Kim Hoo Hang
Haibin Wang

Vol. 10 | Mathematical Olympiad Series

Solving Problems in Geometry

Insights and Strategies for Mathematical Olympiad and Competitions

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Solving Problems in Geometry

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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Names: Hang, Kim Hoo, author. | Wang, Haibin, author.

Title: Solving problems in geometry : insights and strategies for mathematical Olympiad and competitions / by Kim Hoo Hang (NTU, Singapore), Haibin Wang (NUS High School of Mathematics and Science, Singapore).

Description: New Jersey : World Scientific, [2017] | Series: Mathematical olympiad series ; volume 10

Identifiers: LCCN 2017021633| ISBN 9789814590723 (hardcover : alk. paper) | ISBN 9789814583749 (pbk : alk. paper)

Subjects: LCSH: Geometry--Problems, exercises, etc.

Classification: LCC QA459 .H25 2017 | DDC 516.076--dc23

LC record available at https://lccn.loc.gov/2017021633

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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Preface

Elementary geometry is a foundational and important topic not only in Mathematics competitions, but also in mainstream pre-university Mathematics education. Indeed, this is the first axiomatic system most learners encounter: definitions, theorems, proofs and counterexamples. While beginners find the basic theorems and illustrations intuitive, they may encounter difficulties and frequently become clueless when solving problems. For example, the concept of congruent triangles is the most straightforward and easy to understand, but many beginners find it difficult to identify congruent triangles in a diagram, not mentioning constructing congruent triangles intentionally to solve the problem. In particular, drawing auxiliary lines is perceived by many learners as a mysterious skill.

Geometry problems which appear at higher level Mathematics competitions are of course more challenging and require deeper skills. Even the most experienced contestant may spend an hour or so to solve one such problem − while the final solution may be elegantly written down in half a page. In this case, a beginner cannot learn much from merely reading the solution. Such obstacles, with insufficient scaffolding and the lack of guidance, hinder many learners when studying problem-solving in geometry.

In this book, we focus on showing the readers **how** to seek clues and acquire the geometric insight. One may find a few paragraphs named "*Insight*" for almost every problem, where we illustrate how to start tackling the problem, which clues could be found, and how to link the clues leading to the conclusion. Note that such a process is inevitably a lengthy one, during which the reader could attempt a number of strategies and *fail* repeatedly before reaching the final conclusion. A formal proof, usually much shorter, will be presented after we obtain the insight. Occasionally, if sufficient clues have been revealed, we will leave it to the reader to complete the proof.

In the first few chapters, we introduce the basic properties of triangles, quadrilaterals and circles. Proofs and explanatory notes are written down so that the learners will gain the geometric insight of those results, instead of memorizing the literal expression of the theorems. Examples, which range from easy and straightforward to difficult, are used to elaborate how these properties are applied in problem-solving.

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In the later chapters, we give a list of commonly used facts, useful skills and problem-solving strategies which could help readers tackle challenging geometry problems at high-level Mathematics competitions. Such a collection of facts, skills and strategies are seldom found in any mainstream textbooks as these are not standard theorems. They essentially focus on ideas and methodology. We illustrate these skills and strategies using geometry problems from recent-year competitions. The following is a list of these competitions.

1. International, Regional and Invitational Competitions

- IMO International Mathematical Olympiad (including shortlist problems)
- APMO Asia Pacific Mathematical Olympiad
- EGMO European Girls' Mathematical Olympiad
- CMO China Mathematical Olympiad
- CGMO China Girls' Mathematical Olympiad
- CWMO China West Mathematical Olympiad (Invitation)
- CZE-SVK Czech and Slovak Mathematical Olympiad
- IWYMIC Invitational World Youth Mathematics Intercity Competition

2. National Competitions and Selection Tests

- AUT Austria
- BLR Belarus
- BRA Brazil
- BGR Bulgaria
- CAN Canada
- CHN China
- HRV Croatia
- GER Germany

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- HEL Greece
- HUN Hungary
- IND India
- IRN Iran
- ITA Italy
- JPN Japan
- ROU Romania
- RUS Russia
- SVN Slovenia
- TUR Turkey
- UKR Ukraine
- USA U.S.A.
- VNM Vietnam

Elementary geometry is a beautiful area of mathematics. Upon the mastery of the basic knowledge and skills, one will always find solving a geometry problem an exciting experience. We wish the readers a pleasant experience with the time spent on this book. Enjoy Mathematics and enjoy problem-solving!

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Contents

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Chapter 1

Congruent Triangles

We assume the reader knows the following basic geometric concepts, which we will not define:

- Points, lines, rays, line segments and lengths
- Angles, right angles, acute angles, obtuse angles, parallel lines (//) and perpendicular lines (⊥)
- Triangles, isosceles triangles, equilateral triangles, quadrilaterals, polygons
- Height (altitudes) of a triangle, area of a triangle
- Circles, radii, diameters, chords, arcs, minor arcs and major arcs

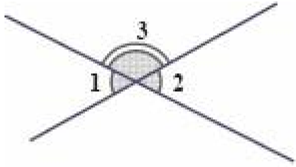
1.1 Preliminaries

We assume the reader is familiar with the fundamental results in geometry, especially the following, the illustration of which can be found in any reasonable secondary school textbook.

(1) For any two fixed points, there exists a unique straight line passing through them (and hence, if two straight lines intersect more than once, they must coincide).

(2) For any given straight line ℓ and point P, there exists a unique line passing through P and parallel to ℓ.

(3) Opposing angles are equal to each other. (Refer to the diagram below. $\angle 1$ and $\angle 2$ are opposing angles. We have $\angle 1 = 180° - \angle 3 = \angle 2$.)



Opposing Angles

(4) In an isosceles triangle, the angles which correspond to equal sides are equal. (Refer to the diagram below.)

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The inverse is also true: if two angles in a triangle are the same, then they correspond to the sides which are equal.

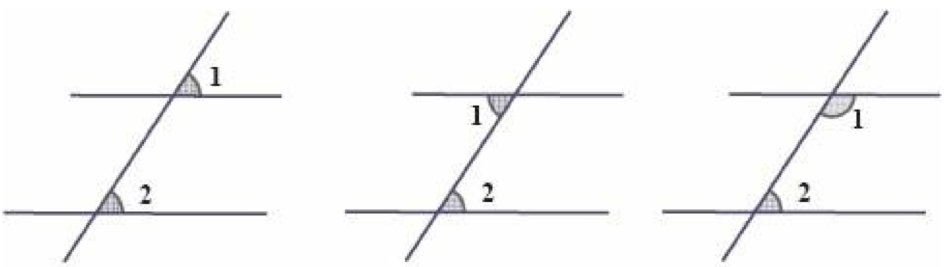
Equal angles correspond to equal sides.

(5) Triangle Inequality: In any triangle ΔABC, $AB + BC > AC$.

(A straight line segment gives the shortest path between two points.)

(6) If two parallel lines intersect with a third, we have:

- The corresponding angles are the same.
- The alternate angles are the same.
- The interior angles are *supplementary* (i.e., their sum is 180°). (Refe to the diagrams below.)



Corresponding Angles Alternate Angles Interior Angles

Its inverse also holds: equal corresponding angles, equal alternate angles or supplementary interior angles imply parallel lines.

One may use (6) to prove the following well-known results.

Theorem 1.1.1 *The sum of the interior angles of a triangle is* 180°.

