$

4.4.9

We first prove the following result.

Lemma. *If a triangle MNP is contained in a parallelepiped $ABCD.A'B'C'D'$ of the size $a \times b \times c$, then its area satisfies the following inequality:*

$$S \le \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

Indeed, denote by S_1, S_2, S_3 the areas of the projections of MNP on the planes $(ABCD), (ABB'A'), (DAA'D')$ respectively, and by α, β, γ the angles between (MNP) and $(ABCD), (ABB'A'), (DAA'D')$ respectively. Then

$$\begin{aligned} S_1^2 + S_2^2 + S_3^2 &= S^2\cos^2\alpha + S^2\cos^2\beta + S^2\cos^2\gamma \\ &= S^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) \\ &= S^2. \end{aligned}$$

Note also that any triangle within a rectangle of size $x \times y$ cannot have an area larger than $\frac{xy}{2}$. So

$$S^2 = S_1^2 + S_2^2 + S_3^2 \le \frac{a^2b^2 + b^2c^2 + c^2a^2}{4},$$

from which the result follows.

Now divide the cube into 36 parallelepipeds of size $\frac{1}{6} \times \frac{1}{3} \times \frac{1}{2}$. Then by the Pigeonhole principle, there exists a parallelepiped which contains at least 3 points M, N, P among 75 given points. By Lemma

$$S_{MNP} \le \frac{1}{2}\sqrt{\left(\frac{1}{18}\right)^2 + \left(\frac{1}{6}\right)^2 + \left(\frac{1}{12}\right)^2} = \frac{7}{12}.$$

Remark: If we divide a cube into $3^3 = 27$ parellelepipeds of the size $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$, then we still have $S_{MNP} < \frac{7}{12}$, but need only 55 points given.

4.4.10

Since $1 \le |a_{i+1} - a_i| \le 2n-1$, the set of numbers $|a_{i+1} - a_i|$ is exactly the set $\{1, \ldots, 2n-1\}$. Let

$$S = \sum_{i=1}^{2n-1} |a_{i+1} - a_i| + a_1 - a_{2n}.$$

We can write S as

$$S = \sum_{i=1}^{2n} \varepsilon_i a_i,$$

where $\varepsilon_i \in \{-2, 0, 2\}$ for all i. In particular, $\varepsilon_1 \in \{0, 2\}$. It is easy to check that

(1) $\displaystyle\sum_{i=1}^{2n} \varepsilon_i = 0$, and

(2) if we delete all numbers 0 in the sequence $\varepsilon_1, \ldots, \varepsilon_{2n}$ then we get an alternating sequence of $-2, 2$.

Now let $b_1, \ldots, b_n$, respectively $c_1, \ldots, c_n$ be the sequence of numbers greater than n, respectively not greater than n in the sequence $a_1, \ldots, a_{2n}$. By (1), we have

$$S = \sum_{i=1}^{2n} (a_i - n) \le 2\sum_{i=1}^{n} (b_i - n) - 2\sum_{i=1}^{n} (c_i - n)$$

$$= 2\sum_{i=1}^{n} b_i - 2\sum_{i=1}^{n} c_i = 2n^2.$$

Therefore,

$$a_1 - a_{2n} = S - \sum_{i=1}^{2n-1} |a_{i+1} - a_i| \le 2n^2 - (1 + \cdots + (2n-1)) = n.$$

Thus $a_{i+1} - a_i = n$ if and only if $\varepsilon_i = 2$ for all i with $a_i > n$, and $\varepsilon_i = -2$ for all i with $a_i \le n$. Since $\varepsilon_{2n} \in \{-2, 0\}$ and due to (1), this is equivalent to the condition $\varepsilon_i = -2$ for $i = 2, 4, \ldots, 2n$ which means that $1 \le a_{2k} \le n$ for $k = 1, \ldots, n$.

4.4.11

We prove by induction along $n \ge 2$.

For $n = 2$ it is obviously true. Suppose that the problem is true for $n = m \ge 2$. For $n = m + 1$, we have to prove that for each integer k with $2m - 1 \le k \le \frac{m(m+1)}{2}$, there exist $m+1$ distinct real numbers $a_1, \ldots, a_{m+1}$ such that among all numbers of the form $a_i + a_j$ $(1 \le i < j \le m+1)$ there are exactly k distinct numbers.

Indeed, there are two cases:

Case 1: If $3m - 3 \le k \le \frac{m(m+1)}{2}$: then $2m - 3 \le k - m \le \frac{m(m-1)}{2}$, and by the inductive hypothesis, there exist m distinct real numbers $a_1, \ldots, a_m$ such that among all numbers of the form $a_i + a_j$ $(1 \le i < j \le m)$ there are exactly $k - m$ distinct numbers. Put

$$a_{m+1} = \max_{1 \le i < j \le m} (a_i + a_j) + 1.$$

We see that numbers $a_1 + a_{m+1}, \ldots, a_m + a_{m+1}$ are distinct and not in the set $\{a_i + a_j, 1 \le i < j \le m\}$.

So $a_1, \ldots, a_{m+1}$ are $m + 1$ real numbers such that among all sums $a_i + a_j, 1 \le i < j \le m + 1$ there are exactly $(k - m) + m = k$ distinct numbers.

Case 2: If $2m - 1 \le k \le 3m - 3$: in this case real numbers $1, \ldots, m, k - m + 2$ are $m + 1$ those that satisfy the claim. Indeed:

i) Since $k \ge 2m - 1$, $k > 2m - 2 \implies k - m + 2 > m$, and so $m + 1$ real numbers said above are distinct.

ii) Denote $M = \{1, \ldots, m, k - m + 2\}$. We see that if $a, b \in M, a \ne b$, then $3 \le a + b \le k + 2$. Conversely, for each integer $c \in [3, k + 2]$ there exist $a, b \in M, a \ne b$ such that $c = a + b$. So among all sums $a + b$ with $a \ne b, a, b \in M$ there are $(k + 2) - 3 + 1 = k$ distinct numbers.

This completes the proof.

4.4.12

Let T be such a subset and s the common value for $S(i,j)$. Since there are $\binom{8}{2}$ pairs (i,j) with $1 \le i < j \le 8$, $\binom{8}{2}s$ is the number of points of intersections of the diagonals of the quadrilaterals if we allow multiple quadrilaterals. Since for every point $P \in T$ there are $\binom{4}{2}$ pairs of vertices of the octagon which are vertices of a quadrilateral having P as the intersection of its diagonals, we have

$$|T| = \frac{\binom{8}{2}}{\binom{4}{2}}s = \frac{14s}{3}.$$

Then $|T| \ge 14$. If we choose T to be the set of the intersection points of the diagonals of the 14 quadrilaterals with the following indices of vertices:

$$1234,\ 1256,\ 1278,\ 1357,\ 1368,\ 1458,\ 1467,$$

$$2358,\ 2367,\ 2457,\ 2468,\ 3456,\ 3478,\ 5678,$$

then we can check that $S(i,j) = 3$ for all $1 \le i < j \le 8$. Thus 14 is the smallest possible of $|T|$.

4.4.13

Observe that we can put 2 groups of 4 balls of the last two configurations on a 4×2 table so that every square has one ball. As we can subdivide the 2004×2006 table into 4×2 tables, we can put groups of 4 balls into this table that satisfies requirements of the problem.

We now prove that for the 2005×2006 table it is impossible. If we color all odd rows black and all even rows white, then we obtain 1003×2006 black squares and 1002×2006 white squares. Furthermore, 2 black squares and 2 white squares receive balls whenever we put a groups of 4 balls into the table. Therefore, the numbers of balls on black squares and on white squares are always the same. Assume in contrary that we can do as required, and k is a such equal number of balls on each square. Then we must have $1003 \cdot 2006 \cdot k = 1002 \cdot 2006 \cdot k$, which is impossible.

4.4.14

Let $A_1, \ldots, A_{2007}$ be vertices of the given polygon. Note that a quadrilateral satisfies the requirements of the problem if and only if its four vertices are adjacent vertices of the polygon.

Denote by $M := \{A_1, A_2, A_3, A_5, A_6, A_7, \dots, A_{2005}, A_{2006}\}$, that is, removed vertices $A_{4i}, i = 1, \dots, 501$ and A_{2007}. Obviously, $|M| = 1505$ and M does not contains any four adjacent vertices of the polygon. Also any subset of M has the same property. So $k \geq 1506$.

We prove that for any choice of 1506 vertices there will be four adjacent vertices of the polygon. Indeed, let A be a set consisting of 1506 vertices. We consider the following partition of polygon's vertices

$$\begin{aligned}
&B_1 = \{A_1, A_2, A_3, A_4\};\\
&B_2 = \{A_5, A_6, A_7, A_8\};\\
&\dots \quad \dots \quad \dots\\
&B_{501} = \{A_{2001}, A_{2002}, A_{2003}, A_{2004}\};\\
&B_{502} = \{A_{2005}, A_{2006}, A_{2007}\}.
\end{aligned}$$

Assume that A does not contains any four adjacent vertices. In this case for each $i = 1, \dots, 501$, the set B_i is not in A, that is each B_i has at least one vertex not belonging to A. Then $|A| \leq 3 \cdot 502 = 1506$. Since $|A| = 1506$, $B_{502} \subset A$ and each B_i contains exactly three elements of A.

We have $A_{2005}, A_{2006}, A_{2007} \in A$, which implies that

$$A_1 \notin A \implies A_2, A_3, A_4 \in A \implies A_5 \notin A \implies A_6, A_7, A_8 \in A$$

$$\implies \dots\dots\dots \implies A_{2002}, A_{2003}, A_{2004} \in A.$$

Then four adjacent vertices $A_{2002}, A_{2003}, A_{2004}, A_{2005} \in A$: a contradiction.

Thus $k = 1506$.

4.4.15

Let X be a set of numbers satisfying the requirements of the problem. Denote

$$\begin{aligned}
&A^* := \{a \in \mathbb{N} : \ a \text{ has no more than 2008 digits}\},\\
&A := \{a \in A^* : a \equiv 0 \pmod 9\},\\
&A_k := \{a \in A : \ \text{among digits of } a \text{ there are exactly } k \text{ digits } 9\},\\
&\quad 0 \leq k \leq 2008.
\end{aligned}$$

Consider an arbitrary $a \in A^*$. Suppose that a has m digits. Adjoining, in front of a, $2008 - m$ zero digits (which does not change the number

a at all), we obtain a representation of a with 2008 digits, denoted by $\overline{a_1a_2\dots a_{2008}}$. In this case A_k can be written as

$$\begin{aligned}A_k \;=\; &\{\overline{a_1a_2\dots a_{2008}}:\ \text{among digits } a_1,a_2,\dots,a_{2008} \text{ there are exactly}\\ &k \text{ digits 9, and } \sum_{i=1}^{2008} a_i \equiv 0 \pmod 9\},\ 0\le k\le 2008.\end{aligned}$$

Now we have $X = A\setminus(A_0\cup A_1)$. Note that $A_0, A_1\subset A$ and $A_0\cap A_1=\emptyset$, we can deduce that

$$|X| = |A| - (|A_0| + |A_1|). \tag{1}$$

Also please note the following.

Note 1. $|A_0| = 9^{2007}$.

Indeed, from definition of A_0 it follows that $\overline{a_1a_2\dots a_{2008}}\in A_0$ if and only if $a_i\in\{0,1,\dots,8\}$, $\forall i=1,\dots,2007$, and $a_{2008}=9-r$, where r is an integer from the interval $[1,9]$ with $r\equiv\sum_{i=1}^{2007} a_i \pmod 9$. So $|A_0|$ is exactly equal to the number of possible sequences of length 2007 consisting from $\{0,1,\dots,8\}$, that is $|A_0| = 9^{2007}$.

Note 2. $|A_1| = 2008\cdot 9^{2006}$.

Indeed, numbers (of the form $\overline{a_1a_2\dots a_{2008}}$) in the above-mentioned presentation from A_1 can be formed by doing consecutively two steps:

Step 1. From $\{0,1,\dots,8\}$ form a sequence of length 2007 so that the sum of its digits is divisible by 9.

Step 2. For such a sequence, write 9 either right before the first digit, or right after the last digit, or in between two consecutive digits of the considered number.

By the same argument as in the proof of Note 1, we see that there are 9^{2006} ways to do the first step, while there are 2008 ways to do the second step. Thus there are 2008.9^{2006} ways to do the two consecutive steps. Each resulting number belongs to A_1 and two resulting numbers are distinct. Therefore, $|A_1| = 2008\cdot 9^{2006}$.

From those claims, taking into account that $|A| = 1 + \dfrac{10^{2008}-1}{9}$, it follows from (1) that

$$|X| = \frac{10^{2008} - 2017\cdot 9^{2007} + 8}{9}.$$

4.5 Geometry

Plane Geometry

4.5.1

By the law of sines, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a+b+c}{\sin A + \sin B + \sin C} = \frac{2p}{\sin A + \sin B + \sin C}.$$

Hence, taking into account that

$$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2},$$
$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

we obtain

$$a = \frac{2p\sin A}{\sin A + \sin B + \sin C} = \frac{p\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{p\sin\frac{A}{2}}{\cos\frac{B}{2}\sin\frac{A+B}{2}}. \qquad (1)$$

Also the triangle's area S is computed as follows:

$$S = \frac{1}{2}ca\sin B = \frac{1}{2}a\sin B \cdot \frac{a\sin C}{\sin A} = \frac{a^2\sin B\sin C}{2\sin A}. \qquad (2)$$

Substituting the value of a from (1) into equation (2), we get

$$\begin{aligned} S &= \left(\frac{p\sin\frac{A}{2}}{\cos\frac{B}{2}\sin\frac{A+B}{2}}\right)^2 \cdot \frac{\sin B\sin(A+B)}{2\sin A} \\ &= \left(\frac{p\sin\frac{A}{2}}{\cos\frac{B}{2}\sin\frac{A+B}{2}}\right)^2 \cdot \frac{2\sin\frac{B}{2}\cos\frac{B}{2}\cdot 2\sin\frac{A+B}{2}\cos\frac{A+B}{2}}{4\sin\frac{A}{2}\cos\frac{A}{2}} \\ &= p^2\tan\frac{A}{2}\tan\frac{B}{2}\cot\frac{A+B}{2}. \end{aligned}$$

Then the numerical result is $S \approx 101$ unit area.

4.5.2

1) Suppose that at time t, the navy ship is at O', while the enemy ship is at A'. Then the navy ship already travelled a distance $OO' = ut$ and the

enemy ship did $AA' = vt$. The distance between two ships is $O'A' = d$. We have (see Fig. 4.1)

$$\begin{aligned} d^2 &= O'N^2 + NA'^2 = (OA - OM)^2 + (AA' - AN)^2 \\ &= (a - ut\cos\varphi)^2 + (vt - ut\sin\varphi)^2 \\ &= (u^2 + v^2 - 2uv\sin\varphi)t^2 - (2au\cos\varphi)t + a^2. \end{aligned}$$

So since $d > 0$, it attains minimum if and only if d^2 attains the minimum.

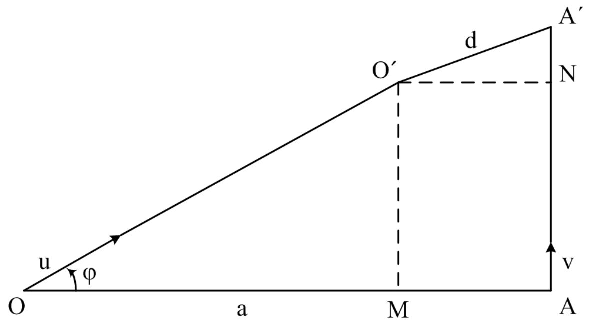

Figure 4.1:

Note that d^2 is a quadratic function in t, and the coefficient of t^2 is $u^2 + v^2 - 2uv\sin\varphi = (u - v)^2 + 2uv(1 - \sin\varphi) \geq 2uv(1 - \sin\varphi) > 0$ (as $0 < \varphi < \frac{\pi}{2}$), then d^2 attains its minimum at

$$t = \frac{au\cos\varphi}{u^2 + v^2 - 2uv\sin\varphi}$$

and the minimum value is

$$d^2_{\min} = \frac{-\Delta'}{u^2 + v^2 - 2uv\sin\varphi} = \frac{a^2(u\sin\varphi - v)^2}{u^2 + v^2 - 2uv\sin\varphi} = (d_{\min})^2,$$

which implies that

$$d_{\min} = \frac{a|u\sin\varphi - v|}{\sqrt{u^2 + v^2 - 2uv\sin\varphi}}.$$

From this it follows that $d = 0 \iff u\sin\varphi - v = 0$. As $\sin\varphi < 1$, we must have $v < u$, that is the navy ship's speed must be greater than the enemy's one. This condition is also sufficient, as in this case we suffice to choose $\sin\varphi = \frac{v}{u}$.

2) If d does not vanish, then we have $u \leq v$.

* If $u = v$: in this case

$$(d_{\min})^2 = \frac{a^2}{2}(1 - \sin\varphi) = \frac{a^2}{2}\left[1 - \cos\left(\frac{\pi}{2} - \varphi\right)\right] = a^2 \sin^2\left[\frac{1}{2}\left(\frac{\pi}{2} - \varphi\right)\right].$$

Therefore,

$$d_{\min} = a \sin\left[\frac{1}{2}\left(\frac{\pi}{2} - \varphi\right)\right].$$

It is clear that $d_{\min}$ cannot vanish, but can be as small as possible, if we choose φ close to $\frac{\pi}{2}$.

* If $u < v$: in this case

$$[(d_{\min})^2]'_\varphi = \frac{2a^2u^2\cos\varphi(v - u\sin\varphi)}{(u^2 + v^2 - 2uv\sin\varphi)^2}(u - v\sin\varphi).$$

Since $0 < \varphi < \frac{\pi}{2}$ and $v > u > u\sin\varphi$, $[(d_{\min})^2]'_\varphi = 0 \Longleftrightarrow v\sin\varphi - u = 0$, that is $\sin\varphi = \frac{u}{v}$, or $\varphi = \arcsin\frac{u}{v} := \varphi_0$.

We can see that $(d_{\min})^2$ attains its minimum when $\varphi = \varphi_0$. Then

$$t_{\min} = \frac{au\cos\varphi_0}{u^2 + v^2 - 2uv\sin\varphi_0} = \frac{au\sqrt{1 - \frac{u^2}{v^2}}}{u^2 + v^2 - 2u^2} = \frac{au}{v\sqrt{v^2 - u^2}}.$$

At this time $AA' = vt_{\min} = \frac{au}{\sqrt{v^2-u^2}} = a\tan\varphi_0$. This shows that a position A' of the enemy ship is on the line OO', along which the navy ship is running.

4.5.3

1) Let AB tangents (I, r) at F. For a right triangle AFI, we have

$$IA = \frac{IF}{\sin\frac{\alpha}{2}} = \frac{r}{\sin\frac{\alpha}{2}} = \text{const}.$$

So A is the intersection of the line x and the circle centered at I of the radius $\frac{r}{\sin\frac{\alpha}{2}}$. Then the construction is as follows (see Fig. 4.2):

* Draw an arc centered at I with radius $\frac{r}{\sin\frac{\alpha}{2}}$, that intersects the line x at A.

* Draw two tangents of the circle (I, r) from A that meet the line y at B and C.

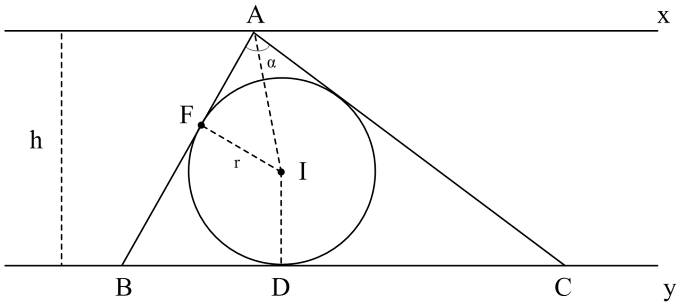

Figure 4.2:

We can have a solution if and only if the circle $(I, \frac{r}{\sin\frac{\alpha}{2}})$ intersects line x, that is, $\frac{r}{\sin\frac{\alpha}{2}} \geq h - r$, or $\sin\frac{\alpha}{2} \leq \frac{r}{h-r}$ with $h > 2r$.