Proof. Refer to the diagram below. Draw a line passing through A which is parallel to BC. We have $\angle B = \angle 1$ and $\angle C = \angle 2$.

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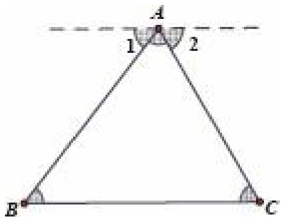
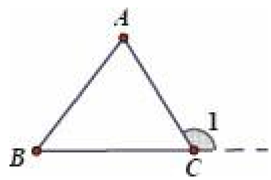


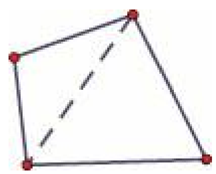
Hence, $\angle A + \angle B + \angle C = \angle A + \angle 1 + \angle 2 = 180°$.

□

An immediate and widely applicable corollary is that an *exterior* angle of a triangle equals the sum of two non-neighboring interior angles. Refer to the diagram below. We have $\angle 1 = 180° - \angle C = \angle A + \angle B$.

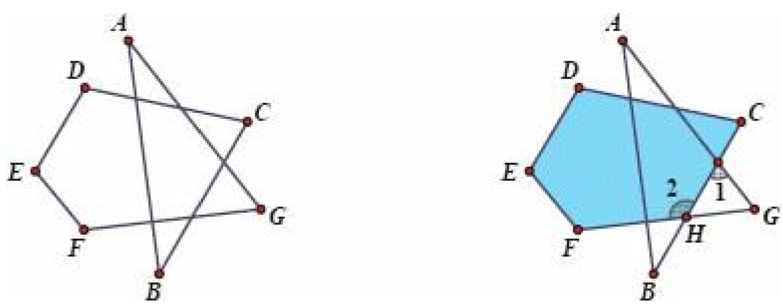


It is also widely known that the sum of the interior angles of a quadrilateral is 360°. Notice that a quadrilateral could be divided into two triangles. Refer to the diagram below.



One sees that similar arguments apply to a general n-sided (convex) polygon: the sum of the interior angles is $180° \times (n - 2)$.

Example 1.1.2 Find $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$ in the left diagram below.



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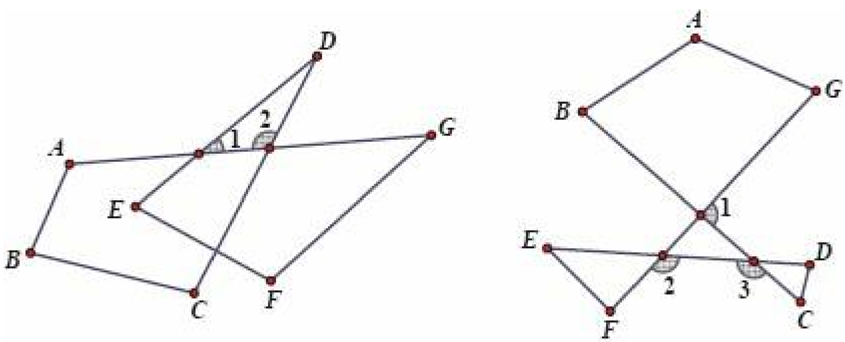
Ans. Refer to the right diagram above. Let *BC* and *FG* intersect at *H*. Notice that $\angle A + \angle B = \angle 1$ and $\angle 1 + \angle G = \angle 2$.

Now $\angle 2 + \angle C + \angle D + \angle E + \angle F = 540°$, as this is the sum of the interior angles of the convex pentagon (i.e., a 5-sided polygon) *CDEFH*.

In conclusion, $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G = 540°$.

□

Note: Using the exterior angles of a triangle is an effective method to solve this type of questions. Refer to the following diagrams. Can you see $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G = 540°$ in both cases?



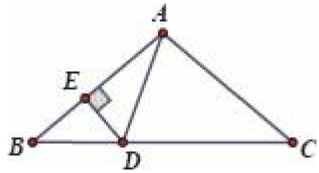
(i) (ii)

Hint:

(i) Connect *EG*. Can you see $\angle E + \angle F + \angle G = 180° + \angle 1$? A similar argumer applies to $\angle A + \angle B + \angle C$.

(ii) Connect *BG*. Can you see $\angle A + \angle B + \angle G = 180° + \angle 1$? Can you see $\angle E$ $\angle F = \angle 2$? Can you find $\angle 1 + \angle 2 + \angle 3$? (Consider their supplementary angles.)

Example 1.1.3 Refer to the diagram below. ΔABC is an isosceles triangle where $AB = AC$. *D* is a point on *BC* such that $AB = CD$. Draw $DE \perp AB$ at *E*. Show that $2\angle ADE = 3\angle B$.



Insight. We are not given the exact value of $\angle BAC$ or $\angle B$, but if we know either of them, then the positions of *D* and *E* are uniquely determined, according to the construction of the diagram. Let $\angle B = x$. Can you express $\angle ADE$ in term of *x*?

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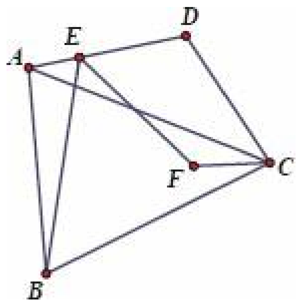
Proof. Let $\angle B = \angle C = x$. We have $\angle BAC = 180° - 2x$. Notice that ΔCAD is an isosceles triangle, where $AC = AB = CD$. It follows that $\angle ADC = \angle CAD = \frac{1}{2}(180° - x) = 90° - \frac{x}{2}$.

Now $\angle BAD = \angle BAC - \angle CAD = (180° - 2x) - \left(90° - \frac{x}{2}\right) = 90° - \frac{3}{2}x$.

Hence, $\angle ADE = \frac{3}{2}x$ (because $\angle BAD + \angle ADE = 90°$ in the right angled triangle ΔAED). The conclusion follows.

□

Example 1.1.4 Given a quadrilateral *ABCD, E* is a point on *AD*. *F* is a point inside *ABCD* such that *CF*, *EF* bisects $\angle ACB$ and $\angle BED$ respectively. Show that $\angle CFE = 90° + \frac{1}{2}(\angle CAD + \angle CBE)$. (**Note**: an *angle bisector* divides the angle into two equal halves.)



Insight. Refer to the diagram below. One sees that $\angle CAD$ and $\angle CBE$ are NOT related. For example, if $\angle CAD$ is given, one may move *E* along *AD* and $\angle CBE$ will vary. On the other hand, if $\angle CBE$ is given, one may choose *A*' on *DA* extended so that $\angle CA'D$ is smaller than $\angle CAD$.
Hence, if we let $\angle CAD = \alpha$, we cannot express $\angle CBE$ in α (and vice versa). How about letting $\angle CBE = \beta$? We **should** be able to express $\angle CFE$ in α and β.
Notice that $\angle CFE$ is constructed via angle bisectors *EF* and *CF*. Let $\angle BED = 2x$ and $\angle ACB = 2y$. Refer to the diagram below. Let *AC* and *EF* intersect at *G*. In ΔCFG, one sees that $\angle CFE = 180° - y - \angle CGF$, where $\angle CGF = \angle AGE = \angle DEF - \angle EAG = x - \alpha$.
Hence, $\angle CFE = 180° - x - y + \alpha$. (1)

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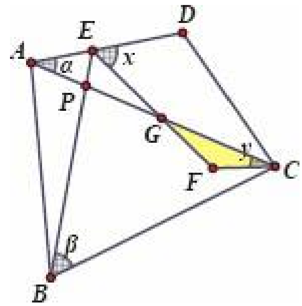


We are to show $\angle CFE = 90° + \frac{1}{2}(\angle CAD + \angle CBE) = 90° + \frac{1}{2}(\alpha + \beta)$.

How are *x, y* related to α,β ? Let *AC* and *BE* intersect at *P*. Consider Δ*AEP* and Δ*BCP*. One sees that ∠*PAE* + ∠*PEA* = 180° – ∠*APE* = 180° – ∠*BPC* = ∠*PBC* + ∠*PCB*.

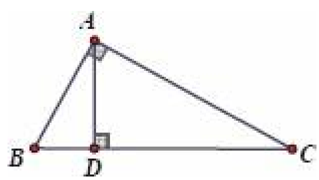
Hence, $\alpha + (180° - 2x) = \beta + 2y$, which implies $x + y = 90° + \frac{1}{2}(\alpha - \beta)$.

Now (1) gives $\angle CFE = 180° - \left(90° + \frac{1}{2}(\alpha - \beta)\right) + \alpha = 90° + \frac{1}{2}(\alpha + \beta)$.

Note: This is not an easy problem, but it could be solved by elementary knowledge. When solving problems purely about angles, it is a useful technique to set an unknown angle as a variable and apply algebraic manipulations. If one variable is not enough (to express the other angles), one may set more variables, but remember to work towards cancelling out those variables, simply because they should **not** appear in the conclusion. In order to cancel out the variables, one should seek for equalities among angles. Useful clues include right angles, isosceles triangles, exterior angles and angle bisectors.

The following examples give standard results which are frequently used in problem-solving. One should be very familiar with these results.

Example 1.1.5 In Δ*ABC*, ∠*A* = 90° and *AD* ⊥ *BC* at *D*. Show that ∠*BAD* = ∠*C* and ∠*CAD* = ∠*B*.



Proof. Refer to the diagram below. We have ∠ *BAD* = ∠90° – ∠*B* = ∠*C* and similarly, ∠*CAD* = 90° – ∠*C* = ∠*B*.

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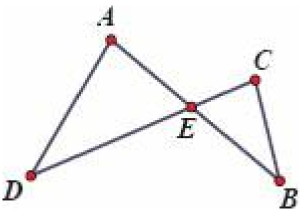
□

Example 1.1.6 Refer to the diagram below. AB and CD intersect at E. If $\angle B = \angle D$, show that $\angle A = \angle C$.



Proof. We have $\angle A = 180° - \angle D - \angle AED$ and $\angle C = 180° - \angle B - \angle BEC$. Since $\angle B = \angle D$ and $\angle AED = \angle BEC$, it follows that $\angle A = \angle C$.

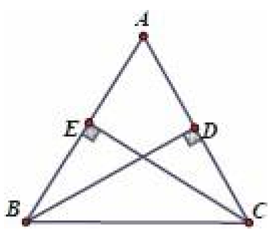
□

Notice that $\angle A + \angle D = \angle B + \angle C$ always holds even if we do not have $\angle B = \angle D$. We used this fact in Example 1.1.4.

Example 1.1.7 In an acute angled triangle ΔABC, BD, CE are heights. Show that $\angle ABD = \angle ACE$.

Proof. Refer to the diagram below. We have $\angle ABD = 90° - \angle A = \angle ACE$.

□



One may also see this as a special case of Example 1.1.6, where $\angle BEC = \angle BDC = 90°$.

Example 1.1.8 In ΔABC, M is the midpoint of BC. Show that if $AM = \frac{1}{2}BC$, then $\angle A = 90°$.

Proof. Refer to the diagram below. Since $AM = BM = CM$, we have $\angle 1 = \angle B$ and $\angle 2 = \angle C$, i.e., $\angle A = \angle B + \angle C$ Since $\angle A + \angle B + \angle C = 180°$, $\angle A = 90°$.

□

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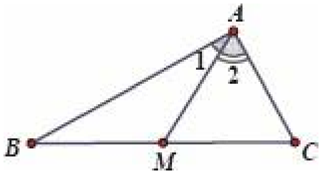
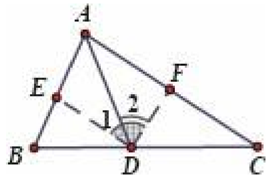


Example 1.1.9 In ΔABC, D is on BC. Show that the angle bisectors of $\angle ADB$ and $\angle ADC$ are perpendicular to each other.

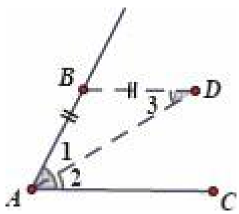
Proof. Refer to the diagram below. Let DE, DF be the angle bisectors of $\angle ADB$ and $\angle ADC$ respectively.



Since $\angle 1 = \frac{1}{2}\angle ADB$, $\angle 2 = \frac{1}{2}\angle ADC$ and $\angle ADB + \angle ADC = 180°$, we have $\angle 1 + \angle 2 = 90°$, and hence the conclusion.

□

Example 1.1.10 Refer to the diagram below. Let AD bisect $\angle A$. If $BD \parallel AC$, show that $AB = BD$.



Proof. We are given $\angle 1 = \angle 2$. Since $BD \parallel AC$, $\angle 2 = \angle 3$. Now $\angle 1 = \angle 3$ and it follows that $AB=BD$.

□

Note:

(1) It is a commonly used technique to construct an isosceles triangle from an angle bisector and parallel lines. Besides giving equal angles, angle bisectors have many other useful properties, which we will see in later chapters.

(2) Notice that the inverse also holds:

- If we are given that $AB = BD$ and AD bisects $\angle A$, then we must have $BD \parallel AC$.

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- If we are given that $AB = BD$ and $BD \parallel AC$, then AD must be the angle bisector of $\angle A$.

Example 1.1.11 Given lines $\ell_1 \parallel \ell_2$ and a point P, draw $PA \perp \ell_1$ at A and $PB \perp \ell_2$ at B, then P,A,B are collinear (i.e., the three points lie on the same line).

Proof. Refer to the diagrams below. Suppose otherwise that P,A,B are not collinear. Let AP extended intersect ℓ_2 at C. Now $\angle PCB = 90°$ and ΔPBC has two 90° interior angles. This is absurd.

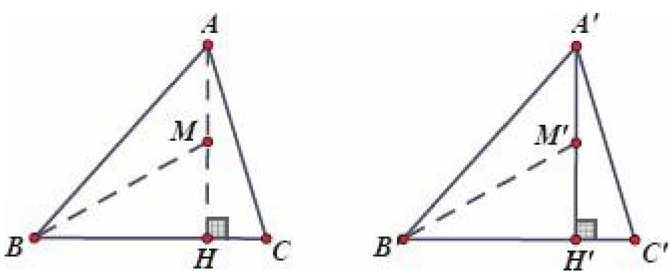
Notice that the argument holds even if ℓ_1,ℓ_2 are on the same side of P. Refer to the diagram above on the right.

□

1.2 Congruent Triangles

Congruent triangles are the cornerstones of elementary geometry. We say two triangles ΔABC and $\Delta A'B'C'$ are congruent if they are exactly the same: $AB = A'B'$, $AC = A'C'$, $BC = B'C'$, $\angle A = \angle A'$, $\angle B = \angle B'$ and $\angle C = \angle C'$. We denote this by $\Delta ABC \cong \Delta A'B'C'$.