2) We have $2S = ah = 2pr$, where $BC = a$, p is the half-perimeter of ΔABC. Then

$$\frac{2p}{a} = \frac{h}{r} \Longleftrightarrow \frac{a+b+c}{a} = \frac{h}{r} \Longleftrightarrow \frac{b+c}{a} = \frac{h-r}{r}.$$

Furthermore, by the law of sines

$$\frac{b+c}{a} = \frac{\sin B + \sin C}{\sin A}.$$

Therefore,

$$\begin{aligned} \frac{b+c}{a} = \frac{h-r}{r} \quad &\Longleftrightarrow \quad \frac{\sin B + \sin C}{\sin A} = \frac{h-r}{r} \\ &\Longleftrightarrow \quad \sin B + \sin C = \frac{h-r}{r} \sin A \\ &\Longleftrightarrow \quad \sin\frac{B+C}{2}\cos\frac{B-C}{2} = \frac{h-r}{r}\sin\frac{A}{2}\cos\frac{A}{2} \\ &\Longleftrightarrow \quad \cos\frac{B-C}{2} = \frac{h-r}{r}\sin\frac{A}{2}. \end{aligned}$$

That is

$$\cos\frac{B-C}{2} = \frac{h-r}{r}\sin\frac{\alpha}{2}.$$

Put $\cos\frac{B-C}{2} = \cos\varphi$ with $0 \leq \varphi \leq 90°$, we find that $\frac{B-C}{2} = \pm\varphi$. Taking into account that $B + C = 180° - A = 180° - \alpha$, we arrive at $B = 90° - \frac{\alpha}{2} \pm \varphi, C = 90° - \frac{\alpha}{2} \mp \varphi$.

3) We can easily see that $DB = p - b, DC = p - c$, and so

$$DB \cdot DC = (p-b)(p-c).$$

Note that the area S of the triangle ABC can also be computed by the Hero's formula. We then have

$$S^2 = p(p-a)(p-b)(p-c) = p^2 r^2,$$

from which it follows that

$$DB \cdot DC = \frac{pr^2}{p-a} = r^2 \frac{p}{p-a}.$$

On the other hand, since $2S = ah$, we have

$$ah = 2pr \Longleftrightarrow \frac{p}{h} = \frac{a}{2r} = \frac{p-a}{h-2r},$$

which gives

$$\frac{p}{p-a} = \frac{h}{h-2r}.$$

Thus

$$DB \cdot DC = r^2 \frac{h}{h-2r},$$

which is a constant.

4.5.4

1) Note that (see Fig. 4.3) the quadrilaterals $HMQS$ and $HMRP$ are cyclic, as $\widehat{HMS} = \widehat{HQS} = 90°$ and $\widehat{HMR} = \widehat{HPR} = 90°$, respectively. So $\widehat{HSQ} = \widehat{HRP}$.

But $\widehat{HRP} = 180° - \widehat{ARH}$, and hence $\widehat{HSQ} = 180° - \widehat{ARH}$, or equivalently, $\widehat{HSQ} + \widehat{ARH} = 180°$.

Thus the quadrilateral $ARHS$ is cyclic and the claim follows.

2) The quadrilateral AR_1HS_1 is cyclic (see Fig. 4.4), and so $\widehat{AR_1H} = \widehat{AS_1H}$.

Consider two similar right triangles HPR_1 and HQS_1, we have

$$\frac{PR_1}{QS_1} = \frac{HP}{HQ}.$$

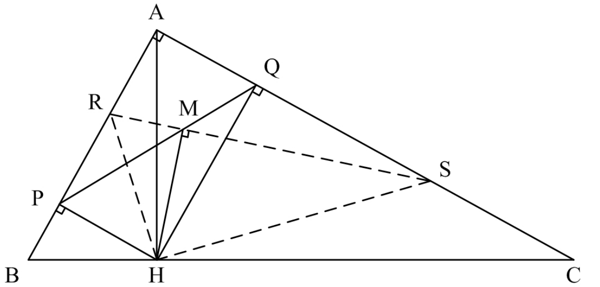

Figure 4.3:

Similarly, we also have

$$\frac{PR}{QS} = \frac{HP}{HQ}.$$

Therefore,

$$\frac{HP}{HQ} = \frac{PR_1}{QS_1} = \frac{PR}{QS} = \frac{PR_1 - PR}{QS_1 - QS} = \frac{RR_1}{SS_1}.$$

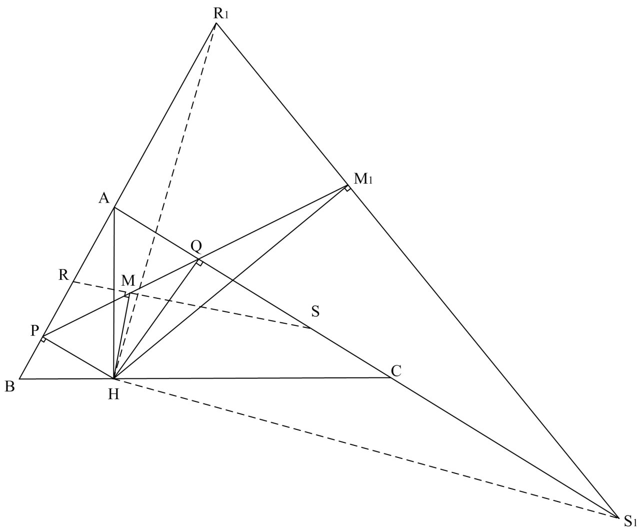

Figure 4.4:

Note that $\widehat{A} = 90^\circ$ and the triangle ABC is fixed, then H, P, Q are fixed. So we see that the ratio $\frac{HP}{HQ}$ is constant, that is, the ratio $\frac{RR_1}{SS_1}$ is constant too.

3) By symmetry, we have $\widehat{RKS} = \widehat{RHS}$ (see Fig. 4.5).

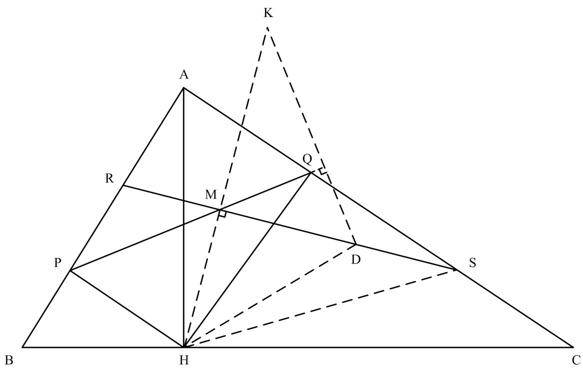

Figure 4.5:

Note that for a cyclic quadrilateral $ARHS$ we have $\widehat{RHS} = \widehat{RAS} = 90°$, which shows that K is on the circum-circle the quadrilateral $ARHS$. So the quadrilateral $ARHK$ is cyclic. We then see that $\widehat{PRH} = \widehat{AKH}$. But $\widehat{PRH} = \widehat{PMH}$ as the quadrilateral $HPRM$ is cyclic, and so $\widehat{PMH} = \widehat{AKH}$. This implies that $AK \parallel PM$.

By the assumption, $KD \perp PQ$ as $Q \in PM$, and so $KD \perp AK$, or $\widehat{AKD} = 90°$, that is $\widehat{DKR} + \widehat{AKR} = 90°$. We also have $\widehat{BHR} + \widehat{AHR} = 90°$. As noted before, the quadrilateral $ARHK$ is cyclic and hence $\widehat{AKR} = \widehat{AHR}$. Therefore, $\widehat{DKR} = \widehat{BHR}$. Note also that K is symmetric to H with respect to the line RS and so $\widehat{DKR} = \widehat{DHR}$. Thus $\widehat{DHR} = \widehat{BHR}$.

Similarly, as $\widehat{BHR} + \widehat{CHS} = 90°$, and so $\widehat{DHR} + \widehat{CHS} = 90°$. Taking into account the earlier result that the quadrilateral $ARHS$ is cyclic, which gives that $\widehat{DHR} + \widehat{DHS} = 90°$, we arrive to $\widehat{DHS} = \widehat{CHS}$.

4.5.5

We have

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{12}{7}$$

and

$$\sin A \sin B \sin C = 8 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{12}{25}.$$

These equalities give

$$\begin{cases}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=0.1\\ \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=0.6.\end{cases}$$

Furthermore, since

$$\sin\frac{C}{2}=\cos\frac{A+B}{2}=\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2},$$

multiplying both sides by $\sin\frac{C}{2}\cos\frac{C}{2}$ we get

$$\sin^2\frac{C}{2}\cos\frac{C}{2}=0.6\sin\frac{C}{2}-0.1\cos\frac{C}{2},$$

or equivalently,

$$(1-t^2)t=0.6\sqrt{1-t^2}-0.1t\Longleftrightarrow 11t-10t^3=6\sqrt{1-t^2},$$

where $t=\cos\frac{C}{2}$. This equation gives three values of $\cos\frac{C}{2}$: $\sqrt{\frac{1}{2}},\sqrt{\frac{4}{5}},\sqrt{\frac{3}{10}}$, and so the corresponding values of $\sin C$ are $1, 0.8, 0.6$.

Thus we obtain infinitely many of the “Egyptian triangles”.

4.5.6

Let AX be the desired segment line (see Fig. 4.6).

We have

$$\frac{S_1}{S_2}=\frac{2p_1}{2p_2},$$

where S_1,p_1 and S_2,p_2 are area and half-perimeter of triangles ABX and ACX, respectively. Applying the formula $S=pr$ gives $r_1=r_2$. So the problem reduces to constructing AX so that radii of in-circles of ΔABX and ΔACX are equal. The centers I_1,I_2 of those circles are on the bisectors of $\widehat{B},\widehat{C}$ respectively, and $I_1I_2\parallel BC$ (as $r_1=r_2$). Moreover, $\widehat{I_1AI_2}=\frac{1}{2}\widehat{BAC}$. From this it follows the construction:

* Let I be the intersection point of the bisectors of $\widehat{B}$ and $\widehat{C}$. In the triangle BIC draw arbitrarily $I_1'I_2'\parallel BC$, where $I_1'\in BI, I_2'\in CI$.

* Draw the arc of $\frac{1}{2}\widehat{BAC}$ over $I_1'I_2'$. Join IA to meet this arc at A'. Note that two triangles $AI_1'I_2'$ and $A'I_1I_2$ have the property that $\frac{IA}{IA'}=\frac{II_1}{II_1'}=\frac{II_2}{II_2'}$, that is these triangles are in the homothety with center I, and so draw a triangle AI_1I_2.

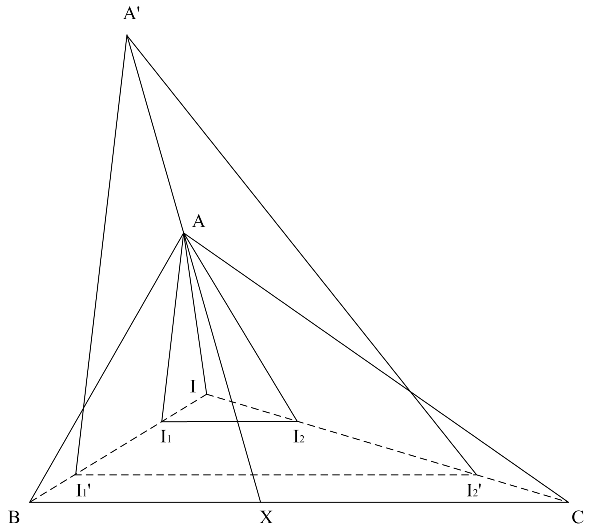

Figure 4.6:

* Draw two circles centered at I_1, I_2 tangent to AB, AC, respectively. The tangent AX to the first circle gives $\widehat{I_1AX} = \frac{1}{2}\widehat{BAX}$.

Therefore,

$$\widehat{XAI_2} = \widehat{I_1AI_2} - \widehat{I_1AX} = \frac{1}{2}\widehat{BAC} - \frac{1}{2}\widehat{BAX} = \frac{1}{2}\widehat{XAC},$$

which shows that AI_2 is the bisector of $\widehat{XAC}$, that is AX tangents to the second circle. So the point X is desired one.

4.5.7

First we can easily prove that Δ and Δ' are equilateral triangles.

Next consider a triangle O_1AO_2 (see Fig. 4.7), we have

$$\begin{aligned} O_1O_2^2 &= O_1A^2 + O_2A^2 - 2O_1A \cdot O_2A \cdot \cos\widehat{O_1AO_2} \\ &= \left(\frac{c\sqrt{3}}{3}\right)^2 + \left(\frac{b\sqrt{3}}{3}\right)^2 - 2\frac{c\sqrt{3}}{3} \cdot \frac{b\sqrt{3}}{3} \cdot \cos(A + 60^\circ) \\ &= \frac{1}{3}[c^2 + b^2 - 2bc\cos(A + 60^\circ)]. \end{aligned}$$

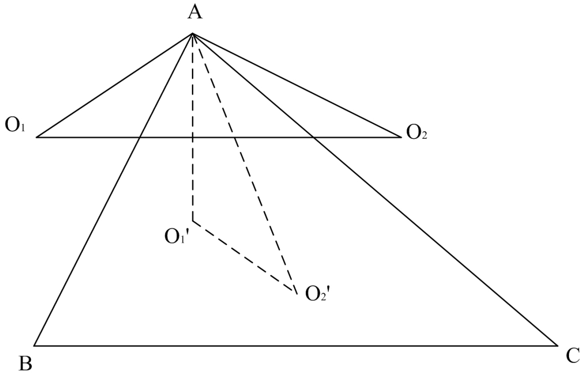

Figure 4.7:

So the area of Δ is

$$S_{\Delta} = \frac{\sqrt{3}}{12}[c^2 + b^2 - 2bc\cos(A + 60°)].$$

Similarly, the area of Δ' is

$$S_{\Delta'} = \frac{\sqrt{3}}{12}[c^2 + b^2 - 2bc\cos(A - 60°)].$$

Hence,

$$\begin{aligned} S_{\Delta} - S_{\Delta'} &= \frac{\sqrt{3}}{12} \cdot 2bc \cdot [\cos(A - 60°) - \cos(A + 60°)] \\ &= \frac{\sqrt{3}}{12} \cdot 2bc \cdot 2\sin A \sin 60° \\ &= \frac{1}{2} bc \sin A \\ &= S_{ABC}. \end{aligned}$$

4.5.8

Let S and S_1 be the areas of triangles ABC and DEF respectively. Note that the three quadrilaterals $AFME, BDMF, CEMD$ are cyclic (see Fig. 4.8). We then have

$$S_1 = \frac{1}{2} FE \cdot FD \cdot \sin \widehat{DFE} = \frac{1}{2}(MA \cdot \sin A) \cdot (MB \cdot \sin B) \cdot \sin \widehat{DFE}. \quad (1)$$

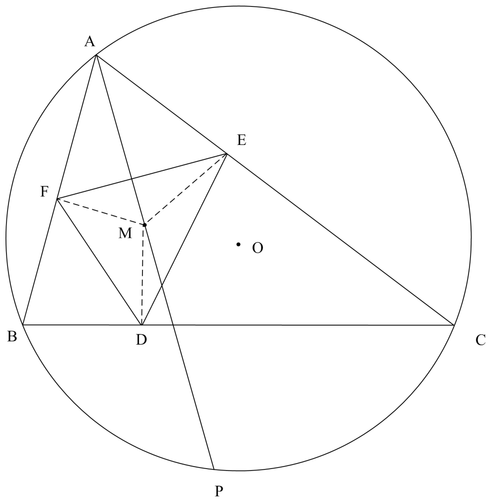

Figure 4.8:

Let P be the intersection of AM and the circum-circle of ABC. Consider the circum-circles of $BDMF, AFME$ and ABC. We have

$$\widehat{MFE} = \widehat{MAE} = \widehat{PBC},$$

and

$$\widehat{MFD} = \widehat{MBD}.$$

From these equalities it follows that

$$\widehat{DFE} = \widehat{MFE} + \widehat{MFD} = \widehat{PBC} + \widehat{MBD} = \widehat{MBP}.$$

Also in the triangle MBP we have

$$\frac{MB}{\sin \widehat{MPB}} = \frac{MP}{\sin \widehat{MBP}},$$

or equivalently,

$$\frac{MB}{\sin C} = \frac{MP}{\sin \widehat{MBP}}.$$

Hence,

$$\sin \widehat{DFE} = \frac{MP \cdot \sin C}{MB}.$$

So (1) becomes

$$\begin{aligned} S_1 &= \frac{1}{2} MA \cdot MP \cdot \sin A \sin B \sin C \\ &= \frac{1}{2}\big(- \mathcal{P}_M(O)\big) \cdot \frac{S}{2R^2} \\ &= \frac{S}{4R^2}(R^2 - OM^2) \\ &= k \quad \text{(a given constant)}, \end{aligned}$$

where S is the area of ABC.

Thus

$$\frac{S}{4}\left(1 - \frac{OM^2}{R^2}\right) = k,$$

which implies that OM^2 is constant. So M is on the circle centered at O.

* If $k = \frac{S}{4}$, then $OM = 0$, we get one point.

* If $k > \frac{S}{4}$, then $OM^2 < 0$, in which there is no such point.

* If $k < \frac{S}{4}$, then $OM^2 > 0$, the locus of M is the circle of radius $OM = R\sqrt{\frac{S-4k}{S}}$.

4.5.9

Consider two circles (c) and (C) inscribed in and circumscribed about the square $ABCD$ (see Fig. 4.9).

It is obvious that their centers are the same as the center of the square, and the radii are 1 and $\sqrt{2}$ respectively. When A moves on (C) counter-clockwise, B also moves on (C) by the same direction, and so AB always tangents to (c).

Let S be the area of the region bounded by (c) and (C), S_0 the area of the region bounded by BC and the small arc $\frown BC$. Then we have

$$S = \pi(\sqrt{2})^2 - \pi.1^2 = 2\pi - \pi = \pi,$$

and

$$S_0 = \frac{1}{4}\left(\pi(\sqrt{2})^2 - 4\right) = \frac{1}{4}(2\pi - 4) = \frac{\pi}{2} - 1.$$

When $A \equiv C$ then $B \equiv D$. In this case the area S^* of the region that AB formed by moving satisfies

$$S^* < S - S_0 = \pi - \left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2} + 1 < \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}.$$

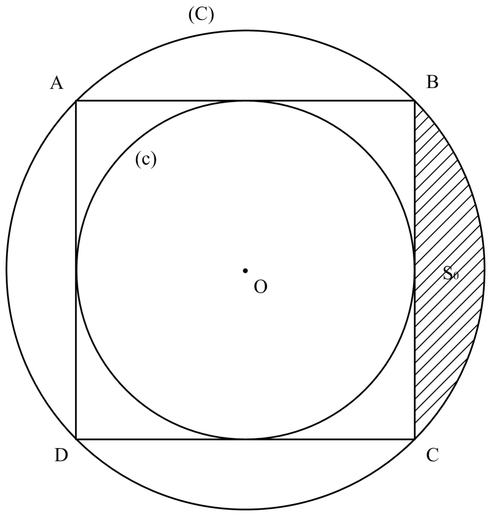

Figure 4.9:

4.5.10

Consider the case ABC is acute triangle (see Fig. 4.10). We have $\frac{MA}{MB} = \frac{MB'}{MA'}$, which shows that two triangles MAB' and MBA' are similar. Then $\widehat{MAB'} = \widehat{MBA'}$. Similarly, $\widehat{MBC'} = \widehat{MCB'}, \widehat{MCA'} = \widehat{MAC'}$.

From these three equalities, it follows, by the law of sines, that circum-circles of the triangles MBC, MCA and MAB are equal. These circles coincide if M is outside of ABC, and they are distinct if M is inside of ABC.

In the first case, M is on the circum-circle of ABC. Let's consider the second case. Denote by O_1, O_2 and O_3 the centers of the three circles said above. Then quadrilaterals MO_2AO_3, MO_3BO_1 and MO_1CO_2 are rhombuses, and so quadrilaterals BCO_2O_3, CAO_3O_1 and ABO_1O_2 are parallelograms.

Furthermore, note that AM, BM and CM are perpendicular to O_2O_3, O_3O_1 and O_1O_2, respectively. So AM, BM, CM are perpendicular to the sides BC, CA and AB of the triangle ABC, respectively. This shows that M is the orthocenter H of ABC.

Conversely, if either M is on circum-circle of ABC, or is the orthocenter of ABC, then we can easily verify that $MA{\cdot}MA' = MB{\cdot}MB' = MC{\cdot}MC'$.

The cases when ABC is right or obtuse we also have the same conclusion.

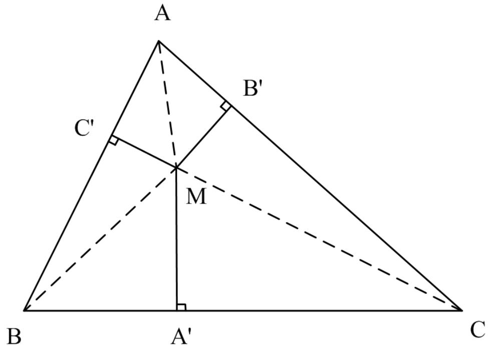

Figure 4.10:

So we conclude that the locus of all points M consists of the circum-circle of the triangle ABC and a separate point which is the orthocenter H of ABC. Note that this point H belongs to the circum-circle of ABC if and only if ABC is the triangle right angled at H.

4.5.11

1) Let M be the midpoint of the side BC. Put $AM = m_a$ and consider the power of M with respect to the circum-circle (O) (see Fig. 4.11). We have

$$MA \cdot MD = MB \cdot MC \Longleftrightarrow m_a \cdot MD = \frac{a^2}{4} \Longleftrightarrow MD = \frac{a^2}{4m_a}.$$

From this it follows, by the Arithmetic-Geometric Mean inequality, that

$$GD = GM + MD = \frac{m_a}{3} + \frac{a^2}{4m_a} \geq 2\sqrt{\frac{m_a}{3} \cdot \frac{a^2}{4m_a}} = \frac{a}{\sqrt{3}},$$

or equivalently,

$$\frac{1}{GD} \leq \frac{\sqrt{3}}{a} = \frac{\sqrt{3}}{BC}.$$

Similarly,

$$\frac{1}{GE} \leq \frac{\sqrt{3}}{CA}, \ \frac{1}{GF} \leq \frac{\sqrt{3}}{AB}.$$

Summing up the three inequalities yields

$$\frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \leq \sqrt{3}\left(\frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}\right).$$

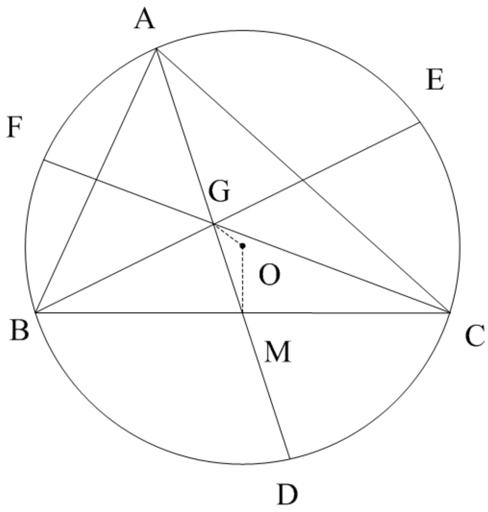

Figure 4.11:

The equality occurs if and only if the triangle ABC is equilateral.

2) Let R be the radius of circum-circle (O) of ABC. We always have

$$GA^2 + GB^2 + GC^2 = 3(R^2 - GO^2).$$

Indeed, from $\overrightarrow{GA} = \overrightarrow{GO} + \overrightarrow{OA}$ if follows that $GA^2 = GO^2 + R^2 + 2\overrightarrow{GO} \cdot \overrightarrow{OA}$. It is similar for $\overrightarrow{GB}$ and $\overrightarrow{GC}$. Hence,

$$\begin{aligned} GA^2 + GB^2 + GC^2 &= 3GO^2 + 3R^2 + 2\overrightarrow{GO} \cdot (\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) \\ &= 3GO^2 + 3R^2 + 2\overrightarrow{GO} \cdot 3\overrightarrow{OG} \\ &= 3GO^2 + 3R^2 - 6GO^2 \\ &= 3(R^2 - GO^2). \end{aligned}$$

Also, we note that $GA \cdot GD = GB \cdot GE = GC \cdot GF = -\mathcal{P}_G(O) = R^2 - GO^2$. Therefore

$$\frac{GA^2}{GA \cdot GD} + \frac{GB^2}{GB \cdot GE} + \frac{GC^2}{GC \cdot GF} = \frac{GA^2 + GB^2 + GC^2}{R^2 - OG^2} = 3,$$

that is

$$\frac{GA}{GD} + \frac{GB}{GE} + \frac{GC}{GF} = 3 \Longleftrightarrow \frac{AD}{GD} + \frac{BE}{GE} + \frac{CF}{GF} = 6.$$

But all $AD, BE, CF \leq 2R$, and so from the last equality it follows that

$$\frac{2R}{GD} + \frac{2R}{GE} + \frac{2R}{GF} \geq 6,$$

or equivalently,

$$\frac{1}{GD} + \frac{1}{GE} + \frac{1}{GF} \geq \frac{3}{R}.$$

The equality occurs if and only if the triangle ABC is equilateral.

4.5.12

Suppose that $\mathcal{H}$ has $AB = CD = a < AD = BC = b$. Denote by I the intersection point of diagonals AC and BD, put $\widehat{CID} = \varphi < 45^\circ$, by the assumption of the problem. It is easy to verify that

$$2a < b. \tag{1}$$

Denote by A', B', C', D' the images of A, B, C, D under the rotation $\mathcal{R}_I^x$ centered at I by an angle x. We show that the common area $S(x)$ between $\mathcal{H}$ and $\mathcal{H}_x$ is minimum when $x = 90^\circ$ or $x = 270^\circ$, and the minimum value is a^2.

Note that a rectangle has two symmetric axes, and so $S(x + 180^\circ) = S(x) = S(x - 180^\circ)$. Then it suffices to consider $S(x)$ with $0 \leq x \leq 90^\circ$.

There are three cases.

1) $x = 0^\circ$: then $\mathcal{H}_x = ABCD$ and $S(x) = ab > a^2$.

2) $0^\circ < x < \varphi$: in this case note that $\mathcal{H}_x \cap \mathcal{H}$ is the octagon $MNPQRSUV$ (see Fig. 4.12).

Since $\mathcal{R}_I^{180^\circ}$ maps the pairs $AB, A'B'$ into the pairs $CD, C'D'$, it also maps M into R. This implies that M, I and R are collinear, which give

$$\begin{aligned}
S(x) &= S(MNPQRSUV) = 2S(MNPQR) \\
&= 2S(IMN) + 2S(INP) + 2S(IPQ) + 2S(IQR) \\
&= \frac{b}{2}MN + \frac{a}{2}NP + \frac{a}{2}PQ + \frac{b}{2}QR \\
&= \frac{b-a}{2}MN + \frac{a}{2}(MN + NP + PQ) + \frac{b}{2}QR \\
&> \frac{a}{2}(MN + NP + PQ) > \frac{a}{2}(BN + NP + PC) \\
&= \frac{ab}{2} > a^2.
\end{aligned}$$

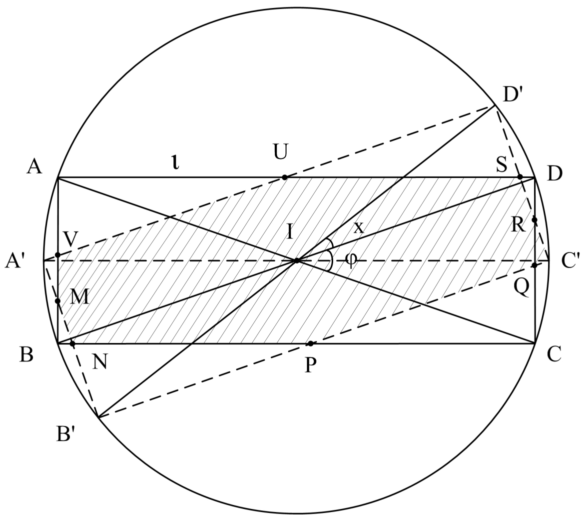

Figure 4.12:

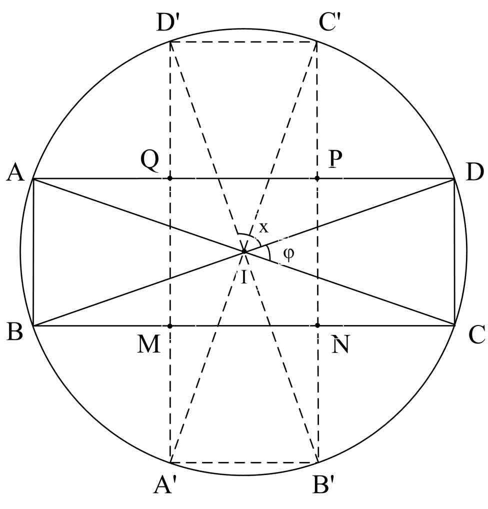

Figure 4.13:

3) $\varphi \leq x \leq 90^\circ$: in this case $\mathcal{H}_x \cap \mathcal{H}$ is a parallelogram $MNPQ$ (see Fig. 4.13).

Then we have

$$S(x) = S(MNPQ) = AB \cdot MN \geq a^2.$$

The equality occurs if and only if $MN = A'B'$, which in turn is equivalent to $x = 90^\circ$.

Taking into account the two symmetric axes said above, we conclude that $S(\mathcal{H}_x \cap \mathcal{H}) \geq a^2$ always and the minimum a^2 is attained if and only if either $x = 90^\circ$ or $x = 270^\circ$.

4.5.13

Note that $\widehat{B'AC'}$ is equal to either $3\widehat{A}$, or $2\pi - 3\widehat{A}$, or $3\widehat{A} - 2\pi$. It is similar to $\widehat{C'BA'}$ and $\widehat{A'CB'}$. In all cases we always have

$$\cos\widehat{B'AC'} = \cos 3A,\ \cos\widehat{C'BA'} = \cos 3B,\ \cos\widehat{A'CB'} = \cos 3C.$$

For the triangles $A'B'C$ and ABC, by the law of cosines, we have (see Fig. 4.14)

$$\begin{aligned} A'B'^2 &= a^2 + b^2 - 2ab\cos 3C \\ &= c^2 + 2ab\cos C - 2ab\cos 3C \\ &= c^2 + 4ab\sin 2C\sin C \\ &= c^2 + 8ab\cos C\sin^2 C \end{aligned}$$

$$\begin{aligned} &= c^2 + 4(a^2 + b^2 - c^2)\sin^2 C \\ &= c^2 + (a^2 + b^2 - c^2) \cdot \frac{4R^2\sin^2 C}{R^2} \\ &= \frac{c^2}{R^2}(R^2 + a^2 + b^2 - c^2). \end{aligned}$$

Similarly,

$$B'C'^2 = \frac{a^2}{R^2}(R^2 + b^2 + c^2 - a^2);\ C'A'^2 = \frac{b^2}{R^2}(R^2 + c^2 + a^2 - b^2).$$

So the triangle $A'B'C'$ is equilateral if and only if

$$a^2(R^2 + b^2 + c^2 - a^2) = b^2(R^2 + c^2 + a^2 - b^2) = c^2(R^2 + a^2 + b^2 - c^2)$$

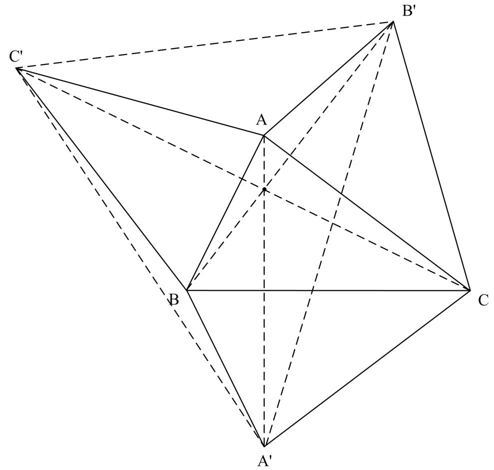

Figure 4.14:

$$\iff \begin{cases} (a^2-b^2)(R^2+c^2-a^2-b^2)=0 \\ (b^2-c^2)(R^2+a^2-b^2-c^2)=0, \end{cases}$$

which give either ABC is equilateral, or ABC is isosceles having two 75° angles, or ABC is isosceles having two 15° angles.

4.5.14

Let p be the perimeter of a quadrilateral $ABCD$. Put $\widehat{AOB}=2x, \widehat{BOC}=2y, \widehat{COD}=2z, \widehat{DOA}=2t$ (see Fig. 4.15), we have $0<x,y,z,t<\frac{\pi}{2}$ and $x+z=y+t=\frac{\pi}{2}$.

By the law of sines applying to the triangle ABC, we see that $AB = 2a\sin\widehat{ACB} = 2a\sin x$. It is similar to BC, CD and DA. So we obtain

$$p = 2a(\sin x + \sin y + \sin z + \sin t),$$

where x and y are related by $\sin 2x \cdot \sin 2y = \frac{b^2}{a^2}$. The problem is reduced to finding the maximum and minimum of

$$f(x,y) = f(y,x) = \sin x + \cos x + \sin y + \cos y.$$

Note that $PA\cdot PC = PB \cdot PD = -\mathcal{P}_P(O) = a^2-d^2 := b^2$. After some calculations we can get

$$\max f(x,y) = \sqrt{2} + \sqrt{1+\frac{b^2}{a^2}};\ \min f(x,y) = 2\sqrt{1+\frac{b}{a}},$$

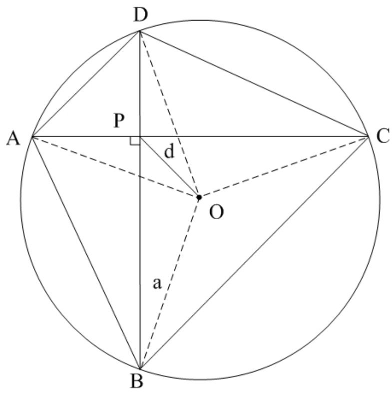

Figure 4.15:

which give

$$p_{\max} = 2(a\sqrt{2} + \sqrt{a^2+b^2}),$$

if and only if $AC = BD = \sqrt{2(a^2+b^2)}$, and

$$p_{\min} = 4\sqrt{a(a+b)},$$

if and only if the longest diagonal of $ABCD$ is the diameter of the circle (O, a) passing through P.

4.5.15

Let I be the in-center of ABC (see Fig. 4.16). Then AA', BB', CC' meet at I. Since $\widehat{C'B'B} = \widehat{C'CB} = \widehat{ACC'}$, a quadrilateral $IQB'C$ is cyclic. Then we have $\widehat{QIB'} = \widehat{ABB'}(= \widehat{ACB'})$, which gives $IQ \parallel AB$. Similarly, $IM \parallel AB$, also $IP \parallel BC$ and $IS \parallel BC$, $IN \parallel CA$ and $IR \parallel CA$.

Thus we get four similar triangles IMN, QIP, RSI and ABC, and three rhombuses $IQAR, ISBM$ and $INCP$. Then we obtain

$$\frac{MN}{BC} = \frac{IM}{AB} = \frac{IN}{AC} = \frac{MN+IM+IN}{BC+AB+CA} = \frac{MN+BM+NC}{2p} = \frac{BC}{2p},$$

which implies that $MN = \frac{BC^2}{2p}$.

Similarly,

$$PQ = \frac{CA^2}{2p}, RS = \frac{AB^2}{2p}.$$

From this the result follows.

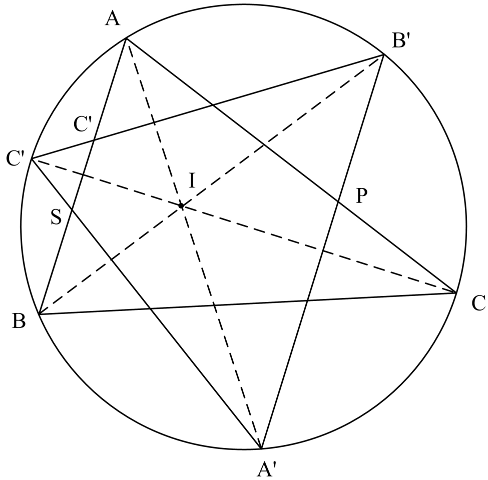

Figure 4.16:

4.5.16

There are two cases:

1) $R_1 = R_2$: in this case $M_1 \equiv O_1, M_2 \equiv O_2$. So N_1, N_2 are the diametrically opposite points of A on $(O_1), (O_2)$ respectively. Then $\widehat{ABN_1} = \widehat{ABN_2} = 90°$, and therefore, the three points N_1, B, N_2 are collinear.

2) $R_1 \neq R_2$, say $R_1 > R_2$ (see Fig. 4.17):

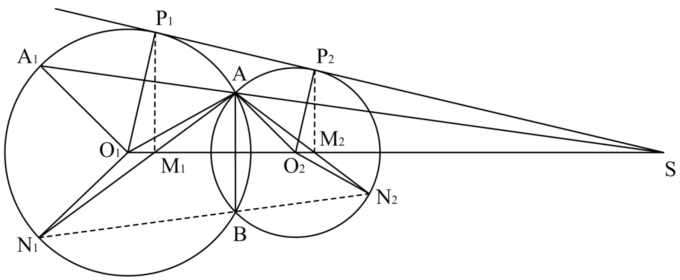

Figure 4.17:

In this case the lines O_1O_2 and P_1P_2 meet at a point S so that O_2 is in between O_1 and S, as well as P_2 is in between P_1 and S. Then we have

$$\widehat{N_1BA}+\widehat{N_2BA}=(180^\circ-\frac{1}{2}\widehat{N_1O_1A})+\frac{1}{2}\widehat{N_2O_2A}, \tag{1}$$

where $\widehat{N_1O_1A}<180^\circ$.

Let A_1 be the second intersection point of SA and (O_1). We see that S is the center of a homothety that maps (O_1) to (O_2), where A_1, O_1, M_1 are mapped to A, O_2, M_2 in such an order. Hence,

$$\widehat{O_1A_1M_1}=\widehat{O_2AM_2}. \tag{2}$$

Note that $SA\cdot SA_1=\mathcal{P}_S(O_1)=SO_1^2-R_1^2=SP_1^2=SO_1\cdot SM_1$, which implies that four points A, M_1, O_1, A_1 lie on the same circle. Then $\widehat{O_1A_1M_1}=\widehat{O_1AM_1}$, which together with (2) gives $\widehat{O_1AM_1}=\widehat{O_2AM_2}$. Thus

$$\widehat{N_1O_1A}=\widehat{N_2O_2A}. \tag{3}$$

From (1) and (3) it follows that $\widehat{N_1BA}+\widehat{N_2BA}=180^\circ$, that is the three points N_1, B, N_2 are collinear.

4.5.17

We consider the case where two circles tangent externally, as the case of internal tangency is similar.

Let xy be a common tangent line of two given circles (see Fig. 4.18).

Since CA and My are tangents to (O_1) at C and M respectively, $\widehat{FCA}=\widehat{CMy}$. Moreover, $\widehat{CMy}=\widehat{FMx}$, and $\widehat{FMx}=\widehat{FAM}$ (as Mx is tangent to (O_2) at M). Hence, $\widehat{FCA}=\widehat{FAM}$. Furthermore, $\widehat{MFA}=\widehat{AFC}$. These equalities show that the two triangles AFC and MFA are similar. Then

$$\frac{FM}{FA}=\frac{FA}{FC} \implies FM\cdot FC=FA^2.$$

Note that $FM\cdot FC=\mathcal{P}_F(O_1)=FO_1^2-R_1^2$, and so $FA^2=FO_1^2-R_1^2$, or equivalently, $FO_1^2-FA^2=R_1^2$.

Similarly, $EO_1^2-EA^2=R_1^2$. Therefore,

$$FO_1^2-FA^2=EO_1^2-EA^2=R_1^2.$$

By the assumption, D is on the line EF. Then we also get $DO_1^2-DA^2=R_1^2$, or equivalently,

$$DO_1^2-R_1^2=DA^2. \tag{1}$$

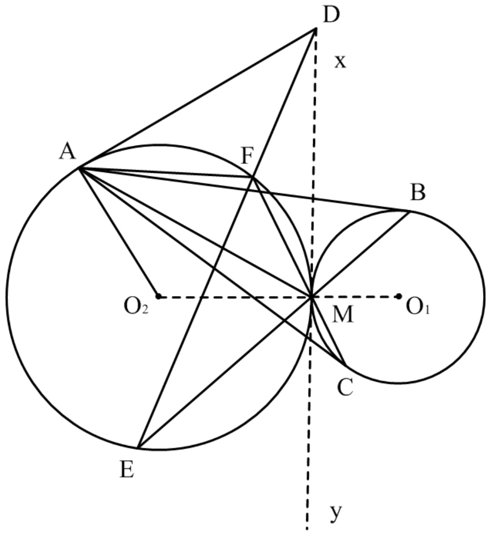

Figure 4.18:

Since DA is tangent to (O_2) at A,

$$DA^2 = \mathcal{P}_D(O_2) = DO_2^2 - R_2^2. \tag{2}$$

From (1) and (2) it follows that

$$DO_1^2 - R_1^2 = DO_2^2 - R_2^2 \iff \mathcal{P}_D(O_1) = \mathcal{P}_D(O_2).$$

This shows that D is on the line of equal power to the two given circles, which is fixed.