Moreover, if $\Delta ABC \cong \Delta A'B'C'$, **<u>all</u>** the corresponding line segments and angles are identical. Refer to the diagrams below for an example: Given $\Delta ABC \cong \Delta A'B'C'$, let AH be the height of ΔABC on the side BC and $A'H'$ be the height of $\Delta A'B'C'$ on the side $B'C'$. Let M,M' be the midpoints of AH, $A'H'$ respectively. We must have $BM = B'M'$ and $\angle BMH = \angle B'M'H'$.



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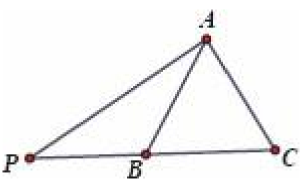
Applying the definition directly could verify a pair of congruent triangles. However, in most of the cases, this is unnecessary. It is taught in most secondary education that one can verify congruent triangles using one of the following criteria:

- S.A.S.: If two pairs of corresponding sides and the angles **between** them are identical, then the two triangles are congruent, i.e., if $AB = A'B'$, $AC = A'C'$ and $\angle A = \angle A'$, then $\Delta ABC \cong \Delta A'B'C'$.
- A.A.S.: If one pair of corresponding sides and any two pairs of corresponding angles are identical, then the two triangles are congruent.
- S.S.S.: If all the corresponding sides are identical, then the two triangles are congruent.

Note:

(1) S.A.S. applies only when two pairs of corresponding sides and the angle **between** them are identical. Otherwise, we **cannot** use this criterion. Refer to the following counter example:
Let ΔABC be an isosceles triangle where $AB = AC$. P is a point on CB extended. Refer to the diagram below. Consider ΔPAC and ΔPAB.
We have $AB = AC$, $\angle P$ is a common angle and AP is a common side. However, one sees clearly that $\Delta PAC \ncong \Delta PAB$ because $\angle PBA > 90° > \angle PCA$.



(2) One may also write A.A.S. as A.S.A. In fact, it does not matter whether the corresponding sides are between the two pairs of corresponding angles, simply because two pairs of equal angles automatically gives the third pair of equal angles: the sum of the interior angles of a triangle is always 180°.

(3) H.L.: If ΔABC and $\Delta A'B'C'$ are right angled triangles, then they are congruent if their hypotenuses and one pair of corresponding legs are identical, i.e., if $\angle A = \angle A' = 90°$, $AB = A'B'$ and $BC = B'C'$, then $\Delta ABC \cong \Delta A'B'C'$.
Indeed, one may place the two right angled triangles together and form an isosceles triangle. Refer to the diagram below. $BC = B'C'$ immediately gives $\angle C = \angle C$ and hence, we have $\Delta ABC \cong \Delta A'B'C$ (A.A.S.).

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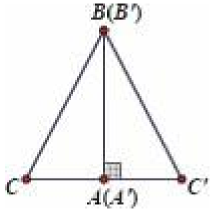
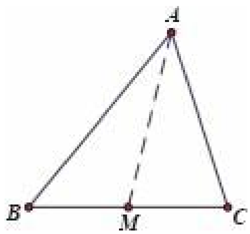


One immediate application of congruent triangles on isosceles triangles is that the angle bisector of the vertex angle, the median on the base and the height on the base of an isosceles triangle coincide.

Definition 1.2.1 In ΔABC, let M be the midpoint of BC such that $BM = CM$, then AM is called the median on the side BC. (Refer to the diagram on the below.)



Theorem 1.2.2 *Let* ΔABC be an isosceles triangle such that $AB = AC$. *Let* M *be the midpoint of BC. We have:*

(1) $AM \perp BC$

(2) *AM bisects* $\angle A$, *i.e.,* $\angle BAM = \angle CAM$.

***Proof*.** The conclusion follows from $\Delta ABM \cong \Delta ACM$ (S.S.S.).

□

Notice that in the theorem above, any point P on the line AM gives an isosceles triangle ΔPBC. Refer to the diagram below. Indeed, AM is the perpendicular bisector of the line segment BC.

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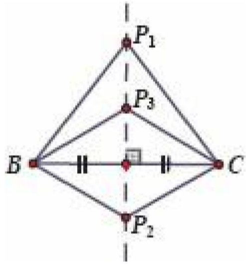
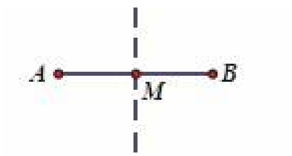


Definition 1.2.3 The perpendicular bisector of a line segment *AB* is a straight line which passes through the midpoint of *AB* and is perpendicular to *AB*.



Perpendicular bisector of *AB*

Theorem 1.2.4 *Given a line segment AB and a point P. We have PA = PB if and only if P lies on the perpendicular bisector of AB. In particular, if P,Q are two points such that PA = PB and QA = QB, then the line PQ is the perpendicular bisector of AB.*

One may show the conclusion easily by using congruent triangles. We leave it to the reader.

Notice that Theorem 1.2.2 states that in an isosceles triangle ΔABC where $AB = AC$, the angle bisector of $\angle A$, the median on *BC*, and the height on *BC* coincide. Moreover, one could show by congruent triangles that the inverse is also true: if any two among these three lines coincide (for example, *AD* bisects $\angle A$ where *D* is the midpoint of *BC*), then ΔABC is an isosceles triangle with $AB = AC$. This is an elementary property of isosceles triangles, but it may apply in a subtle manner in problem-solving, which confuses beginners.

Example 1.2.5 Given ΔABC where $\angle A = 90°$ and $AB = AC$, *D* is a point on *AC* such that *BD* bisects $\angle ABC$. Draw $CE \perp BD$, intersecting *BD* extended at *E*. Show that $BD = 2CE$.

Insight. Apparently, the conclusion does not give us any clue because *BD* and *CE* are not directly related. Perhaps we should seek clues from the conditions.
It is given that *BE* bisects $\angle ABC$ and we see that *BE* is *almost* a height: not a

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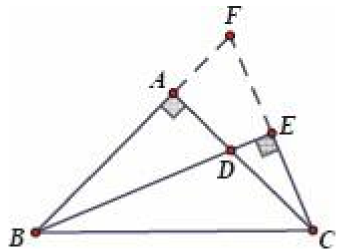
height of any given triangle, but $BE \perp CE$. If we *fill up* the triangle by extending BA and CE, intersecting each other at F, then BE is the height of ΔBCF. Refer to the diagram below.



Can you see ΔBCF is an isosceles triangle? Moreover, E must be the midpoint of CF as well, which implies $CF = 2CE$. Hence, it suffices to show $BD = CF$.

How are BD and CF related? If it is not clear to you, seek clues from the conditions again! Which conditions have we not used yet? We are given $AB = AC$ and $\angle BAC = 90°$. How are they related to BD and CF? We **should** have $\Delta ABD \cong \Delta ACF$ if $BD = CF$. How can we show $\Delta ABD \cong \Delta ACF$? We have a pair of equal sides $AB = AC$ and a pair of right angles. Showing $AD = AF$ may not be easy because we do not know the position of A on BF. Can we find another pair of equal angles?