4.5.18

First we prove that K is on the circle (O) (see Fig. 4.19).

Indeed, from the assumption $\overrightarrow{AD} = \overrightarrow{PC}$ it follows that $ADCP$ is a parallelogram. Then a circum-circles of APC (which is the circum-circle of ABC) and of ADC are symmetric with respect to AC.

Furthermore, as K is the orthocenter of ADC, its symmetric point with respect to AC lies on the circum-circle ADC. This implies that the circum-circles of AKC and of ADC are symmetric with respect to AC.

Therefore, the circum-circle of ABC coincide with the circum-circle of AKC, and so K is on the circum-circle of ABC. Moreover, since ABC is acute, K is on the arc BC that contains A.

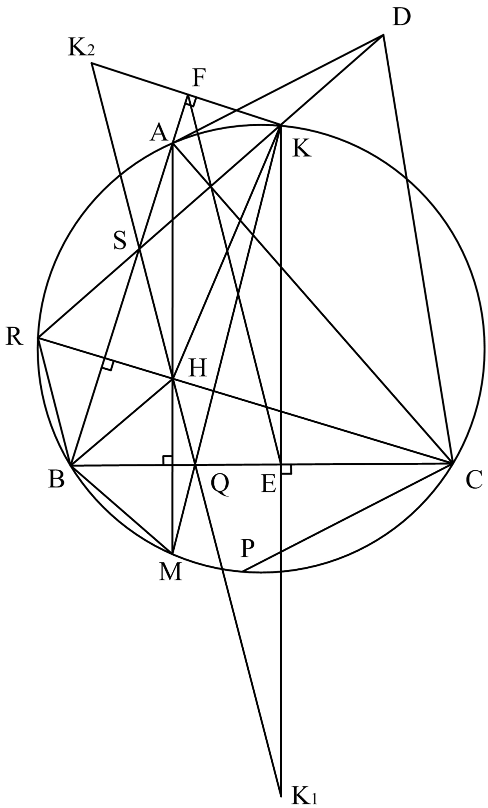

Figure 4.19:

Now let K_1, K_2 be the symmetric points of K with respect to BC, AB, respectively. Denote by M the intersection of AH and (O). Since H and M are symmetric with respect to BC, the quadrilateral HMK_1K is an isosceles trapezia, having BC as the axis of symmetry. So the intersection point Q of the diagonals HK_1 and MK is on BC. Due to symmetry, we have

$$\widehat{BHQ} = \widehat{BMQ}.$$

Similarly, let R be the intersection point of CH and (O), S the intersection point of RK and HK_2. Then S is on AB and we have

$$\widehat{BHS} = \widehat{BRS}.$$

Note also that for the cyclic quadrilateral $BRKM$, we have

$$\widehat{BMQ} + \widehat{BRS} = 180^\circ.$$

So from the last three equalities it follows that

$$\widehat{BHQ} + \widehat{BHS} = 180^\circ,$$

which shows that the three points S, H, Q are collinear. This, in turn, implies that the three points K_1, H, K_2 are collinear too.

Since EF is the median line of the triangle KK_1K_2, it passes the midpoint of the segment HK.

4.5.19

1) By assumptions of the problem the two pairs of the segments O_1C, OO_2 and O_2C, OO_1 are perpendicular to BC and CA respectively. Then OO_1CO_2 is a parallelogram, and so the midpoint I of OC is also the midpoint of O_1O_2 (see Fig. 4.20).

On the other hand, O_1O_2 is perpendicular bisector of CD and meets CD at its midpoint J. Hence, $IJ \parallel OD$ as the median line of the triangle OCD, which implies that OCD is the right triangle at D. Then $CD \leq CO = R$. It is clear that $CD_{\max} = R \Longleftrightarrow D \equiv O \Longleftrightarrow OC \perp AB$.

2) Let P be the midpoint of the arc AB of the circum-circle of AOB that does not contain O, E and F the second points of intersection of PA with (O_1) and PB with (O_2), respectively. It is easy to see that $C \in EF, EF \parallel AB$ and $AE = BF$. Therefore, $PE = PF$ (as $PA = PB$).

From this it follows that

$$PA \cdot PE = PB \cdot PF,$$

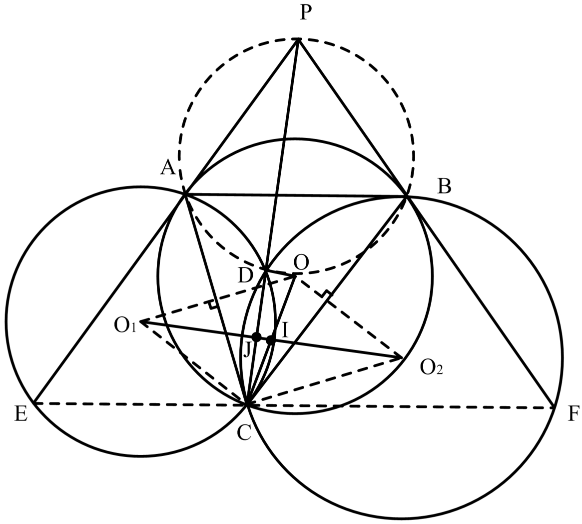

Figure 4.20:

which shows that the point P has the same power with respect to the circles (O_1) and (O_2). Hence, P must belong to the line CD.

Thus CD always passes through the fixed point P.

4.5.20

Let lines AD and BC meet at the point P.

1) If M is on the segment CD, then N is at the same side as M with respect to the line AB (see Fig. 4.21). Since the quadrilaterals $ANMD$ and $BNMC$ are cyclic,

$$\widehat{ANM} = \pi - \widehat{ADM},\ \widehat{BNM} = \pi - \widehat{BCM},$$

and so

$$\widehat{ANB} = 360^\circ - (\widehat{ANM} + \widehat{BNM}) = \widehat{ADM} + \widehat{BCM}.$$

Thus

$$\widehat{ANB} + \widehat{APB} = \pi.$$

This shows that the quadrilateral $APBN$ is cyclic, and hence N is on the fixed circle passing the triple A, B, P.

If M is outside of the segment CD, then N is on different sides of M with respect to the line AB (see Fig. 4.22).

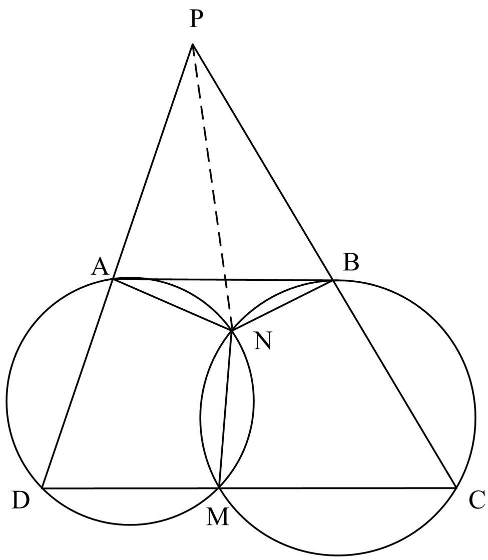

Figure 4.21:

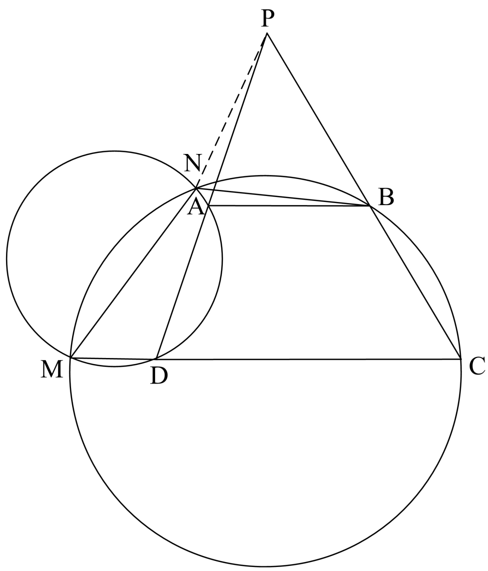

Figure 4.22:

By the similar argument, we have

$$\widehat{ANB} = \pi - (\widehat{ADC} + \widehat{BCD}) = \widehat{APB}.$$

This shows again that N is on the fixed circle passing through the triple A, B, P.

2) Since P is the intersection point of the sides of the trapezia $ABCD$,

$$PA \cdot PD = PB \cdot PC,$$

which means that P is on the axis of equal power to the two circles (AMD) and (BMC). That is the fixed point P is always on the line MN.

4.5.21

Choose the system of coordinates Oxy as follows: O is the midpoint of BC and the x-axis is the line BC (see Fig. 4.23).

Put $BC = 2a > 0$, then the coordinates of the vertices B and C are $B(-a, 0)$ and $C(a, 0)$. Suppose that $A(x_0, y_0)$ with $y_0 \neq 0$. In this case, note that $\overrightarrow{CH} \perp \overrightarrow{AB}$, or equivalently, $\overrightarrow{CH}.\overrightarrow{AB} = 0$, also $AH \perp Ox$, then coordinates of the orthocenter $H(x, y)$ are the solutions of the system

$$\begin{cases} x = x_0 \\ (x - a)(x_0 + a) + yy_0 = 0, \end{cases}$$

which gives $H\left(x_0, \dfrac{a^2 - x_0^2}{y_0}\right)$.

Note that the coordinates of the centroid G are $\left(\dfrac{x_0}{3}, \dfrac{y_0}{3}\right)$, and so the coordinates of the midpoint K of the segment HG are $\left(\dfrac{2x_0}{3}, \dfrac{3a^2 - 3x_0^2 + y_0^2}{6y_0}\right)$. Thus K belongs to the line BC if and only if

$$3a^2 - 3x_0^2 + y_0^2 = 0 \Longleftrightarrow \frac{x_0^2}{a^2} - \frac{y_0^2}{3a^2} = 1 \ (y_0 \neq 0).$$

So the locus of A is the hyperbola of the equation $\frac{x^2}{a^2} - \frac{y^2}{3a^2} = 1$ without two points B, C.

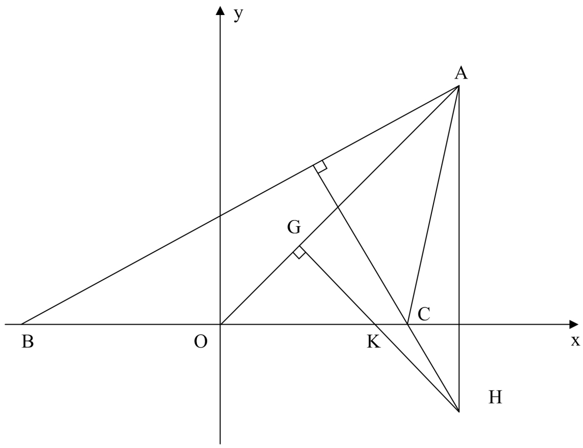

Figure 4.23:

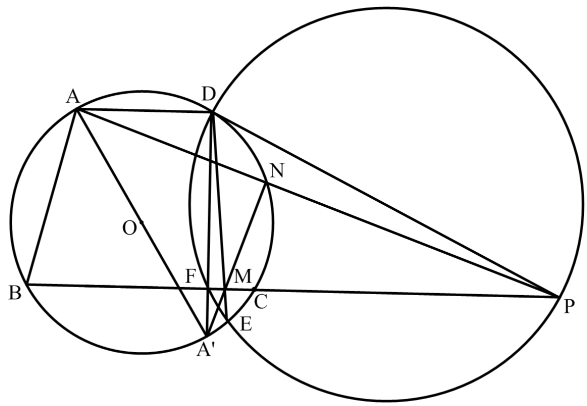

Figure 4.24:

4.5.22

Draw the diameter AA' of the circle (O). We prove that N, M, A' are collinear, from which it follows that MN passes through the fixed point A' (see Fig. 4.24).

Indeed, first note that DE is the axis of equal power of the circle (O) and the circle (C_1) of diameter PD.

Next, since $\widehat{PNA'} = 180^\circ - \widehat{ANA'} = 90^\circ$, N is on the circle (C_2) of diameter PA'. Thus the two points N and A' lie on both circles (O) and (C_2). This shows that NA' must be the axis of equal power of the circles (O) and (C_2).

Finally, denote by F the intersection point of DA' and BC. Note that $\widehat{ADA'} = 90^\circ$ and $AD \parallel BC$, we have $\widehat{PFD} = 90^\circ$, or equivalently, $\widehat{PFA'} = 90^\circ$. Thus the two points P and F lie on both circles (C_1) and (C_2). This means that PF, or the same, BC must be the axis of equal power of the circles (C_1) and (C_2).

Therefore, the three lines DE, BC and NA' meet at the center of equal power M of all the three circles $(O), (C_1)$ and (C_2). That is M, N, A' are collinear.

4.5.23

If $\alpha = 90^\circ$ then $M \equiv C$. In this case $\frac{MC}{AB} = 0 = \cos\alpha$.
Consider the case $\alpha \neq 90^\circ$. Note that if $\alpha < 90^\circ$ then M is outside the segment EC (see Fig. 4.25).

Indeed, for $\alpha < 90^\circ$ we have $AC > BC$. Assume that M belongs to the segment EC. Then M must be inside of this segment. So $\widehat{ECA} = \widehat{BME} = \widehat{ECB} + \widehat{CBM}$, which shows that $\widehat{ECA} > \widehat{ECB}$. If D is the intersection of the bisector of $\widehat{ACB}$ and the side AB, then D must be in between E and A. From this it follows that

$$1 < \frac{CA}{CB} = \frac{DA}{DB} < 1,$$

which is impossible.

Similarly, if $\alpha > 90^\circ$ then M must be in between E and C (see Fig. 4.26).

Now put $\widehat{ECA} = \beta, \widehat{MBC} = \gamma$. By the law of sines for the triangles ACE and BME, we have

$$\frac{AC}{\sin(\pi - \alpha)} = \frac{EA}{\sin\beta} = \frac{EB}{\sin\beta} = \frac{BM}{\sin\alpha},$$

which gives $AC = BM$. Furthermore, by the law of cosines for triangles BCM and ABC, we have

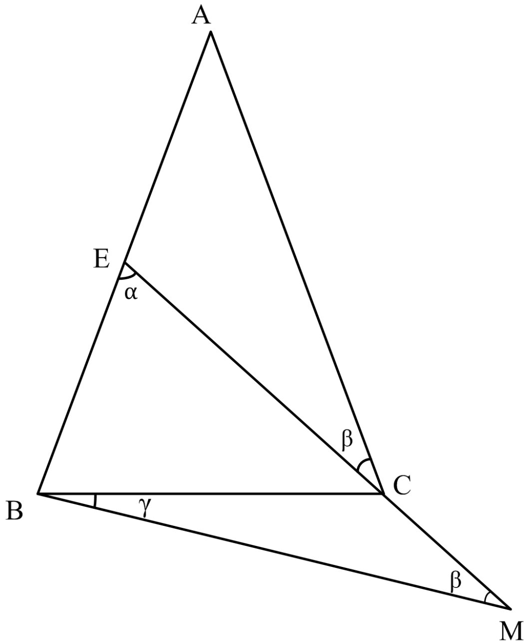

Figure 4.25:

$$\begin{aligned} MC^2 &= BC^2 + BM^2 - 2BC \cdot BM \cdot \cos\gamma \\ &= BC^2 + AC^2 - 2BC \cdot AC \cdot \cos\gamma \\ &= AB^2 + 2BC \cdot AC(\cos\widehat{ACB} - \cos\gamma) \\ &= AB^2 - 4BC \cdot AC \cdot \sin\frac{\widehat{ACB}+\gamma}{2}\sin\frac{\widehat{ACB}-\gamma}{2}. \end{aligned}$$

Note that if M is in between E and C then

$$\frac{\widehat{ACB}+\gamma}{2} = \frac{\beta + \widehat{ECB} + \gamma}{2} = \frac{\beta+\beta}{2} = \beta$$

and

$$\frac{\widehat{ACB}-\gamma}{2}=\frac{(\beta+\widehat{ECB})-(\beta-\widehat{ECB})}{2}=\widehat{ECB},$$

while if M is outside of the segment EC, then

$$\frac{\widehat{ACB}+\gamma}{2}=\frac{(\beta+\widehat{ECB})+(\widehat{ECB}-\beta)}{2}=\widehat{ECB},$$

and

$$\frac{\widehat{ACB}-\gamma}{2}=\frac{\beta+\widehat{ECB}-\gamma}{2}=\frac{\beta+\beta}{2}=\beta.$$

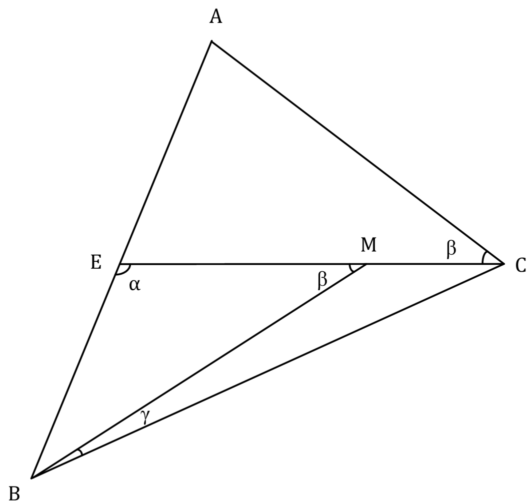

Figure 4.26:

Thus,

$$\begin{aligned} MC^2 &= AB^2-4(AC\cdot\sin\beta)\cdot(BC\cdot\sin\widehat{ECB}) \\ &= AB^2-4(EA\cdot\sin\alpha)\cdot(EB\cdot\sin\alpha) \\ &= AB^2-AB^2\sin^2\alpha \\ &= AB^2\cdot\cos^2\alpha, \end{aligned}$$

which implies that $\frac{MC}{AB}=|\cos\alpha|$.

Solid Geometry

4.5.24

Draw $SF \perp CD$ at F. We first prove that $\widehat{SFO} = \alpha$. Indeed, since OF is a perpendicular projection of SF on the base $ABCD$, $OF \perp CD$ and so $\widehat{SFO}$ is the linear angle of the dihedral of edge DC, that is $\widehat{SFO} = \alpha$ (see Fig. 4.27).

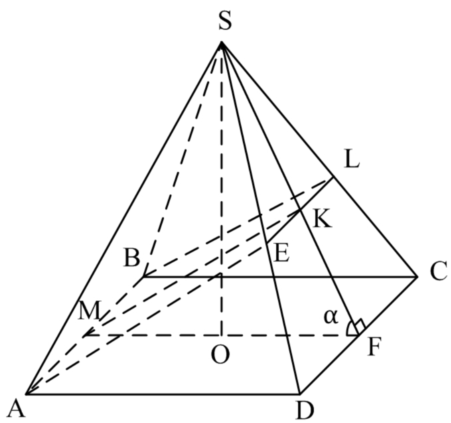

Figure 4.27:

We see that the intersection quadrilateral $ABLE$ is an isosceles trapezia and SK is the altitude of the pyramid $SABLE$. Denote by M the intersection point of FO and AB. Then $MK \perp AB$ and $MK \perp LE$. In this case, the area of the trapezia $ABLE$ is $\dfrac{1}{2}(AB + LE) \cdot MK$, and therefore the volume of the pyramid $SABLE$ is computed by

$$V = \frac{1}{6}(AB + LE) \cdot MK \cdot SK. \tag{1}$$

Also we have

$$OF = h \cot\alpha,\ SF = \frac{h}{\sin\alpha},\ AB = MF = 2OF = 2h\cot\alpha,$$

$$MK = MF \sin\alpha = 2h\cot\alpha \cdot \sin\alpha = 2h\cos\alpha,$$

$$FK = MF\cos\alpha = 2h\cot\alpha \cdot \cos\alpha.$$

Moreover, the altitude

$$SK = SF - FK = \frac{h}{\sin\alpha} - 2h\cot\alpha\cdot\cos\alpha = \frac{h}{\sin\alpha}(1-2\cos^2\alpha)$$
$$= -\frac{h\cos 2\alpha}{\sin\alpha}.$$

Finally, for the two similar triangles SEL and SDC we have

$$EL = \frac{DC\cdot SK}{SF} = \frac{2h\cot\alpha\cdot(-h\cos 2\alpha)\cdot\sin\alpha}{h\sin\alpha} = -2h\cot\alpha\cdot\cos 2\alpha.$$

Substituting all these quantities into (1), we obtain

$$V = -\frac{4}{3}h^3\cos^2\alpha\cos 2\alpha.$$

Note that V must be positive, so we have to show that $\cos 2\alpha < 0$. Indeed, for the triangle SMF we can see that $2\alpha < 180^\circ$, while for the triangle KMF we have $\widehat{KMF} = 90^\circ - \alpha$. Moreover, $\widehat{SMF} > \widehat{KMF}$, which is equivalent to $\alpha > 90^\circ - \alpha \Longleftrightarrow 2\alpha > 90^\circ$. Thus $90^\circ < 2\alpha < 180^\circ$, which shows that $\cos 2\alpha < 0$.

4.5.25

By the assumption, the triangle SBC is equilateral. Since $(SBC)\perp(ABC)$, the altitude SH of the triangle SBC is also the altitude of $SABC$, and $SH = \frac{\sqrt{3}}{2}$. Then (see Fig. 4.28)

$$AC^2 = AH^2 + HC^2 = AH^2 + \frac{1}{4}. \tag{1}$$

On the other hand,

$$AC^2 = SA^2 + SC^2 - 2SA\cdot SC\cdot\cos 60^\circ = SA^2 + 1 - SA. \tag{2}$$

From (1) and (2) it follows that $SA^2 + 1 - SA = AH^2 + \frac{1}{4}$. But

$$AH^2 = SA^2 - SH^2 = SA^2 - \frac{3}{4},$$

and so

$$SA^2 + 1 - SA = SA^2 - \frac{1}{2},$$

which gives $SA = \frac{3}{2}$, and so $AH = \frac{\sqrt{6}}{2}$.

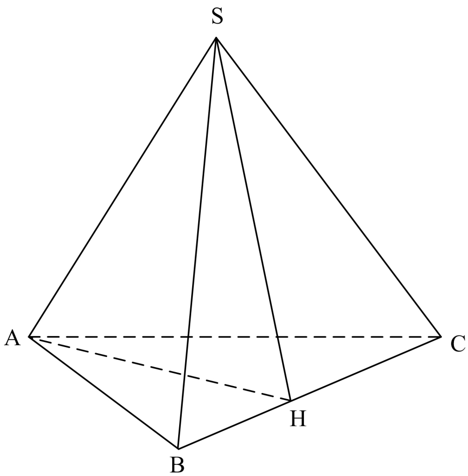

Figure 4.28:

Thus

$$V_{SABC} = \frac{1}{3}S_{ABC} \cdot SH = \frac{1}{3} \cdot \frac{AH \cdot BC}{2} \cdot SH = \frac{1}{3} \cdot \frac{\sqrt{6}}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{8}.$$

4.5.26

1) Let $OB \perp (P)$, $B \in (P)$. Since $AH \perp OH$, by the assumption, and BH is the projection of OH on the plane (P), we get $BH \perp HA$. Thus $\widehat{BHA} = 90°$, and so the locus of H is the circle of diameter BA (see Fig. 4.29).

2) First consider the case when the planes are parallel to (P), for example, $(P') \parallel (P)$. These planes intersect the cone by the regions which are similar to the base (c), that is intersect $\mathcal{C}$ by circles.

Now for the case when the planes are perpendicular to OA. By the proof above, $AH \perp (BOH)$, and so the two planes (OAH) and (OBH) are perpendicular each to other, and OH is their intersection. Thus the generatrix OH of the cone $\mathcal{C}$ can be considered as the intersection of a pair of planes passing through OA, OB and perpendicular each to other. This note shows that two generatrices OA and OB of the cone $\mathcal{C}$ are symmetric. Then if $(P) \perp OB$ and intersects the cone by a circle, then all planes $(Q) \perp OA$ also intersect the cone by circles.

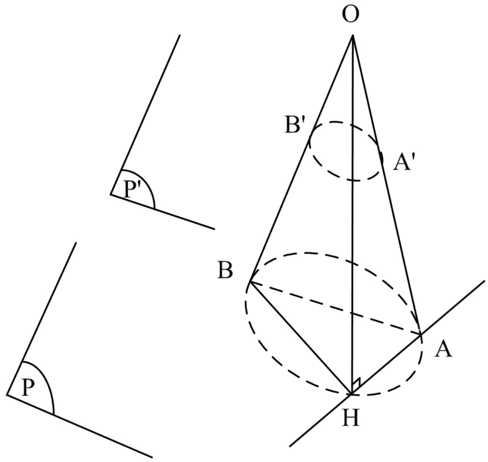

Figure 4.29:

So $\mathcal{C}$ has two symmetric faces AOB and MON passing the bisector OI of $\widehat{AOB}$ and perpendicular to (AOB) (see Fig. 4.30).

The faces AOB and MON intersect $\mathcal{C}$ by $\widehat{AOB} = \alpha$ and $\widehat{MON} = \beta$, respectively. We have, in the right triangle OBA

$$\frac{IB}{IA} = \frac{OB}{OA} = \cos\alpha.$$

On the other hand, from $IM = IN$ and $IM \cdot IN = IA \cdot IB$ it follows that

$$IM^2 = IA \cdot IB \Longleftrightarrow \frac{IM}{IB} = \sqrt{\frac{IA}{IB}} = \sqrt{\frac{1}{\cos\alpha}}.$$

Furthermore, we have in the isosceles triangle OMN

$$\tan\frac{\beta}{2} = \frac{IM}{OI},$$

and in the right triangle OBI

$$\sin\frac{\alpha}{2} = \frac{IB}{IO},$$

which together give

$$\frac{IM}{IB} = \frac{\tan\frac{\beta}{2}}{\sin\frac{\alpha}{2}}.$$

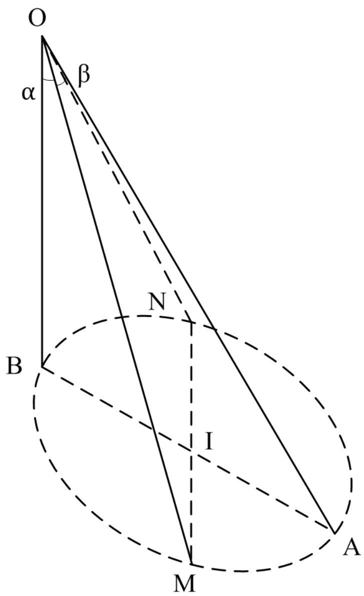

Figure 4.30:

Thus

$$\frac{IM}{IB} = \frac{\tan\frac{\beta}{2}}{\sin\frac{\alpha}{2}} = \sqrt{\frac{1}{\cos\alpha}}.$$

This gives the following relationship between α and β:

$$\tan\frac{\beta}{2} = \frac{\sin\frac{\alpha}{2}}{\sqrt{\cos\alpha}}.$$

4.5.27

1) As (P) makes equal angles with the three edges AB, AD, AE, we have $(BDE) \parallel (P)$. Note that the equilateral triangle BDE has the edge $BD = a\sqrt{2}$, where a is the length of the edge of the cube. Note also that the diagonal AG of the cube is perpendicular to (BDE) and so AG passes through the centroid I of the triangle BDE. Thus we obtain (see Fig. 4.31).

$$BI = \frac{2}{3} \cdot \frac{BD\sqrt{3}}{2} = \frac{a\sqrt{6}}{3}.$$

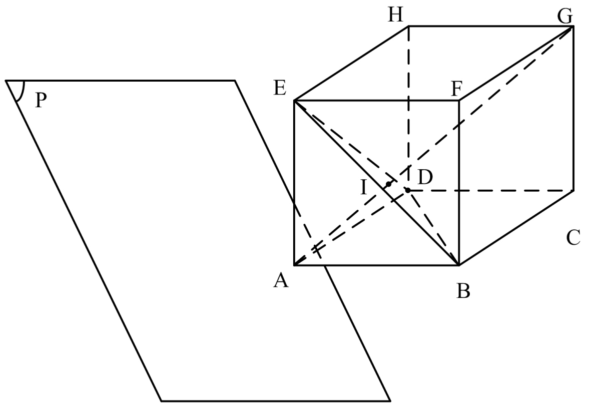

Figure 4.31:

On the other hand, since $(P) \parallel (BDE)$, the angle between AB and (P) is the same as the angle between AB and (BDE), that is $\widehat{ABI}$. We have

$$\cos\widehat{ABI} = \frac{BI}{AB} = \frac{a\sqrt{6}}{3a} = \frac{\sqrt{6}}{3},$$

which is the desired cosine.

Furthermore, since $AG \perp (P)$, the perpendicular projection of G onto (P) coincides with A (see Fig. 4.32). Moreover, the edges of the cube make equal angles with (P), and vertices B, C, D, E, F, H have the same distances to AG, and hence their projections onto (P) form a regular hexagon $B'C'D'E'F'H'$ with the center A and the edge's length $\dfrac{a\sqrt{6}}{3}$.

2) Since $G' \equiv A$ and F', G', D' are collinear, $(FGD) \perp (P)$. Similarly, (BHG) and (CEG) are also perpendicular to (P). So three diagonal faces of the cube are perpendicular to (P).

Note also that the six faces of the cube are equal with the area a^2, and so their projections are equal rhombuses with the area

$$\left(\frac{a\sqrt{6}}{3}\right)^2 \cdot \frac{\sqrt{3}}{2} = \frac{a^2\sqrt{3}}{3}.$$

Thus the faces of the cube make with (P) equal angles the cosine of which is $\frac{\sqrt{3}}{3}$.

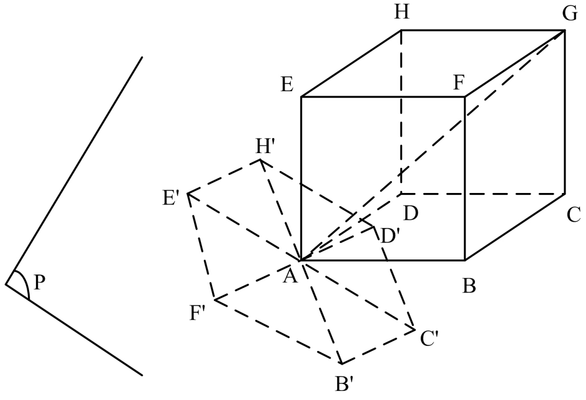

Figure 4.32:

4.5.28

1) We have (see Fig. 4.33)

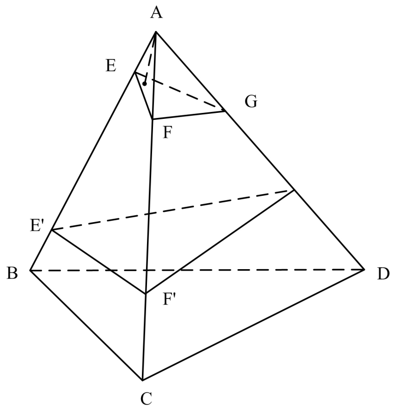

Figure 4.33:

$$V_{EFGE'F'G'} = V_{AE'F'G'} - V_{AEFG}. \tag{1}$$

Moreover,

$$\frac{V_{AE'F'G'}}{V_{ABCD}} = \frac{AE' \cdot AF' \cdot AG'}{AB \cdot AC \cdot AD} = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{5}{12},$$

$$\frac{V_{AEFG}}{V_{ABCD}} = \frac{AE \cdot AF \cdot AG}{AB \cdot AC \cdot AD} = \frac{1}{6} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{72},$$

and

$$V_{ABCD} = \frac{a^3\sqrt{2}}{12}.$$

Substituting these values into (1), we obtain

$$V_{EFGE'F'G'} = \frac{29}{72} V_{ABCD} = \frac{29}{72} \cdot \frac{a^3\sqrt{2}}{12} = \frac{29\sqrt{2}}{864} a^3.$$

2) Let AH be the altitude of the tetrahedron $AEFG$. On one hand, we have

$$V_{AEFG} = \frac{1}{72} V_{ABCD} = \frac{a^3\sqrt{2}}{864}.$$

On the other hand,

$$V_{AEFG} = \frac{1}{3} AH \cdot S_{EFG},$$

and we can easily compute that $S_{EFG} = \frac{5a^2\sqrt{3}}{288}$. So $AH = \frac{a\sqrt{6}}{15}$.

Note that the plane (EFG) makes with the lines AB, AC and AD the angles $\widehat{AEH}, \widehat{AFH}$ and $\widehat{AGH}$, respectively. We see that

$$\sin \widehat{AEH} = \frac{AH}{AE} = \frac{2\sqrt{6}}{5} \approx 0.979,$$

and so $\widehat{AEH} \approx 78°24'$. Similarly, $\widehat{AFH} \approx 41°13', \widehat{AGH} \approx 29°15'$.

4.5.29

First we can easily prove the following claim:

$$\frac{V_{KOBD}}{V_{KOAC}} = \frac{V_{DKAB}}{V_{CKAB}} = \frac{h_D}{h_C} = \frac{KD}{KC},$$

where h_C, h_D are the altitudes from C, D onto the plane (KAB), respectively.

Put $AC = a, AB = b, BD = c, KD = x, KC = y$, we can get (see Fig. 4.34)

$$\begin{cases} x + y = d \\ x - y = \frac{c^2 - a^2}{d}, \end{cases}$$

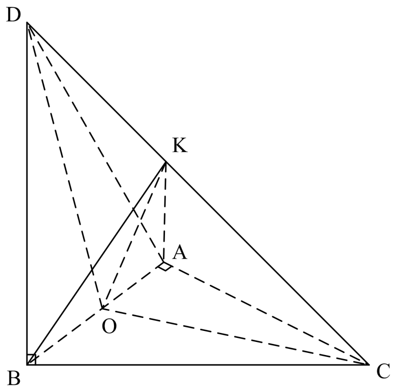

Figure 4.34:

where $d = CD$. Indeed, it is clear that $x + y = d$. Consider the right triangles OKD and OKC, by the Pythagorean theorem, we have

$$\begin{cases} OK^2 + x^2 = OD^2 \\ OK^2 + y^2 = OC^2 \end{cases} \implies x^2 - y^2 = OD^2 - OC^2.$$

Furthermore, for the right triangles OBD and OAC we also have

$$\begin{cases} OD^2 = OB^2 + BD^2 \\ OC^2 = OA^2 + AC^2 \end{cases} \implies OD^2 - OC^2 = BD^2 - AC^2 = c^2 - a^2.$$

From these equalities the desired relations.

Now we have

$$\begin{cases} x = \frac{1}{2}\left(d + \frac{c^2-a^2}{d}\right) \\ y = \frac{1}{2}\left(d - \frac{c^2-a^2}{d}\right) \end{cases},$$

which gives

$$\frac{x}{y} = \frac{d^2 + c^2 - a^2}{d^2 - c^2 + a^2}.$$

Note that BCD is the right triangle, we have $d^2 = c^2 + BC^2 = c^2 + (a^2 + b^2)$, and hence

$$\frac{V_{KOBD}}{V_{KOAC}} = \frac{x}{y} = \frac{2c^2 + b^2}{2a^2 + b^2}. \tag{1}$$

Now suppose that

$$\frac{V_{KOAC}}{V_{KOBD}} = \frac{AC}{BD} = \frac{a}{c}.$$

By (1) we have $\dfrac{2a^2+b^2}{2c^2+b^2}=\dfrac{a}{c}$, or equivalently, $(2ac-b^2)(a-c)=0$. As $a\neq c$, we obtain $2ac=b^2$, that is, $2AC\cdot BD=AB^2$.

Conversely, suppose that $2AC\cdot BD=AB^2$, that is, $2ac=b^2$. Since $a\neq c$, this is equivalent to

$$2ac(a-c)=b^2(a-c)\Longleftrightarrow\frac{2c^2+b^2}{2a^2+b^2}=\frac{c}{a}.$$

Again by (1)

$$\frac{V_{KOBD}}{V_{KOAC}}=\frac{2c^2+b^2}{2a^2+b^2},$$

and so

$$\frac{V_{KOBD}}{V_{KOAC}}=\frac{c}{a}=\frac{BD}{AC}.$$

Remark. If $AC=BD$ then the statement of the problem is not true.

4.5.30

1) Consider a plane $(P)\perp\Delta$ passing through N. Since $MN\perp\Delta$, MN belongs to (P). Let $AA'\perp(P)$, then $AA'\perp NA'$, and so NA' is the distance between the two fixed lines Δ and AA', which is constant (see Fig. 4.35).

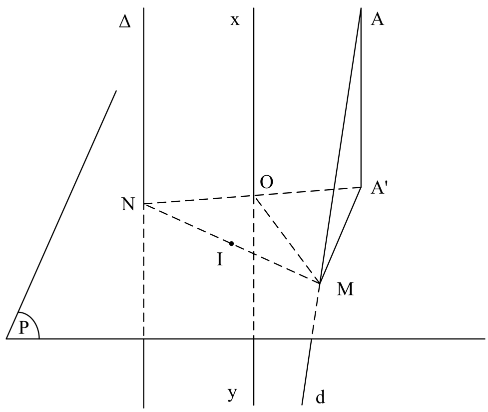

Figure 4.35:

Through the midpoint O of NA' draw a line $xy\perp(P)$, which is parallel to Δ and has the constant distance $NO=\frac{1}{2}NA'$ to Δ.

On the other hand, since $NM \perp MA$, we have $MA' \perp MN$. Then $OM = \frac{1}{2}NA'$ is constant, and $OM \perp xy$, so M is away from xy a constant distance. Thus the locus of M is a rotating cylinder of axis xy and of radius $\frac{1}{2}NA' = a$ (constant).

2) Note that I is the image of M under the homothety centered at N of ratio $\frac{1}{2}$. Since N is on $\Delta \perp MN$, the locus of I is the image of the locus of M under the above-mentioned homothety of axis Δ and ratio $\frac{1}{2}$.

Thus the locus of I is the rotating cylinder of axis xy and of radius $\frac{a}{2}$.

4.5.31

Draw $BB'' \parallel D'D$ and $BB'' = D'D$. Let H, I, K be the perpendicular projections of A', D, A on BD'. From K draw $KN \parallel ID$. We prove that $AN = A'H$, which shows the existence of the triangle AKN of the sides m_1, m_2, m_3 (see Fig. 4.36).

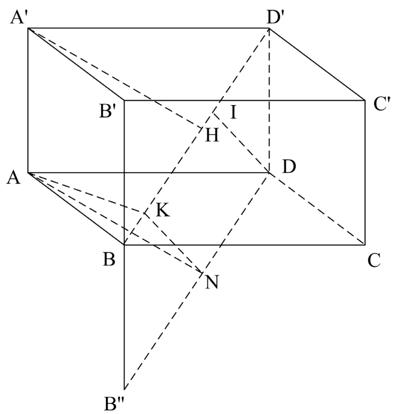

Figure 4.36:

Since $BK \perp AK$ and $BK \perp KN$, $BK \perp (AKN)$. Moreover, since $B''N \parallel BK$, $B''N \perp (AKN)$, and so $B''N \perp AN$. We see that the two triangles $A'BD'$ and $AB''D$ are equal, as $A'B = AB''$, $BD' = B''D$, $D'A' = DA$. Then their corresponding altitudes $A'H$ and AN are equal.

Furthermore, we have

$$V_{A'ABD} = \frac{1}{6}abc = V_{B''ABD},$$

and

$$V_{KAB''D} = \frac{1}{3}S_{AKN}.B''D = \frac{1}{3}S_{AKN} \cdot \sqrt{a^2+b^2+c^2}.$$

But the two pyramids $KAB''D$ and $BAB''D$ have the same base $AB''D$ and the equal altitudes $h_K = h_B$, which imply that their volumes are equal. Then from the last two equalities it follows that

$$2S_{AKN} \cdot \sqrt{a^2+b^2+c^2} = abc.$$

By the Heron's formula,

$$S_{AKN} = \sqrt{p(p-m_1)(p-m_2)(p-m_3)}$$

with $p = \frac{m_1+m_2+m_3}{2}$, and so we obtain the following relation:

$$\frac{abc}{\sqrt{a^2+b^2+c^2}} = 2\sqrt{p(p-m_1)(p-m_2)(p-m_3)}.$$

4.5.32

1) Let $SM = x, SN = y$. By the law of cosines for the triangle MON, we have (see Fig. 4.37) $MN^2 = MO^2 + NO^2 - 2MO \cdot NO \cdot \cos\widehat{MON}$, which gives

$$\begin{aligned}
\cos\widehat{MON} &= \frac{MO^2+NO^2-MN^2}{2MO \cdot NO} \\
&= \frac{(SO^2+SM^2)+(SO^2+SN^2)-(SM^2+SN^2)}{2MO \cdot NO} \\
&= \frac{SO^2}{MO \cdot NO} = \frac{a(x+y)}{MO \cdot NO} = \frac{x}{MO}\cdot\frac{a}{NO} + \frac{y}{NO}\cdot\frac{a}{MO} \\
&= \sin\widehat{SOM}\cdot\cos\widehat{SON} + \sin\widehat{SON}\cdot\cos\widehat{SOM} \\
&= \sin(\widehat{SON}+\widehat{SOM}).
\end{aligned}$$

Thus $\widehat{SON} + \widehat{SOM} + \widehat{MON} = 90°$.