Proof. Let BA extended and CE extended intersect at F. Since BE bisects $\angle ABC$ and $BE \perp CF$, we have $\Delta BEC \cong \Delta BEF$ (A.A.S.) and hence, $CF = 2CE$. It is easy to see $\angle ABD = \angle DCE$ (Example 1.1.6). Since $AB = AC$ and $\angle BAD = 90° = \angle CAF$, we have $\Delta ABD \cong \Delta ACF$ (A.A.S.). It follows that $BD = CF = 2CE$.

□

Note:

(1) One may derive a few conclusions from the proof above. For example, can you see $\angle ADB = \angle BCE$ and $BC = AB + AD$?

(2) How did we see the auxiliary lines? Notice that we basically *reflected* ΔBCE along the angle bisector BE. This is an effective technique which utilizes the symmetry property of the angle bisector.

Recognizing congruent triangles is one of the most fundamental but useful methods in showing equal line segments or angles. In particular, one may seek congruent triangles via the following clues:

- Equilateral triangles and isosceles triangles
- Right angled triangles with the height on the hypotenuse (which gives equal angles, Example 1.1.5)
- Common sides or angles shared by triangles
- Parallel lines

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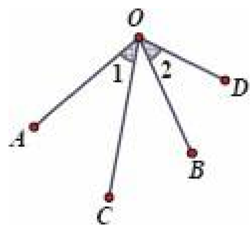
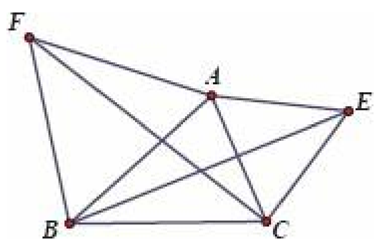
- Medians and angle bisectors
- Opposite angles (Example 1.1.6)

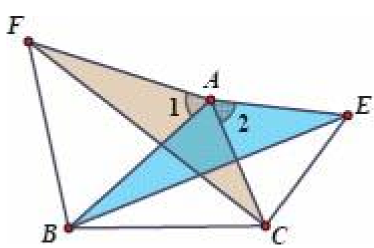


- Equal angles sharing the common vertex: Refer to the diagram on the below. If $\angle 1 = \angle 2$, then $\angle AOB = \angle COD$. Notice that the inverse also holds, i.e., if $\angle AOB = \angle COD$, then $\angle 1 = \angle 2$.

Example 1.2.6 Refer to the diagram below. In ΔABC, draw equilateral triangles ΔABF and ΔACE outwards from AB, CA respectively. Show that $BE = CF$.



Proof. We have equal sides $AF = AB$ and $AC = AE$ due to the equilateral triangles.
Notice that we also have $\angle 1 = \angle 2 = 60°$.
Hence, $\angle 1 + \angle BAC = \angle 2 + \angle BAC$, i.e., $\angle BAE = \angle CAF$.



It follows that $\Delta BAE \cong \Delta FAC$ (S.A.S.), which leads to the conclusion that $BE = CF$.

□

Example 1.2.7 In an acute angled triangle ΔABC, $\angle A = 45°$. AD, BE are heights. If AD, BE intersect at H, show that $AH = BC$.

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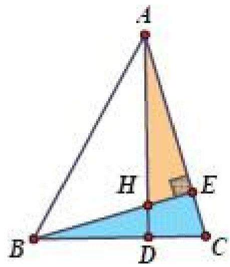
Insight. Can we find a pair of congruent triangles where *AH, BC* are corresponding sides? It is given $\angle A = 45°$ and $BE \perp AC$. Hence, it is easy to see that $AE = BE$.

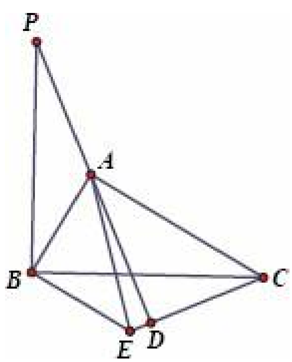
Refer to the diagram below. It *seems* that ΔAEH and ΔBEC are congruent.

Since $\angle AEH = 90° = \angle BEC$, we only need one more condition. Shall we prove $CE = EH$, or find another pair of equal angles? Can you see $\angle CBE = \angle CAH$ (Example 1.1.6)?



We leave it to the reader to complete the proof.

Example 1.2.8 Refer to the diagram below. In ΔABC, $\angle A = 90°$. *P* is a point outside ΔABC such that $PB \perp BC$ and $PB = BC$. *D* is a point on *PA* extended such that $CD \perp PA$. *E* is a point on *CD* extended such that $BE \perp AB$. Show that *AE* bisects $\angle BAC$.



Insight. We are given $PB = PC$ and one can easily see that $\angle ABP = \angle EBC$. Are there any congruent triangles? It *seems* from the diagram that $\Delta ABP \cong \Delta EBC$. Is it true? We are to show *AE* bisects $\angle BAC$, i.e., $\angle BAE = 45°$. Hence, we **should** have ΔABE a right angled isosceles triangle where $AB = BE$, i.e., ΔABP and ΔEBC **should** be congruent. Now can we find another pair of equal sides or angles?

Proof. Notice that $\angle ABP = 90° - \angle ABC = \angle EBC$. We also have $\angle APB = \angle BCE$ (Example 1.1.6, *BC* intersecting *PD*). Since $PB = BC$, we conclude that $\Delta ABP \cong \Delta EBC$. Hence, $AB=BE$, which implies ΔABE is a right angled isosceles

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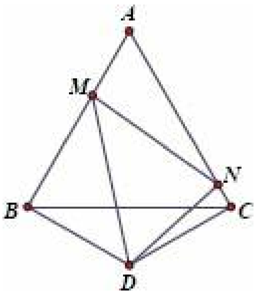
triangle.

Now $\angle BAE = 45° = \frac{1}{2}\angle BAC$, which implies AE bisects $\angle BAC$.

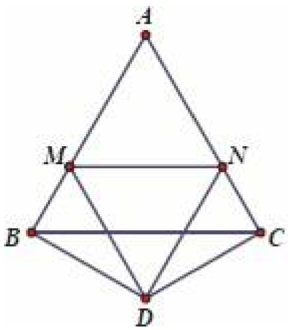
□

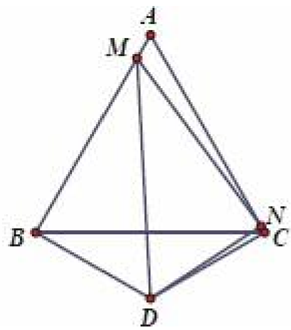
Example 1.2.9 Refer to the diagram below. ΔABC is an equilateral triangle with AB = 10 cm. D is a point outside ΔABC such that $BD = CD$ and $\angle BDC$ = 120°. M, N are on AB, AC respectively such that $\angle MDN$ = 60°. Find the perimeter of ΔAMN.



Insight. The difficulty is that M is arbitrary, i.e., it can be any point on AB. Even though we know $\angle MDN = \frac{1}{2}\angle BDC$, it is hard to apply this condition directly.

What if we choose a special point M, say when ΔDMN is an equilateral triangle? Refer to the left diagram below. Now ΔAMN is also an equilateral triangle. One may show (by studying the property of the right angled triangle ΔBDM) that $AM = \frac{2}{3}AB$. Hence, the perimeter of $\Delta AMN = 2AB = 20$ cm.