2) Let J be the midpoint of MN, and J' the symmetric point of S with respect to J. We can verify that the center of the circum-sphere of the

tetrahedron $OSMN$ is the midpoint of OJ'. Indeed, let I be the midpoint of OJ'. In the right triangle SOJ' we have

$$IS = IO = \frac{1}{2}IJ'. \tag{1}$$

Furthermore, since $IJ \parallel (xSy)$, IJ is perpendicular to the both lines SJ' and MN. Then in the right triangles MIJ and SIJ we have

$$IM^2 = IJ^2 + JM^2 = (IS^2 - JS^2) + JM^2 = IS^2, \tag{2}$$

as in the right triangle SMN there hold $JS = JM = \frac{1}{2}MN$.

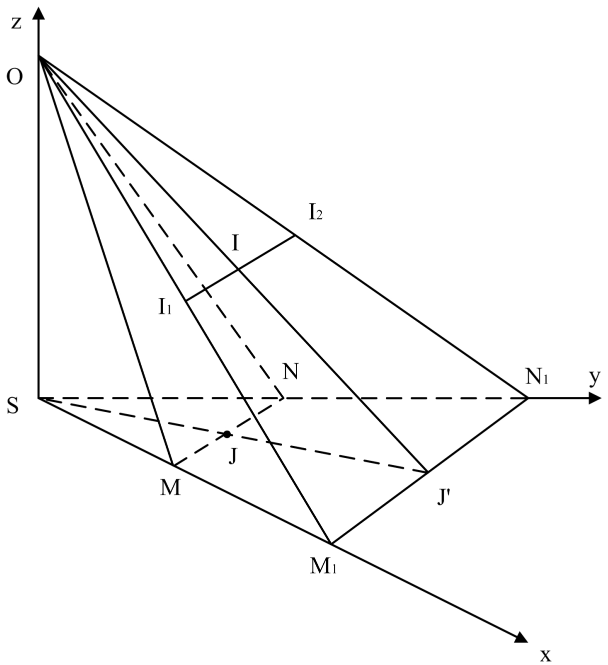

Figure 4.37:

From (1) and (2) it follows that $IO = IS = IM$. Similarly, $IN = IS$, and so I is the circum-center of the tetrahedron $OSMN$.

Thus I is the image of J' under the homothety of the center at O and the ration $\frac{1}{2}$, and so the locus of I is the line I_1I_2 of the equilateral triangle OM_1N_1 of the side $a\sqrt{2}$, where $SM_1 = SN_1 = a$.

4.5.33

First we can prove that $OABC$ is the tetrahedron for which the trihedral angle at the vertex O is right, that is $\widehat{AOB} = \widehat{BOC} = \widehat{COA} = 90^\circ$ (see Fig. 4.38).

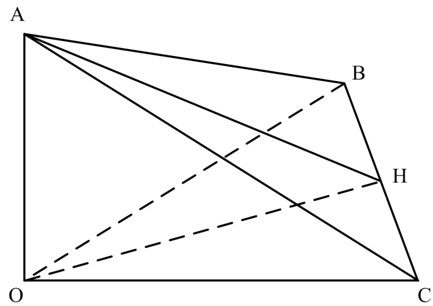

Figure 4.38:

From this it follows that $OA = OB = OC = x$ $(x > 0)$. Draw $OH \perp BC$, then $AH \perp BC$, and so $S = S_{ABC} = \frac{1}{2}AH \cdot BC$.

The triangle BOC is right and isosceles $(OB = OC = x)$, then $BC = x\sqrt{2}$. On the other hand, consider the triangle AOH which is right at O, we see that

$$AH^2 = AO^2 + OH^2 = x^2 + \left(\frac{BC}{2}\right)^2 = x^2 + \frac{x^2}{2} = \frac{3}{2}x^2.$$

This gives $AH = \frac{x\sqrt{6}}{2}$. Then

$$S = \frac{1}{2} \cdot \frac{x\sqrt{6}}{2} \cdot x\sqrt{2} = \frac{x^2\sqrt{3}}{2} \implies x = \sqrt{\frac{2\sqrt{3}}{3}S}.$$

Therefore, the volume V of $OABC$ is

$$V = \frac{1}{3}AO \cdot S_{ABC} = \frac{1}{3}x \cdot S = \frac{S\sqrt{S}\sqrt[4]{12}}{9}.$$

4.5.34

1) Since $ME \perp (ABCD)$, $ME \perp CS$. Also by the assumption, $MP \perp CS$. Then $CS \perp (MEP)$, in particular, $EP \perp CS$. That is $\widehat{EPC} = 90°$ (see Fig. 4.39). So P is on the circle of diameter EC.

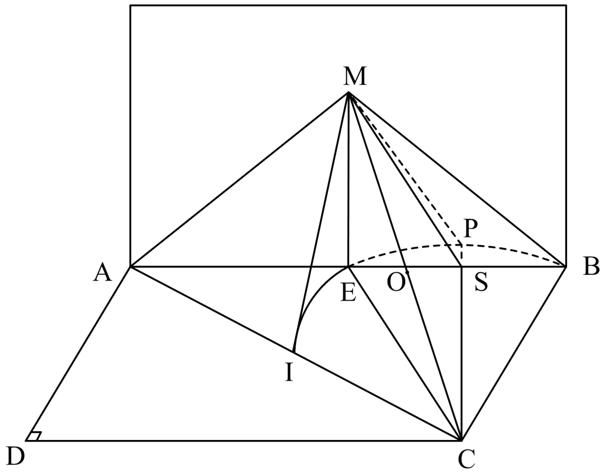

Figure 4.39:

Furthermore, let I be the projection of M onto AC. We note that when $S \equiv A$ then $P \equiv I$, while if $S \equiv B$ then $P \equiv B$ too.

So the locus of P is the arc of the circle of diameter EC without two points B and I.

2) Let the side of the square $ABCD$ is $2a > 0$. We note that $0 \leq x \leq 2a$. Since SO is the median of the triangle SMC, by the law of cosines, we have

$$SO^2 = \frac{2(SM^2 + SC^2) - MC^2}{4}.$$

But

$$SM^2 = x^2 + 4a^2 - 4ax. \cos 60° = x^2 + 4a^2 - 2ax$$
$$SC^2 = 4a^2 + x^2$$
$$MC^2 = 8a^2,$$

then

$$SO = x^2 - ax + 2a^2.$$

From this it follows that

$$SO_{\max} = 2a \text{ (attains at } x = 2a\text{)},$$

$$SO_{\min} = \frac{a\sqrt{7}}{2} \text{ (attains at } x = \tfrac{a}{2}\text{)}.$$

4.5.35

As three faces of the parallelepiped belong to the surface of the tetrahedron, there exists a common vertex, say A. Denote by $AEFLPQHG$ the resulting parallelepiped, as showed in the figure, where the vertex H belongs to the face BCD. The plane $ADHF$ meets BC at K, and the plane $PQHG$ meets DB, DC at M, N, respectively (see Fig. 4.40).

Let h, h_1 be the altitudes of the pyramids $DABC, DPMN$, and V, V_p the volumes of $DABC, AEFLPQHG$, respectively. By the assumptions and Thales's theorem, we have

$$\frac{PM}{AB} = \frac{PN}{AC} = \frac{DP}{DA} = \frac{DH}{DK} = \frac{h_1}{h} = y \ (0 < y < 1)$$

$$\frac{MH}{MN} = \frac{MQ}{MP} = \frac{QH}{PN} = \frac{PG}{PN} = x \ (0 < x < 1).$$

From these equalities it follows that

$$\frac{S_{PMN}}{S_{ABC}} = \frac{PM \cdot PN}{AB \cdot AC} = y^2, \tag{1}$$

$$\frac{S_{MHQ}}{S_{PMN}} = \frac{MH \cdot MQ}{MN \cdot MP} = x^2, \tag{2}$$

and

$$\frac{S_{NHG}}{S_{PMN}} = \frac{NH \cdot NG}{MN \cdot NP} = \frac{(MN - MH) \cdot (NP - PG)}{MN \cdot NP} = (1 - x)^2. \tag{3}$$

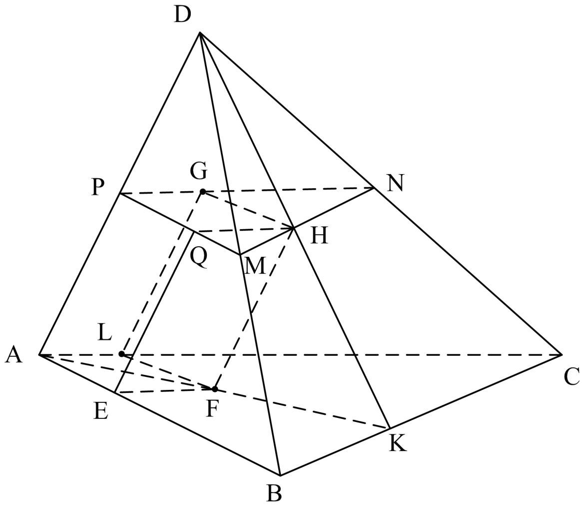

Figure 4.40:

Combining (1), (2) and (3) yields

$$\begin{aligned} S_{PQHG} &= S_{PMN} - S_{MHQ} - S_{NHG} \\ &= S_{PMN} - x^2 S_{PMN} - (1-x)^2 S_{PMN} \\ &= 2x(1-x)S_{PMN} \\ &= 2x(1-x)y^2 S_{ABC}. \end{aligned}$$

Then

$$\frac{V_p}{V} = \frac{(h-h_1)S_{PQHG}}{\frac{1}{3}hS_{ABC}} = 6x(1-x)y^2(1-y). \tag{4}$$

By the Arithmetic-Geometric Mean inequality, we have

$$x(1-x) \leq \frac{1}{4},\ \frac{y}{2} \cdot \frac{y}{2} \cdot (1-y) \leq \frac{1}{27}.$$

So we obtain

$$\frac{V_p}{V} = 6x(1-x)y^2(1-y) \leq 6 \cdot \frac{1}{4} \cdot \frac{4}{27} = \frac{2}{9}.$$

The equality occurs if and only if $x = 1-x$ and $y = 2(1-y)$, or equivalently, $x = \dfrac{1}{2}, y = \dfrac{2}{3}$.

1) Since $\frac{2}{9} < \frac{9}{40}$, it is impossible to have $\frac{V_p}{V} = \frac{9}{40}$.

2) We have

$$\frac{V_p}{V} = \frac{11}{50} \Longleftrightarrow 6x(1-x)y^2(1-y) = \frac{11}{50}.$$

If we choose $y = \frac{2}{3}$, then $400x(1-x) = 99$. This gives $x = \frac{9}{20}$ and $x = \frac{11}{20}$. So we have the following construction.

Take $M \in DB, N \in DC, P \in DA$ so that

$$\frac{DM}{DB} = \frac{DN}{DC} = \frac{DP}{DA} = y = \frac{2}{3}.$$

On MN take H_1, H_2 so that

$$\frac{MH_1}{MN} = \frac{9}{20}, \frac{MH_2}{MN} = \frac{11}{20}.$$

Then we get the two desired parallelepipeds $AEFLPQH_1G$ and $AEFLPQH_2G$.

4.5.36

By the assumptions for the two pyramids we have (see Fig. 4.41)

$$AB \parallel EF, AC \parallel DE, BC \parallel DF \tag{1}$$

and

$$AB = EF = AC = DE = BC = DF. \tag{2}$$

Also S is the intersection point of the three medians DI, EK, FL of the triangle DEF, and R is the intersection point of the three medians AH, BU, CV of the triangle ABC.

1) By (1), the two bases (ABC) and (DEF) are parallel. Also the three line segments AE, BF, CD meet at the midpoint G of SR, and G is the center of symmetry of the part formed by the two pyramids. Therefore, $KS \parallel RH$ and $KS = RH$, $SE \parallel AR$ and $SE = AR$. This shows that $KSHR$ and $SERA$ are parallelograms.

Let M be the intersection point of SA and RK, Q the intersection point of SH and RE. We have (see Fig. 4.42).

$$\frac{SM}{SA} = \frac{SM}{ER} = \frac{KS}{KE} = \frac{1}{3} \implies SM = \frac{1}{3}SA.$$

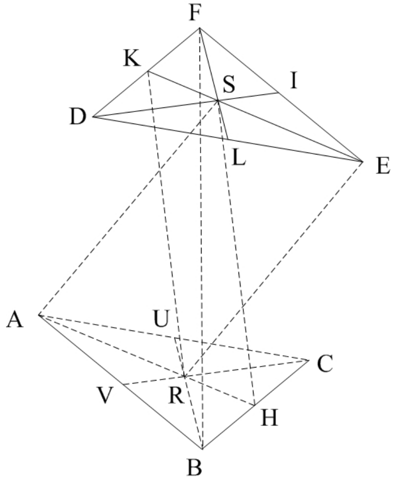

Figure 4.41:

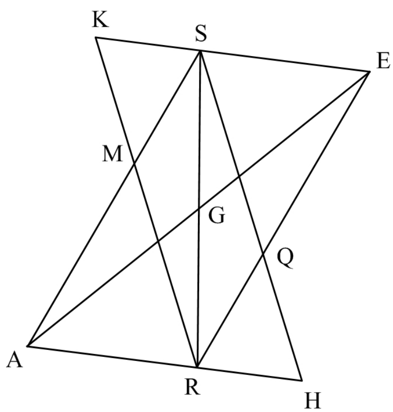

Figure 4.42:

Similarly,

$$RQ = \frac{1}{3}ER \implies RQ = \frac{1}{3}SA.$$

By the same manner, we can determine the intersection points N of SV and RD, T of RI and SC, P of SB and RL, X of SU and RF. So the common part of the two pyramids $SABC$ and $RDEF$ is the solid figure of six faces $SMNPQTXR$.

2) As in 1) we have

$$SM = SP = ST = RQ = RN = RX = \frac{1}{3}SA. \tag{3}$$

Also, $QH = \frac{1}{3}SH$, and so Q is the centroid of the triangle SBC. Let Y be the midpoint of SB, then

$$\frac{YP}{YS} = \frac{YQ}{YC} = \frac{1}{3},$$

which shows that $PQ \parallel SC$ and $PQ = \frac{1}{3}SC = ST$. Thus $STQP$ is a parallelogram, and moreover, as $ST = PQ = SP$, we get that $STQP$ is a rhombus.

Similarly, $SMNP, MXRN, TXRQ, SMXT, RNPQ$ are also rhombuses. Denote by V the volume of the common part of the two given pyramids, we have

$$V = 6V_{SMPT} = 6\left(\frac{1}{3}\right)^3 V_{SABC},$$

or equivalently,

$$\frac{V}{V_{SABC}} = \frac{2}{9}.$$

4.5.37

Let A'', B'', C'' be the midpoints of AA', BB', CC'. From the assumption of the problem it follows that

$$3\overrightarrow{OM} = \overrightarrow{OA'} + \overrightarrow{OB} + \overrightarrow{OC}$$

and

$$3\overrightarrow{OM'} = \overrightarrow{OA} + \overrightarrow{OB'} + \overrightarrow{OC'}.$$

As $\overrightarrow{OS} = \frac{1}{2}(\overrightarrow{OM} + \overrightarrow{OM'})$, we have

$$\begin{aligned} 3\overrightarrow{OS} &= \frac{3}{2}(\overrightarrow{OM} + \overrightarrow{OM'}) \\ &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OA'}) + \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OB'}) + \frac{1}{2}(\overrightarrow{OC} + \overrightarrow{OC'}) \\ &= \overrightarrow{OA''} + \overrightarrow{OB''} + \overrightarrow{OC''}. \end{aligned}$$

Now let H be the perpendicular projection of the center K of the sphere $\mathcal{S}$ onto the plane (Oyz), then HB'', HC'' are projections of KB'', KC'' on this plane, respectively. This means that OK is a diagonal of a right parallelepiped built on the three edges OA'', OB'', OC''. Therefore,

$$\overrightarrow{OK} = \overrightarrow{OA''} + \overrightarrow{OB''} + \overrightarrow{OC''}.$$

Thus we obtain

$$\overrightarrow{OS} = \frac{1}{3}\overrightarrow{OK}.$$

Since K is the center of $\mathcal{S}$, $KA = KB = KC$, and so K is on the ray K_0t perpendicular to the plane (ABC) and passing the circum-center of the triangle ABC, where K_0 is the center of the circum-sphere of the tetrahedron $ABCO$. Hence, S is on the ray S_0t with $\overrightarrow{OS_0} = \frac{1}{3}\overrightarrow{OK_0}$ (S_0t is parallel and has the same direction with K_0t).

4.5.38

Let SH be the altitude of the equilateral tetrahedron $SABC$. Since $SA = SB = SC = a$, $HA = HB = HC$. Denote by O the center of the two concentric spheres. Since $OA = OB = OC$, O is on the ray SH (see Fig. 4.43).

Let D be the midpoint of AB. Then $H \in CD$. Draw $OM \perp SD$, then $OM \perp AB$ (as $AB \perp (SHD)$), and so $OM \perp (SAB)$. Therefore $OM = r$.

Note that the two triangles SOM and SDH are similar. We then have

$$\frac{SO}{OM} = \frac{SD}{DH} = \frac{CD}{DH} = 3,$$

or equivalently,

$$SO = 3r. \tag{1}$$

We can compute that

$$SD = CD = \frac{a\sqrt{3}}{2} \implies CH = \frac{2CD}{3} = \frac{a}{\sqrt{3}}. \tag{2}$$

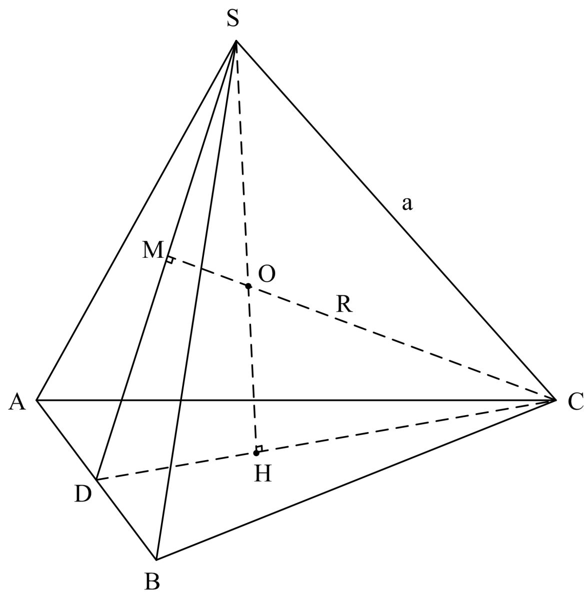

Figure 4.43:

Hence,

$$SH^2 = SC^2 - CH^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3} \implies SH = a\sqrt{\frac{2}{3}}. \qquad (3)$$

From (1), (2), (3) it follows that the relation $CO^2 = CH^2 + OH^2$ can be written as follows:

$$R^2 = \frac{a^2}{3} + \left(a\sqrt{\frac{2}{3}} - 3r\right)^2 \Longleftrightarrow a^2 - 2\sqrt{6}ra + 9r^2 - R^2 = 0,$$

which gives

$$a = r\sqrt{6} \pm \sqrt{R^2 - 3r^2}. \qquad (4)$$

So we must have

$$R^2 - 3r^2 \geq 0 \Longleftrightarrow r \leq \frac{R}{\sqrt{3}}. \qquad (5)$$

From (1) and $SO^2 = SM^2 + OM^2$ it follows that $SM = 2r\sqrt{2}$. Thus for the smaller sphere to tangent with three faces SAB, SBC, SCA, there must be $SM \le SD \iff 4r\sqrt{2} \le a\sqrt{3}$. So by (4) we have

$$r\sqrt{2} \le \sqrt{3(R^2 - 3r^2)} \implies r \le \frac{\sqrt{33}}{11}R. \tag{6}$$

Conversely, if (6) holds then we have (5). Thus for the existence of such a tetrahedron we must have $r \le \frac{\sqrt{33}}{11}R$.

4.5.39

Rotate the triangles CAB, CAD, CBD around the axes AB, AD, BD respectively to become the triangles C_1AB, C_2AD, C_3BD on the plane (ABD), so that C_1 and D are of different sides of the line AB, C_2 and B are of different sides of the line AD, and C_3 and A are of different sides of the line BD. We have (see Fig. 4.44)

$$AC_1 = AC_2 = AC,\ BC_1 = BC_3 = BC,\ DC_2 = DC_3 = DC.$$

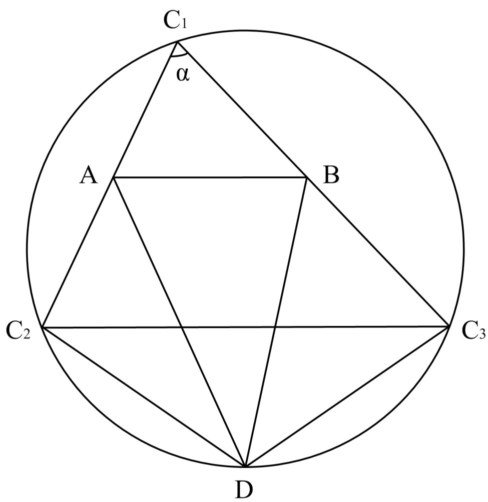

Figure 4.44:

By the assumption 2), C_1, A, C_2 are collinear, and C_1, B, C_3 are collinear too. Furthermore, by the assumption 1), $\widehat{AC_2D} + \widehat{BC_3D} = 180°$, and so the quadrilateral $C_1C_2DC_3$ is cyclic.

Denote by S' the surface area of the tetrahedron $ABCD$, we have

$$S' = S_{C_1C_2C_3} + S_{C_2DC_3}. \tag{1}$$

Let $AC = x, BC = y$. We can see that

$$S_{C_1C_2C_3} = 2xy\sin\alpha.$$

Also

$$\begin{aligned} S_{C_2DC_3} &= \frac{1}{4}(C_2C_3)^2 \cdot \tan\frac{\alpha}{2} \\ &= \big[(x+y)^2 - 2xy(1+\cos\alpha)\big] \cdot \tan\frac{\alpha}{2} \\ &= k^2\tan\frac{\alpha}{2} - 2xy\sin\alpha. \end{aligned}$$

Substituting these values into (1) yields

$$S' = k^2\tan\frac{\alpha}{2}.$$

4.5.40

Let O and R be the center and the radius of the given sphere, respectively. We have

$$AB^2 = \overrightarrow{AB}^2 = (\overrightarrow{OB} - \overrightarrow{OA})^2 = 2R^2 - 2\cdot\overrightarrow{OB}\cdot\overrightarrow{OA}.$$

It is similar for the other terms in the considered sum. Summing up all equalities we obtain

$$\begin{aligned} &AB^2 + AC^2 + AD^2 - BC^2 - CD^2 - DB^2 \\ &= 2(\overrightarrow{OC}\cdot\overrightarrow{OB} + \overrightarrow{OD}\cdot\overrightarrow{OC} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OD}\cdot\overrightarrow{OA}) \\ &= -(OA^2 + OB^2 + OC^2 + OD^2) + (\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} - \overrightarrow{OA})^2 \\ &= -4R^2 + (\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} - \overrightarrow{OA})^2 \geq -R^2. \end{aligned}$$

Thus,

$$AB^2 + AC^2 + AD^2 - BC^2 - CD^2 - DB^2 \geq -R^2. \tag{1}$$

Now we draw the diameter AA' of the sphere. Then we have

$$\widehat{ABA'} = \widehat{ACA'} = \widehat{ADA'} = 90^\circ. \tag{2}$$

Therefore, the equality in (1) occurs if and only if

$$\begin{aligned}
&\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}-\overrightarrow{OA}=\overrightarrow{0}\\
&\iff(\overrightarrow{OA}+\overrightarrow{AB})+(\overrightarrow{OA}+\overrightarrow{AC})+(\overrightarrow{OA}+\overrightarrow{AD})-\overrightarrow{OA}=\overrightarrow{0}\\
&\iff\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{AD}=2\overrightarrow{AO}=\overrightarrow{AA'},
\end{aligned}$$

which gives

$$\begin{cases}\overrightarrow{AC}+\overrightarrow{AD}=\overrightarrow{AA'}-\overrightarrow{AB}=\overrightarrow{BA'}\\ \overrightarrow{AD}+\overrightarrow{AB}=\overrightarrow{AA'}-\overrightarrow{AC}=\overrightarrow{CA'}\\ \overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AA'}-\overrightarrow{AD}=\overrightarrow{DA'}.\end{cases}\tag{3}$$

From (2) and (3) we obtain

$$(\overrightarrow{AC}+\overrightarrow{AD})\overrightarrow{AB}=(\overrightarrow{AD}+\overrightarrow{AB})\overrightarrow{AC}=(\overrightarrow{AB}+\overrightarrow{AC})\overrightarrow{AD}=0,$$

that is

$$\overrightarrow{AC}\cdot\overrightarrow{AD}=\overrightarrow{AD}\cdot\overrightarrow{AB}=\overrightarrow{AB}\cdot\overrightarrow{AC}=0.$$

This implies that the trihedral angle at the vertex A is rectangular.

Conversely, if the trihedral angle at the vertex A is rectangular, then it can be easily seen that the equality in (1) occurs.

4.5.41

Let E be the centroid of the triangle BCD, G the centroid of the tetrahedron $A'BCD$. Take F so that $\overrightarrow{AF}=\overrightarrow{IQ}$, then $QF\parallel IA$. By the assumption, $QA'\parallel IA$, and so the three points Q,F,A' are collinear. This implies that $FA'\perp AA'$ (see Fig. 4.45).

Consider the homothety h with the center at E of the ratio $\frac{1}{4}$. We have

$$h(A)=P,h(A')=G,h(F)=K.$$

This implies that $PK\parallel AF$. Note that $AF\parallel IQ$, and so $PK\parallel IQ$ and

$$PK=\frac{1}{4}AF=\frac{1}{4}IQ.$$

Thus PK is the fixed line segment.

On the other hand, $PG\parallel AA',GK\parallel FA'$, and $FA'\perp AA'$, and so $PG\perp GK$. Consequently, $\widehat{PGK}=90^\circ$, and hence G is on the fixed sphere of diameter PK.

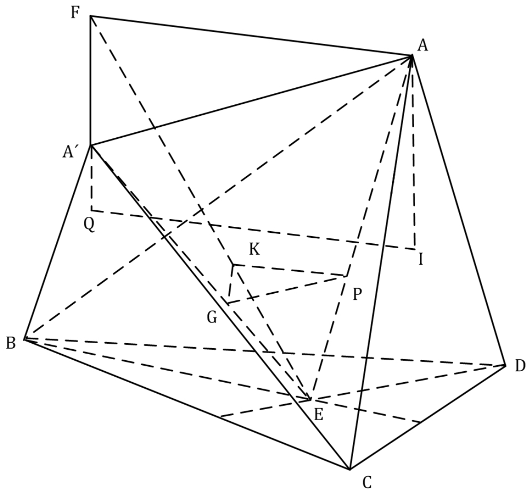

Figure 4.45:

4.5.42

Let O be the center of $\mathcal{S}$ and S' the tangent point of $\mathcal{S}$ and (P). Denote by M the midpoints of SD (see Fig. 4.46).

Draw a plane passing through SD and perpendicular to AB at K. Then AB is perpendicular to the three lines DK, SK, MK. By the assumption, $\Delta DAB = \Delta SAB$, we see that $DK = SK$ as the two coppesponding altitudes. Then in the iscoseless triangle KSD we have $MK \perp SD$ and $\widehat{DKM} = \widehat{SKM}$. This shows that the plane (MAB) is the bisector plane of the dihedral angle of the edge AB and the faces performed by the two half-planes (ABD) and (ABS). Hence, S and D are symmetric with respect to the plane (MAB).

Furthermore, O is equidistance from the two planes (ABC) and (SAB), and the two points O and S are of different sides of the plane (ABC). This shows that O belongs to (ABC). That is, S and D are symmetric with respect to the plane (OAB).

Similarly, E, F are the symmetric images of S with respect to the planes $(OBC), (OCA)$ respectively. Thus we have $OD = OE = OF = OS$.

On the other hand, $OS' \perp (P)$, and so $S'D = S'E = S'F$. Thus S' is the circum-center of ΔDEF.

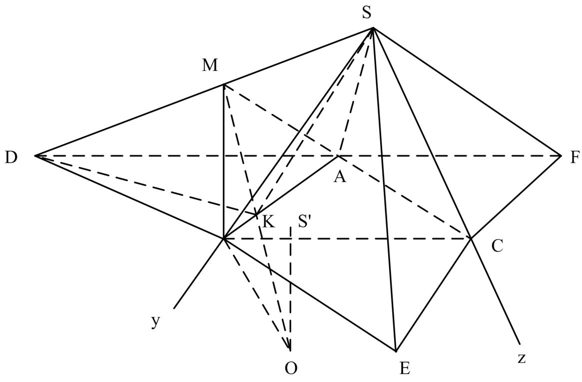

Figure 4.46:

4.5.43

Denote by R the radius of the sphere. Then $OA = OB = OC = OD = R$. Put $\overrightarrow{OA} = \overrightarrow{a}, \overrightarrow{OB} = \overrightarrow{b}, \overrightarrow{OC} = \overrightarrow{c}, \overrightarrow{OD} = \overrightarrow{d}$. From the assumption $AB = AC = AD$ it follows that $\Delta AOB = \Delta AOC = \Delta AOD$, and so $\widehat{AOB} = \widehat{AOC} = \widehat{AOD}$. Then we have (see Fig. 4.47)

$$\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{a} \cdot \overrightarrow{c} = \overrightarrow{a} \cdot \overrightarrow{d}.$$

Note that

$$\begin{aligned} 3\overrightarrow{BG} &= \overrightarrow{BA} + \overrightarrow{BC} + \overrightarrow{BD} \\ &= \overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC} - \overrightarrow{OB} + \overrightarrow{OD} - \overrightarrow{OB} \\ &= \overrightarrow{a} + \overrightarrow{c} + \overrightarrow{d} - 3\overrightarrow{b}. \end{aligned}$$

Also as E and F are the midpoints of BG and AE, respectively, we have

$$\begin{aligned} 12\overrightarrow{OF} &= 6(\overrightarrow{OA} + \overrightarrow{OE}) = 6\overrightarrow{OA} + 3(\overrightarrow{OB} + \overrightarrow{OG}) = 6\overrightarrow{OA} + 3\overrightarrow{OB} + 3\overrightarrow{OG} \\ &= 6\overrightarrow{OA} + 3\overrightarrow{OB} + (\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{OD}) = 7\overrightarrow{OA} + 3\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} \\ &= 7\overrightarrow{a} + 3\overrightarrow{b} + \overrightarrow{c} + \overrightarrow{d}. \end{aligned}$$

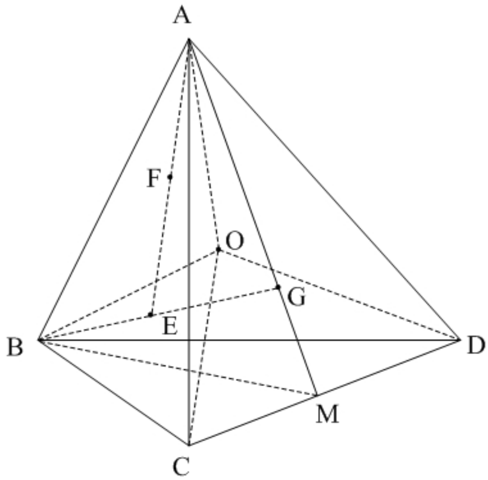

Figure 4.47:

Therefore,

$$\begin{aligned}3\overrightarrow{BG}\cdot 12\overrightarrow{OF} &= (\overrightarrow{a}+\overrightarrow{c}+\overrightarrow{d}-3\overrightarrow{b})\cdot(7\overrightarrow{a}+3\overrightarrow{b}+\overrightarrow{c}+\overrightarrow{d})\\ &= 7\overrightarrow{a}^2-9\overrightarrow{b}^2+\overrightarrow{c}^2+\overrightarrow{d}^2-18\overrightarrow{a}\cdot\overrightarrow{b}+8\overrightarrow{a}\cdot\overrightarrow{c}+8\overrightarrow{a}\cdot\overrightarrow{d}+2\overrightarrow{c}\cdot\overrightarrow{d}\\ &= 7R^2-9R^2+R^2+R^2-18\overrightarrow{a}\cdot\overrightarrow{b}+8\overrightarrow{a}\cdot\overrightarrow{c}+8\overrightarrow{a}\cdot\overrightarrow{d}+2\overrightarrow{c}\cdot\overrightarrow{d}\\ &= -18\overrightarrow{a}\cdot\overrightarrow{d}+8\overrightarrow{a}\cdot\overrightarrow{d}+8\overrightarrow{a}\cdot\overrightarrow{d}+2\overrightarrow{c}\cdot\overrightarrow{d}\\ &= 2\overrightarrow{c}\cdot\overrightarrow{d}-2\overrightarrow{a}\cdot\overrightarrow{d}=2\overrightarrow{d}\cdot(\overrightarrow{c}-\overrightarrow{a})=2\overrightarrow{OD}\cdot\overrightarrow{AC}.\end{aligned}$$

From this it follows that

$$\overrightarrow{BG}\cdot\overrightarrow{OF}=0\iff\overrightarrow{OD}\cdot\overrightarrow{AC}=0,$$

that is $OF\perp BG\iff OD\perp AC$.

4.5.44

1) Let G be the centroid of the tetrahedron $ABCD$. Choose F so that $\overrightarrow{OF}=2\overrightarrow{OG}$. We then have (see Fig. 4.48)

$$\begin{cases}3\overrightarrow{OA_0}=\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}\\ -\overrightarrow{OA_1}=\overrightarrow{OA},\end{cases}$$

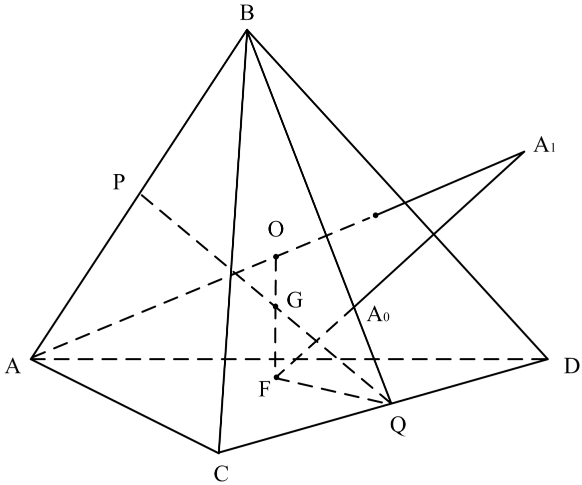

Figure 4.48:

From this it follows that

$$3\overrightarrow{OA_0} - \overrightarrow{OA_1} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4\overrightarrow{OG}.$$

Thus we obtain

$$\overrightarrow{OA_0} - \overrightarrow{OA_1} = 4\overrightarrow{OG} - 2\overrightarrow{OA_0} = 2(2\overrightarrow{OG} - \overrightarrow{OA_0}) = 2(\overrightarrow{OF} - \overrightarrow{OA_0}),$$

or equivalently,

$$\overrightarrow{A_1A_0} = 2\overrightarrow{A_0F}.$$

This shows that A_0A_1 passes through F.

Similarly, B_0B_1, C_0C_1, D_0D_1 also pass through F. This is the point said in the problem.

2) Let P and Q be the midpoints of AB and CD respectively. Since G is the centroid of $ABCD$, G is the midpoint of PQ.

On the other hand, from 1) we see that $\overrightarrow{OF} = 2\overrightarrow{OG}$, which means that G is also the midpoint of OF.

Thus $PFQO$ is a parallelogram, and hence $FP \parallel OQ$. Note that $OQ \perp CD$ (as the triangle OCD is isosceles), and so $FP \perp CD$.

4.5.45

1) Let E be the midpoint of BC. Draw a line $Ex \parallel PA$. Then $Ex \perp (PBC)$ and all points on Ex have the same distance to P, B, C. Choose a point $Q \in Ex$ so that $\overrightarrow{EQ} = \frac{1}{2}\overrightarrow{PA}$. Then the triangle QAP is isosceles ($QA = QP$), and hence $QA = QP = QC = QB$. That is Q is the circum-center of the sphere $\mathcal{S}$ (see Fig. 4.49).

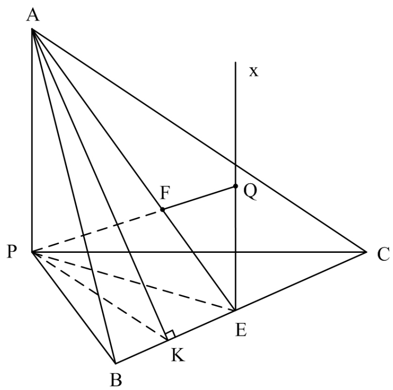

Figure 4.49:

Let $F \in (APEQ)$ be the intersection point of AE and PQ. Since $EQ \parallel PA$,

$$\frac{PF}{FQ} = \frac{PA}{EQ} = 2 \implies PF = 2FQ,$$

which means that

$$\overrightarrow{PF} = \frac{2}{3}\overrightarrow{PQ}.$$

Thus F is the fixed point and, as $F \in AE$, it belongs to the plane (ABC).

2) Put $PA = a, PB = b, PC = c$. Draw $AK \perp BC$, then $PK \perp BC$. We have

$$S = S_{ABC} = \frac{1}{2} \cdot BC \cdot AK,$$

which implies that

$$\begin{aligned} S^2 &= \frac{1}{4}\cdot BC^2\cdot AK^2 \\ &= \frac{1}{4}\cdot BC^2(PA^2+PK^2) \\ &= \frac{1}{4}\left[(PB^2+PC^2)PA^2+(BC\cdot PK)^2\right] \\ &= \frac{1}{4}\left[b^2a^2+c^2a^2+(2S_{PBC})^2\right] \\ &= \frac{1}{4}(b^2a^2+c^2a^2+b^2c^2). \end{aligned}$$

Note that

$$a^2+b^2+c^2=a^2+BC^2=4QE^2+4BE^2=4(QE^2+BE^2)=4QB^2=4R^2.$$

Now applying the inequality

$$xy+yz+zx\le\frac{(x+y+z)^2}{3},$$

(which is equivalent to $(x-y)^2+(y-z)^2+(z-x)^2\ge 0$), we have

$$4S^2=a^2b^2+b^2c^2+c^2a^2\le\frac{(a^2+b^2+c^2)^2}{3}=\frac{16R^4}{3}.$$

Hence,

$$S^2\le\frac{4}{3}R^4 \Longleftrightarrow S\le\frac{2}{\sqrt{3}}R^2.$$

The equality occurs if and only if $a^2=b^2=c^2=\dfrac{4}{3}R^2 \Longleftrightarrow a=b=c=\dfrac{2}{\sqrt{3}}R.$

4.5.46

Choose on each given ray one unit vector, denoted by $\overrightarrow{OA_i}=\overrightarrow{e_i}$ $(i=1,2,3,4)$. Let φ be the angle between any two rays.

1) It is clear that the isosceles triangles OA_iA_j with $i,j\in\{1,2,3,4\}$, $i\neq j$, are equal. We deduce that the tetrahedron $A_1A_2A_3A_4$ is regular. Therefore, $\sum\limits_{i=1}^{4}\overrightarrow{e_i}=\overrightarrow{0}$, which implies that

$$0=\left(\sum_{i=1}^{4}\overrightarrow{e_i}\right)^2=4+12\cos\varphi,$$

or equivalently, $\cos\varphi = -\frac{1}{3}$.