What if we choose M to be very close to A? Refer to the diagram above on the right. ΔAMN seems to become very narrow. AM is approaching to zero

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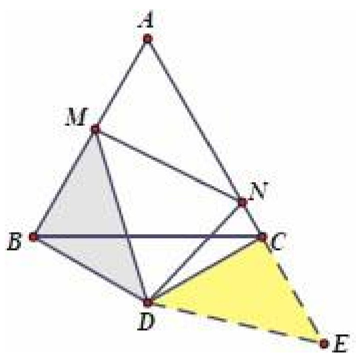
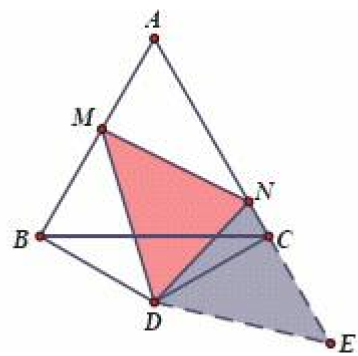
length while AN and MN are very close to AC. In this case, we may expect the perimeter of ΔAMN to be $0 + AC + AC = 2AC = 20$ cm.

It seems that we shall prove $AM + AN + MN = AB + AC$, i.e., $MN = BM + CN$ However, it may not be easy to show this directly as BM and CN are far apart. Notice that we encounter the same difficulty: given that $\Delta MDN = \frac{1}{2}\angle BDC$, how to handle the remaining portions of $\angle BDC$? If we can put those portions together, an equal angle of $\angle MDN$ would appear. How can we put $\angle BDM$ and $\angle CDN$, as well as BM and CN together? Cut and paste!

Ans. Extend AC to E such that $CE = BM$. Connect DE. Notice that $\angle DBC = \angle DCB = \frac{1}{2}(180° - 120°) = 30°$. Hence, $\angle DBM = \angle DCE = 90°$ and we have $\Delta DBM \cong \angle DCE$ (S.A.S.). This implies $\angle BDM = \angle CDE$ and $DM = DE$ Refer to the following left diagram.





In order to show $MN = BM + CN = CE + CN = NE$ it suffices to show $\Delta DNM \cong \Delta DNE$. Since $\angle MDN = 60°$, $\angle BDM + \angle CDN = 60°$.
Hence, $\angle EDN = \angle CDE + \angle CDN = 60° = \angle MDN$. Since $DM = DE$, it follows that $\Delta DNM \cong \Delta DNE$ (S.A.S.).
In conclusion, $AM + MN + AN = AB + AC = 20$ cm.

□

One may apply congruent triangles to prove the following useful properties. These are not the standard theorems, but one familiar with these results could have a better understanding of the basic geometrical facts and seek clues during problem-solving more effectively.

Example 1.2.10 Given a line segment AB and two points P, Q such that line PQ intersects AB at C, if $\angle APC = \angle BPC$ and $\angle AQC = \angle BQC$, then PQ is the perpendicular bisector of AB.

Proof. Case I: P, Q are on the same side of AB. Refer to the diagram

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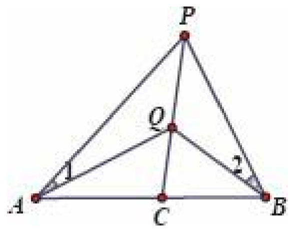
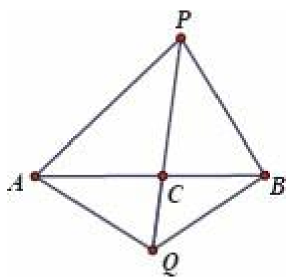
below. We have $\angle 1 = \angle AQC - \angle APC = \angle BQC - \angle BPC = \angle 2$.

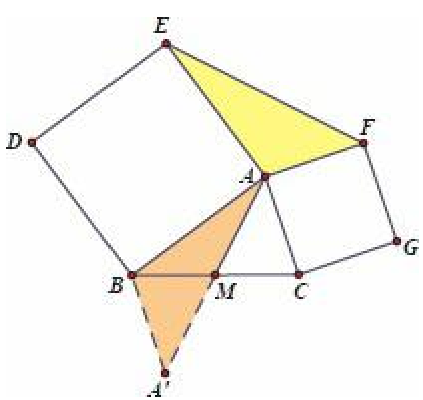


It follows that $\Delta APQ \cong \Delta BPQ$ (A.A.S.), which implies $AP = BP$.
Since PC is the angle bisector of the isosceles triangle ΔPAB, it is also the perpendicular bisector of AB.
Case II: P, Q are on different sides of AB. Refer to the diagram below. It is easy to see that $\Delta APQ \cong \Delta BPQ$ (A.A.S.). Hence, $PA = PB$ and $QA = QB$ The conclusion follows by Theorem 1.2.4.

□



Example 1.2.11 Given ΔABC, draw squares $ABDE$ and $CAFG$ outwards based on AB, CA respectively. Let M be the midpoint of BC. Show that $AM = \frac{1}{2}EF$.



Insight. We see that ΔABC and ΔAEF have two equal pairs of sides: AB =

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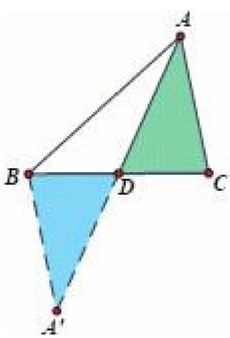
AE and $AC = AF$. However, it is clear that $\Delta ABC \ncong \Delta AEF$ because $\angle BAC \neq \angle EAF$. In fact, $\angle BAC$ and $\angle EAF$ are supplementary. (Can you see it?)
Since M is the midpoint of BC, a commonly used technique is to double AM. Refer to the diagram above on the right. If we extend AM to A' such that $AM = A'M$, can you see that $\Delta BAA' \cong \Delta AEF$?

Proof. Extend AM to A' such that $AM = A'M$. Since $BM = CM$ and $\angle A'MB = \angle AMC$, we have $\Delta A'MB \cong \Delta AMC$ (S.A.S.), which implies $\angle CAM = \angle BA'M$ and hence, $AC \parallel A'B$. It follows that $\angle ABA' = 180° - \angle BAC$. Since $\angle EAF + \angle BAC = 360° - 90° - 90° = 180°$, we must have $\angle EAF = 180° - \angle BAC = \angle ABA'$. Since $AB = AE$, $BA' = AC = AF$, we conclude that $\Delta BAA' \cong \Delta AEF$ (S.A.S.). It follows that $EF = AA' = 2AM$.

□

Note: It is an important technique to extend and double the median of a triangle because this immediately gives congruent triangles.



Refer to the diagram below where AD is a median of ΔABC and we have $\Delta ACD \cong \Delta A'BD$.
After this *rotation* of ΔACD, we may put together lengths and sides which are previously far apart and perhaps obtain useful conclusions.

Example 1.2.12 In ΔABC, D is the midpoint of BC. E is a point on AC such that BE intersects AD at P and $BP = AC$. Show that $AE = PE$.

Insight. We are given $BP = AC$, which should be an important condition. However, BP and AC are far apart and it seems not clear how one could use this condition. How about the median AD doubled? Refer to the diagram below. If we extend AD and take $A'D = AD$, one sees immediately that $\Delta ACD \cong A'BD$. In particular, we have $AC = A'B$.
In fact, we rotated ΔACD and hence, moved AC to $A'B$. Now $A'B$ and BP are connected: we can apply the condition $BP = AC$.