2) Choose on the ray Or the unit vector $\overrightarrow{e}$, and relabel the angles $\alpha, \beta, \gamma, \delta$ by $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. In this case we have

$$p := \sum_{i=1}^{4} \cos\varphi_i = \sum_{i=1}^{4} \overrightarrow{e}.\overrightarrow{e_i} = \overrightarrow{e}.\sum_{i=1}^{4} \overrightarrow{e_i} = 0,$$

and

$$q := \sum_{i=1}^{4} \cos^2\varphi_i = \sum_{i=1}^{4} (\overrightarrow{e}.\overrightarrow{e_i})^2.$$

Representing $\overrightarrow{e}$ in a form

$$\overrightarrow{e} = \sum_{i=1}^{4} x_i \overrightarrow{e_i},$$

we get for each $i = 1, 2, 3, 4$

$$\overrightarrow{e_i} \cdot \overrightarrow{e} = x_i - x_i \cos\varphi + \cos\varphi \sum_{i=1}^{4} x_i = \frac{4}{3} x_i - \frac{1}{3} \sum_{i=1}^{4} x_i.$$

Then

$$\sum_{i=1}^{4} (\overrightarrow{e_i} \cdot \overrightarrow{e})\overrightarrow{e_i} = \frac{4}{3}\overrightarrow{e},$$

which gives

$$\sum_{i=1}^{4} (\overrightarrow{e} \cdot \overrightarrow{e_i})^2 = \frac{4}{3}(\overrightarrow{e})^2 = \frac{4}{3}.$$

Thus $p = 0$ and $q = \dfrac{4}{3}$.

4.5.47

Denote the faces BCD, CDA, DAB and ABC by numbers $1, 2, 3$ and 4, respectively. For $X \in \{A, B, C, D\}$ and $i \in \{1, 2, 3, 4\}$ denote by X_i the linear angle at the vertex X of the face i. We have (see Fig. 4.50)

$$\sum_{i=2}^{4} A_i + \sum_{i=1, i\neq 2}^{4} B_i + \sum_{i=1, i\neq 3}^{4} C_i + \sum_{i=1}^{3} D_i = 4\pi. \tag{1}$$

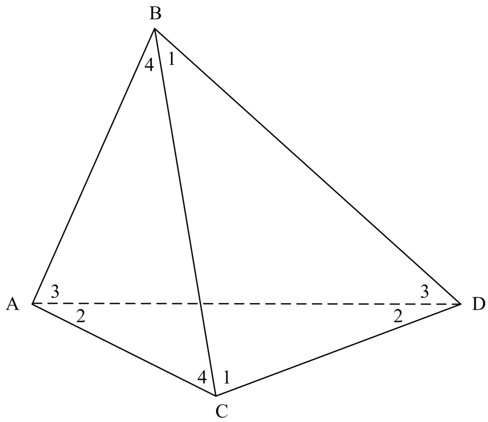

Figure 4.50:

Without lost of generality we can assume that

$$\sum_{i=1}^{3} D_i = \min\left\{\sum_{i=2}^{4} A_i, \sum_{i=1,i\neq 2}^{4} B_i, \sum_{i=1,i\neq 3}^{4} C_i, \sum_{i=1}^{3} D_i\right\}.$$

In this case, from (1) it follows that

$$\sum_{i=1}^{3} D_i \leq \pi. \tag{2}$$

Suppose that $D_1 = \max\{D_1, D_2, D_3\}$, then $2D_1 < D_1 + D_2 + D_3 \leq \pi \implies D_1 < \frac{\pi}{2}$, which means that all linear angles at the vertex D are acute.

From the assumptions of the problem, by the law of sines, we have

$$\begin{cases} \sin D_1 = \sin A_4 \\ \sin D_2 = \sin B_4 \\ \sin D_3 = \sin C_4. \end{cases} \tag{3}$$

Without lost of generality we can assume that $A_4 = \max\{A_4, B_4, C_4\}$. Then A_4 must be acute. Indeed, if $A_4 \geq \frac{\pi}{2}$, then from (3) we get

$$D_1 = \pi - A_4,\ D_2 = B_4,\ D_3 = C_4,$$

which imply that $D_2 + D_3 = B_4 + C_4 = \pi - A_4 = D_1$. This contradicts the properties of the trihedral angle.

Thus $A_4 < \frac{\pi}{2}$, and so the triangle ABC is acute. Then from (3) it follows that $D_1 = A_4,\ D_2 = B_4,\ D_3 = C_4$, which give $D_1 + D_2 + D_3 = \pi$.

Combining the last fact and (1), (2) yields that the sum of all linear angles at each vertex of the tetrahedron is π. Now put all faces BCD, CDA, DAB on the plane ABC, we can easily see that $AB = CD$, $BC = AD$ and $AC = BD$.

4.5.48

First note that for $n = 4$ the four vertices of a regular tetrahedron satisfy the problem.

For $n \geq 5$ we show that there is no n satisfying the problem. Assume in contrary that such an n exists. We denote by R the radius of the circles passing either three of these points. There are two cases:

1) If n points are on the same plane: since there are only two circles of the same radius R passing some two fixed points among the given points, by Pigeonhole principle, there exist two among $n - 2 \geq 3$ remained points lying on the same circle of radius R. Thus there are four points which are on the same circle, which contradicts the second condition of the problem.

2) If n points are not on the same plane: then a plane, passing through at least three points A, B, C of the given n points, divides a space into two half-spaces, one of which contains at least two points D, E of the given n points.

Consider the tetrahedrons $ABCD$ and $ABCE$. Applying the result of the previous problem, we get that those tetrahedrons have the property that opposite edges of each tetrahedron are equal. So as D, E are on the same side of ABC, E must coincide with D, again, a contradiction.

Thus $n = 4$ is the only answer to the problem.

Chapter 5

Olympiad 2009

2009-1. Solve the system

$$\begin{cases} \frac{1}{\sqrt{1+2x^2}} + \frac{1}{\sqrt{1+2y^2}} = \frac{1}{\sqrt{1+2xy}}, \\ \sqrt{x(1-2x)} + \sqrt{y(1-2y)} = \frac{2}{9}. \end{cases}$$

2009-2. Let a sequence (x_n) be defined by

$$x_1 = \frac{1}{2},\ x_n = \frac{\sqrt{x_{n-1}^2 + 4x_{n-1}} + x_{n-1}}{2},\ n \geq 2.$$

Prove that a sequence (y_n) defined by $y_n = \sum\limits_{i=1}^{n} \frac{1}{x_i^2}$ converges and find its limit.

2009-3. In the plane given two fixed points $A \neq B$ and a variable point C satisfying condition $\widehat{ACB} = \alpha$ ($\alpha \in (0^\circ, 180^\circ)$ is constant). The in-circle of the triangle ABC centered at I is tangent to AB, BC and CA at D, E and F respectively. The lines AI, BI intersect the line EF at M, N respectively.

1) Prove that a line segment MN has a constant length.

2) Prove that the circum-circle of a triangle DMN always passes through some fixed point.

2009-4. Three real numbers a, b, c satisfy the following condition: for each positive integer n, the sum $a^n + b^n + c^n$ is an integer. Prove that there exist three integers p, q, r such that a, b, c are the roots of the equation $x^3 + px^2 + qx + r = 0$.

2009-5. Let n be a positive integer. Denote by T the set of the first $2n$ positive integers. How many subsets S are there such that $S \subset T$ and there are no $a, b \in S$ with $|a - b| \in \{1, n\}$? (Remark: the empty set $\emptyset$ is considered as a subset that has such a property).

SOLUTIONS

2009-1. The conditions of the system are as follows

$$\begin{cases} 1 + 2xy > 0 \\ x(1-2x) \geq 0 \\ y(1-2y) \geq 0 \end{cases} \iff \begin{cases} 0 \leq x \leq \frac{1}{2} \\ 0 \leq y \leq \frac{1}{2}. \end{cases}$$

Then $0 \leq xy \leq \dfrac{1}{4}$.

Note that for $a, b \leq 0$ and $ab < 1$, we always have

$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} \leq \frac{2}{\sqrt{1+ab}}, \tag{1}$$

the equality occurs if and only if $a = b$.

Indeed, (1) is equivalent to

$$\begin{aligned} &\left(\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}}\right)^2 \leq \left(\frac{2}{\sqrt{1+ab}}\right)^2 \\ \iff \quad &\frac{1}{1+a^2} + \frac{1}{1+b^2} + \frac{2}{\sqrt{(1+a^2)(1+b^2)}} \leq \frac{4}{1+ab}. \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} &1 + ab \leq \sqrt{(1+a^2)(1+b^2)} \\ \iff \quad &\frac{2}{\sqrt{(1+a^2)(1+b^2)}} \leq \frac{2}{1+ab}. \end{aligned}$$

Furthermore, since $a, b \leq 0$ and $ab < 1$,

$$\begin{aligned} &\frac{1}{1+a^2} + \frac{1}{1+b^2} - \frac{2}{1+ab} = \frac{(a-b)^2(ab-1)}{(1+ab)(1+a^2)(1+b^2)} \leq 0 \\ \iff \quad &\frac{1}{1+a^2} + \frac{1}{1+b^2} \leq \frac{2}{1+ab}. \end{aligned}$$

Therefore,

$$\frac{2}{\sqrt{(1+a^2)(1+b^2)}}+\frac{1}{1+a^2}+\frac{1}{1+b^2}\le\frac{4}{1+ab},$$

the equality occurs if and only if $a=b$.

Return back to the problem. Applying (1) for $a=x\sqrt{2}, b=y\sqrt{2}$, we have

$$\frac{1}{\sqrt{1+2x^2}}+\frac{1}{\sqrt{1+2y^2}}\le\frac{1}{\sqrt{1+2xy}},$$

the equality occurs if and only if $x=y$.

Thus the given system is equivalent to

$$\begin{cases} x=y \\ \sqrt{x(1-2x)}+\sqrt{y(1-2y)}=\frac{2}{9} \end{cases} \iff \begin{cases} x=y \\ 162x^2-81x+1=0, \end{cases}$$

which gives two solutions of the problem:

$$\left(\frac{81+\sqrt{5913}}{324},\frac{81+\sqrt{5913}}{324}\right) \text{ and } \left(\frac{81-\sqrt{5913}}{324},\frac{81-\sqrt{5913}}{324}\right).$$

2009-2. From the assumptions of the problem it follows that $x_n>0$ for all n. Then

$$\begin{aligned} x_n-x_{n-1} &= \frac{\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}}{2}-x_{n-1} \\ &= \frac{\sqrt{x_{n-1}^2+4x_{n-1}}-x_{n-1}}{2} \\ &= \frac{\left(\sqrt{x_{n-1}^2+4x_{n-1}}-x_{n-1}\right)\cdot\left(\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}\right)}{2\left(\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}\right)} \\ &= \frac{2x_{n-1}}{\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}}>0,\ \forall n\ge 2. \end{aligned}$$

This shows that (x_n) is increasing.

If there exists $\lim\limits_{n\to\infty} x_n=L$, then $L>0$. Letting $n\to\infty$ in the given formula of (x_n) yields

$$L=\frac{\sqrt{L^2+4L}+L}{2}\Longrightarrow L=0,$$

which is a contradiction. So $x_n \to \infty$ as $n \to \infty$.

Now we note that for all $n \geq 2$

$$\begin{aligned} & x_n = \frac{\sqrt{x_{n-1}^2 + 4x_{n-1}} + x_{n-1}}{2} \\ \Longrightarrow \quad & 2x_n - x_{n-1} = \sqrt{x_{n-1}^2 + 4x_{n-1}} \\ \Longrightarrow \quad & (2x_n - x_{n-1})^2 = x_{n-1}^2 + 4x_{n-1} \\ \Longrightarrow \quad & x_n^2 = (x_n + 1)x_{n-1} \\ \Longrightarrow \quad & \frac{1}{x_{n-1}} - \frac{1}{x_n} = \frac{1}{x_n^2}. \end{aligned}$$

Then we have for all $n \geq 2$

$$\begin{aligned} y_n &= \sum_{i=1}^{n} \frac{1}{x_i^2} \\ &= \frac{1}{x_1^2} + \left(\frac{1}{x_1} - \frac{1}{x_2}\right) + \left(\frac{1}{x_2} - \frac{1}{x_3}\right) + \cdots + \left(\frac{1}{x_{n-1}} - \frac{1}{x_n}\right) \\ &= \frac{1}{x_1^2} + \left(\frac{1}{x_1} - \frac{1}{x_n}\right) = 6 - \frac{1}{x_n}. \end{aligned}$$

Note also that since $x_n > 0,\ \forall n \geq 1,\ y_n < 6,\ \forall n \geq 1$. Moreover,

$$y_n = y_{n-1} + \frac{1}{x_n} > y_n.$$

So (y_n) is increasing, bounded from above, and therefore, converges. Its limit is

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(6 - \frac{1}{x_n}\right) = 6.$$

2009-3. 1) We have (see Fig. 5.1)

$$\widehat{NFA} = \widehat{CFE} = \frac{180^\circ - \widehat{ACB}}{2} = \frac{\widehat{BAC}}{2} + \frac{\widehat{ABC}}{2} = \widehat{AIN},$$

which implies that the quadrilateral $ANFI$ is cyclic. In this case $\widehat{INF} = \widehat{IAF} = \widehat{IAD}$ and $\widehat{ANI} = \widehat{AFI} = 90^\circ$.

Note that the two triangles NIM and AIB are similar, as they have two equal angles, and so

$$\frac{NM}{AB} = \frac{NI}{AI}. \tag{1}$$

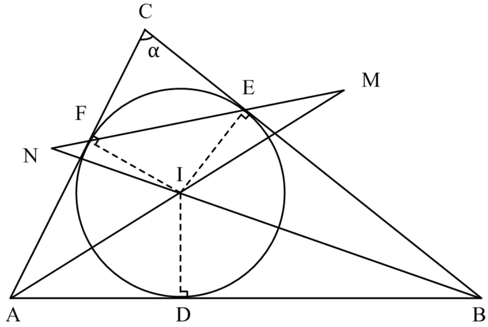

Figure 5.1:

Furthermore, since $\widehat{ANI} = \widehat{AFI} = 90°$, in the right triangle ANI we have

$$\frac{NI}{AI} = \cos\widehat{AIN}. \tag{2}$$

Finally,

$$\widehat{AIN} = \widehat{AFN} = \widehat{CFE} = \frac{180° - \alpha}{2}. \tag{3}$$

From (1), (2) and (3) it follows that

$$MN = AB \cdot \cos\frac{180° - \alpha}{2},$$

which is constant.

2) Let K be the midpoint of AB. As noted above, $\widehat{INA} = \widehat{IMB} = 90°$, and so D, M, N are feet of perpendiculars from the vertices of the triangle ABI. This shows that a circle passing through the three points D, M, N is exactly the Euler's circle of the triangle ABI, and therefore, this circle must pass through the midpoint K of AB. Since AB is fixed, K is fixed too.

2009-4. By Viète formula for the cubic equation we have

$$\begin{cases} a + b + c = -p, \\ ab + bc + ca = q, \\ abc = -r. \end{cases}$$

Thus we suffice to prove that $a + b + c, ab + bc + ca, abc$ are integer.

1) First it is obvious, by the assumption of the problem for $n = 1$, that

$$a + b + c \in \mathbb{Z}. \tag{1}$$

2) Next we prove that abc is integer. The follwing identities will be used very often in the sequel: for any real numbers x, y, z

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx),$$
$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)[(x^2 + y^2 + z^2) - (xy + yz + zx)].$$

Note that $a^n + b^n + c^n$ is integer, in particular, for $n = 2, 3, 4$ and 6. Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) \in \mathbb{Z},$$

and

$$2(a^2b^2 + b^2c^2 + c^2a^2) = (a^2 + b^2 + c^2)^2 - (a^4 + b^4 + c^4) \in \mathbb{Z},$$

we have

$$2(a^3 + b^3 + c^3) - 6abc = (a + b + c)[2(a^2 + b^2 + c^2) - 2(ab + bc + ca)],$$

which implies that

$$6abc = 2(a^3 + b^3 + c^3) - (a + b + c)[2(a^2 + b^2 + c^2) - 2(ab + bc + ca)] \in \mathbb{Z}.$$

Furthermore,

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 = (a^2 + b^2 + c^2)(a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2),$$

and so

$$2(a^6+b^6+c^6)-6a^2b^2c^2 = (a^2+b^2+c^2)[2(a^4+b^4+c^4)-2(a^2b^2+b^2c^2+c^2a^2)],$$

which gives

$$6a^2b^2c^2 = 2(a^6+b^6+c^6)-(a^2+b^2+c^2)[2(a^4+b^4+c^4)-2(a^2b^2+b^2c^2+c^2a^2)] \in \mathbb{Z}.$$

Thus we obtain that both numbers $6abc$ and $6a^2b^2c^2$ are integers. From this it follows that abc is integer too.

3) We prove finally that $ab + bc + ca$ is integer. Indeed, as

$$(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c),$$

we have

$$2(ab + bc + ca)^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) + 4abc(a + b + c) \in \mathbb{Z}.$$

Then from the fact that both numbers $2(ab+bc+ca)$ and $2(ab+bc+ca)^2$ are integers, it follows that $ab+bc+ca$ is integer.

2009-5. We consider the following problem: Given the two rows of points, $A_1, \ldots, A_n$ at the upper row, and $B_1, \ldots, B_n$ at the lower row. We join the pairs of points (A_i, A_{i-1}), (B_i, B_{i-1}), (A_i, B_i), and also the pair (A_1, B_n). Our target is to determine there are how many ways of choosing some points that no two of them are joined.

Let S_n be a number of ways satisfying the requirement said above, but may contain both A_1 and B_n. Denote by x_n the number of ways satisfying the requirement that do not contain any of A_1, B_1, A_n, B_n, by y_n the number of ways satisfying the requirement that contain exactly one of those four points, by z_n the number of ways satisfying the requirement that contain exactly two points A_1, A_n or B_1, B_n, and finally by t_n the number of ways satisfying the requirement that contain exactly two points A_1, B_n or A_n, B_1.

In this case, we have

$$S_n = x_n + y_n + z_n + t_n \qquad (1)$$

and the number of ways satisfying the problem is

$$S_n - \frac{t_n}{2}.$$

It is easy to see that the sequence (S_n) can be defined as follows:

$$S_0 = 1, S_1 = 3,\ S_{n+1} = 2S_n + S_{n-1},\ \forall n \geq 2. \qquad (2)$$

We also have

$$x_n = S_{n-2}, \qquad (3)$$

$$y_n = 2(S_{n-1} - S_{n-2}), \qquad (4)$$

$$z_n = t_{n-1} + \frac{1}{2}y_{n-2},\ t_n = z_{n-1} + \frac{1}{2}y_{n-2}. \qquad (5)$$

By (1), (3) and (4), we have

$$\begin{aligned} z_n + t_n &= S_n - x_n - y_n \\ &= S_n - S_{n-2} - 2(S_{n-1} - S_{n-2}) = S_n - 2S_{n-1} + S_{n-2}, \end{aligned}$$

which, by (2), is equivalently to

$$z_n + t_n = 2S_{n-2}. \qquad (6)$$

Furthermore, by (5) we have

$$z_n - t_n = -(z_{n-1} - t_{n-1}),$$

which implies that

$$z_n - t_n = 2(-1)^{n-1}. \tag{7}$$

Combining (6) and (7) yields

$$t_n = \frac{(z_n + t_n) - (z_n - t_n)}{2} = S_{n-2} + (-1)^n,$$

and hence

$$S_n - \frac{1}{2}t_n = \frac{2S_n - S_{n-2} + (-1)^{n-1}}{2}.$$

This gives the following explicit result

$$S_n = \frac{(5+4\sqrt{2}) \cdot (1+\sqrt{2})^{n-1} + (5-4\sqrt{2}) \cdot (1-\sqrt{2})^{n-1} + 2 \cdot (-1)^{n-1}}{4}.$$

Now return to the given problem. Assigning the number $n+i$ to the point A_i and the number i to the point B_i $(i = 1, \ldots, n)$, the problem is completely solved.

Vietnam has actively organized the National Competition in Mathematics and since 1962, the Vietnamese Mathematical Olympiad (VMO). On the global stage, Vietnam has also competed in the International Mathematical Olympiad (IMO) since 1974 and constantly emerged as one of the top ten.

To inspire and further challenge readers, we have gathered in this book problems of various degrees of difficulty of the VMO from 1962 to 2009.

The book is highly useful for high school students and teachers, coaches and instructors preparing for mathematical olympiads, as well as non-experts simply interested in having the edge over their opponents in mathematical competitions.

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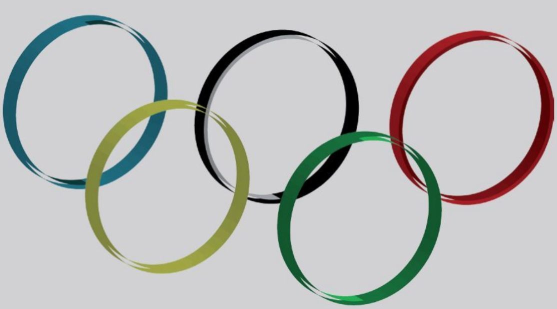

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