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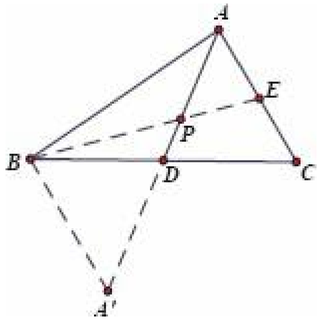


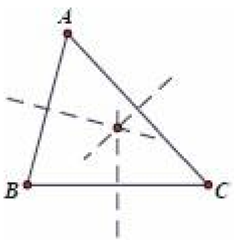
Proof. Extend AD to A' such that $AD = A'D$. It is easy to see that $\Delta ACD \cong \Delta A'BD$ (S.A.S.). Hence, $\angle A' = \angle CAD$ and $BP = AC = A'B$, i.e., $\Delta BA'P$ is an isosceles triangle.
Now we have $\angle APE = \angle BPA' = \angle A' = \angle CAD$, which implies $AE = PE$.

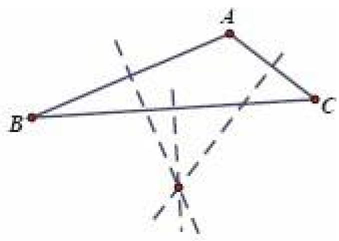
□

1.3 Circumcenter and Incenter of a Triangle

Given a triangle, there are many interesting points in it.

Recall the definition of the perpendicular bisector of a line segment. Since each triangle has three sides, one may draw three perpendicular bisectors. Note that these perpendicular bisectors are *concurrent*, i.e., they pass through the same point. Refer to the following diagrams.





This particular point is called the **circumcenter** of the triangle. Notice that each triangle has exactly one circumcenter and it could be outside the triangle. Refer to the right diagram above.

Now we use congruent triangles to show the existence of the circumcenter of a triangle.

Theorem 1.3.1 *The perpendicular bisectors of a triangle are concurrent*.

Proof. Refer to the left diagram below. Let the perpendicular bisectors of AB, BC intersect at O. We are to show that the perpendicular bisector of AC passes through O as well.

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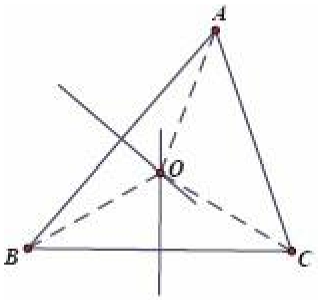
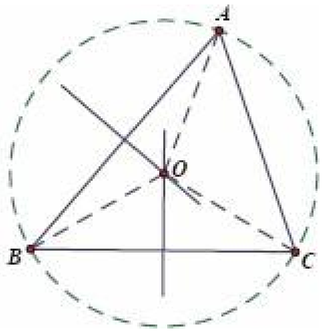
Since O lies on the perpendicular bisector of AB, we have $AO = BO$ (Theorem 1.2.4). Similarly, $BO = CO$. It follows that $AO = CO$. Hence, O lies on the perpendicular bisector of AC (Theorem 1.2.4).

□





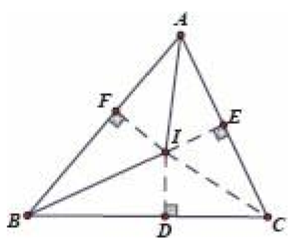
Note:

(1) This common point of intersection is called the circumcenter as it is the center of the circumcircle of ΔABC. Refer to the right diagram above. A circle centered at O with radius OA passes through A, B and C, since $OA = OB = OC$.

(2) In the proof above, we assume the two perpendicular bisectors interse at O and show that this point lies on the third perpendicular bisector. This is a common method to show three lines passing through the same point.

Theorem 1.3.2 *The angle bisectors of a triangle are concurrent*.

Proof. Refer to the diagram on the below. Let the angle bisector of $\angle A$ and $\angle B$ intersect at I. We show that the angle bisector of $\angle C$ passes through I as well, i.e., $\angle ACI = \angle BCI$.



Draw $ID \perp BC$ at D, $IE \perp AC$ at E and $IF \perp AB$ at F. Since AI is the angle bisector of $\angle A$, it is easy to see $\Delta AIF \cong \Delta AIE$ (A.A.S.). Hence, $IF = IE$ Similarly, $ID = IF$. It follows that $ID = IE$.
Now it is easy to see that $\Delta CID \cong \Delta CIE$ (H.L.), which leads to the conclusion that $\angle ACI = \angle ABI$.

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□

Note:

(1) *I* is called the **incenter** of ΔABC.

(2) Since $\Delta AIF \cong \Delta AIE$, one sees that $AE = AF$, i.e., *A* (and similarly *I*) lie on th perpendicular bisector of *EF*. Hence, *AI* is the perpendicular bisector of *EF*. A similar argument applies for *BI* and *CI* as well.

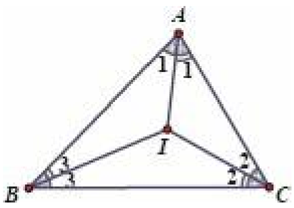
Theorem 1.3.3 *Let I be the incenter of* ΔABC. $\angle BIC = 90° + \frac{1}{2}\angle A$.

Proof. Refer to the diagram below. Since *I* is the incenter of ΔABC, *AI*, *BI*, *CI* are angle bisectors. Since $2(\angle 1 + \angle 2 + \angle 3) = 180°$, we must have $\angle 1 + \angle 2 + \angle 3 = 90°$.

Now $\angle BIC = 180° - \angle 2 - \angle 3 = 180° - (90° - \angle 1)$

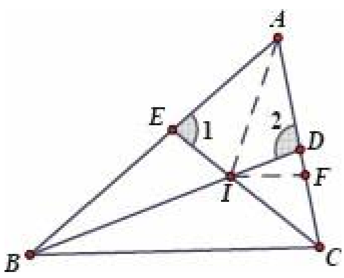
$= 90° + \angle 1 = 90° + \frac{1}{2}\angle A$.

□



Example 1.3.4 Given ΔABC where $\angle A = 60°$, *D*, *E* are on *AC*, *AB* respectively such that *BD*, *CE* bisects $\angle B$, $\angle C$ respectively. If *BD* and *CE* intersect at *I*, show that $DI = EI$.

Insight. Refer to the diagram on the below. Since *I* is the incenter of ΔABC, *AI* bisects $\angle A$. If we can show $\Delta AEI \cong \Delta ADI$, then it follows immediately that $DI = EI$.



However, it seems from the diagram that ΔAEI and ΔADI **cannot** be congruent:

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$\angle 1$ is acute but $\angle 2$ is obtuse, i.e., ΔAEI and ΔADI are **not** symmetric about AI. Why not reflect ΔAEI about AI and construct congruent triangles? Let us choose F on AC such that $AF = AE$ Now $\Delta AEI \cong \Delta AFI$ and we have $EI = FI$. Can we show $DI = FI$? Notice that ΔIDF **should** be an isosceles triangle. How can we show it? Since $\angle AFI = \angle 1$, it suffices to show that $\angle 1 = 180° - \angle 2$, or equivalently, $\angle 1 + \angle 2 = 180°$. This may not be difficult because both $\angle 1$ and $\angle 2$ can be expressed using $\angle B$ and $\angle C$ (using exterior angles) and we know $\angle B + \angle C = 180° - \angle A = 120°$!

Proof. Choose F on AC such that $AE = AF$. Notice that I is the incenter of ΔABC, i.e., AI bisects $\angle A$. Now we have $\Delta AEI \cong \Delta AFI$ (S.A.S.) and hence, $EI = FI$.

Since $\angle 1 = \angle B + \frac{1}{2}\angle C$ and $\angle 2 = \angle C + \frac{1}{2}\angle B$, we have

$$\angle 1 + \angle 2 = \frac{3}{2}(\angle B + \angle C) = \frac{3}{2}(180° - \angle A) = 180°, \text{ i.e., } \angle 1 = 180° - \angle 2.$$

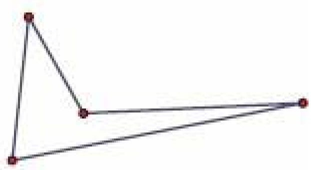
It follows that $\angle DFI = \angle 1 = \angle FDI$. Now $DI = FI = EI$.

□

1.4 Quadrilaterals

A quadrilateral is a polygon with four sides. In this book, we focus on convex quadrilaterals only. Refer to the following diagrams for examples.

Convex quadrilateral



Concave quadrilateral

There are two important types of quadrilaterals: parallelograms (including rectangles, rhombus and squares) and trapeziums. We will study their properties in this section.

Definition 1.4.1 A parallelogram is a quadrilateral with both pairs of opposing sides parallel to each other.

We give a list of equivalent ways to define a parallelogram.

(1) A parallelogram is a quadrilateral with two pairs of equal opposite sides

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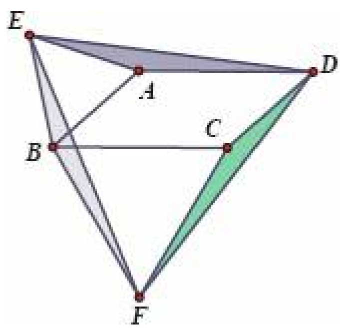
(2) A parallelogram is a quadrilateral with a pair of opposite sides equal and parallel to each other.
(3) A parallelogram is a quadrilateral with both pairs of opposite angles equal.
(4) A parallelogram is a quadrilateral with two diagonals bisecting each other.

One may show that all these definitions are equivalent by the techniques of congruent triangles.

Note that these definitions also describe the properties of a parallelogram. One may pay particular attention to (4), which is less frequently mentioned in textbooks, but widely applicable in problem-solving.

Example 1.4.2 Given a parallelogram $ABCD$, draw equilateral triangles ΔABE and ΔBCF outwards from AB, BC respectively. Show that ΔDEF is an equilateral triangle.

Insight. Refer to the diagram on the below. Given a parallelogram and equilateral triangles, one shall seek congruent triangles. Apparently, ΔADE, ΔCFD, ΔBFE **should** be congruent. It is easy to show equal sides, while a bit of calculation might be needed to show equal angles.



Proof. We have $AE = AB = CD$ and $AD = BC = CF$. Notice that $\angle DAE = 360° - \angle BAD - \angle BAE$ and $\angle FCD = 360° - \angle BCF - \angle BCD$. Since $\angle BAD = \angle BCD$ and $\angle BCF = 60° = \angle BAE$, we have $\angle DAE = \angle FCD$. Hence, $\Delta ADE \cong \Delta CFD$ (S.A.S.) and $DE = DF$.
Similarly, $BE = AB = CD$ and $BF = CF$. Notice that $\angle EBF = \angle ABE + \angle CBF + \angle ABC = 60° + 60° + 180° - \angle BCD = 300° - \angle BCD = 360° - \angle BCF - \angle BCD = \angle FCD$. Now $\Delta BEF \cong \Delta CFD$ (S.A.S.) and hence, $DF = EF$. This completes the proof.

□

Notice that the techniques for solving problems on quadrilaterals are still

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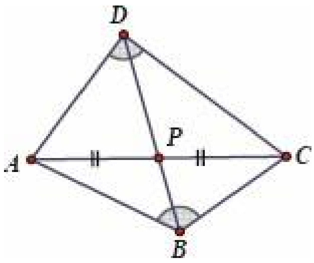
mainly through congruent triangles.

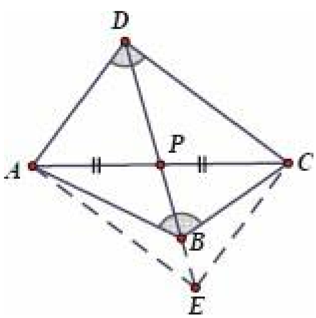
Example 1.4.3 Let $ABCD$ be a quadrilateral such that $\angle B = \angle D$. AC and BD intersect at P. If $AP = CP$, show that $ABCD$ is a parallelogram.

Insight. It is not easy to show the conclusion using congruent triangles directly. Although there are pairs of equal angles and identical lengths, they do not form congruent triangles. Refer to the diagram on the below.



Since DP is the median on AC, the median doubled could help to construct congruent triangles.
Moreover, among all the criteria to determine a quadrilateral, we may use (4): two diagonals bisecting each other. This is because we are given $AP = CP$ and we only need to show $DP = BP$. Bingo! This coincides with our strategy to double the median DP.

Proof. We claim that $BP = DP$, which leads to the conclusion immediately.



Suppose otherwise, say $BP < DP$, without loss of generality. We extend PB to E such that $DP = EP$. Now $AECD$ is a parallelogram, which implies $\angle D = \angle E$.
However, $\angle B = \angle D$ and we must have $\angle B = \angle E$. This is impossible! Notice that $\angle B = \angle ABD + \angle CBD$, where $\angle ABD = \angle AED + \angle EAB > \angle AED$.
Similarly, $\angle CBD > \angle CED$. We have $\angle B > \angle AED + \angle CED = \angle E$.
In conclusion, we must have $BP = DP$ and hence, $ABCD$ must be a parallelogram.

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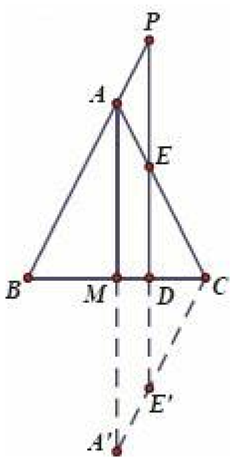
□

Example 1.4.4 Given an isosceles triangle ΔABC where $AB = AC$, M is the midpoint of BC. P *is a* point on BA extended and $PD \perp BC$ at D. If PD intersects AC at E, show that $PD + DE = 2AM$.

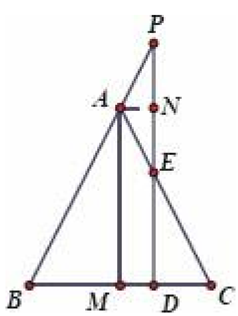
Insight. AM is a median of ΔABC and we need $2AM$ in the conclusion. Hence, it is natural to extend and double the median AM. Refer to the diagram below. Extend AM *to* A' such that $AM = A'M$. We are to show $PE +$ $PD = AA'$. Can you see a line segment equal to DE?

Proof. Extend AM to A' such that $AM = A'M$. Since $AB = AC$, one sees that $\Delta ABM \cong \Delta A'CM \cong \Delta ACM$. Let PD extended intersect $A'C$ at E'. We have $\Delta CDE \cong \Delta CDE'$ (A.A.S.) and hence, $DE = DE'$. We also conclude that $PD // AM$ and $AP // A'C$. Now $AA'E'P$ is a parallelogram and $AA' = PE'$. It follows that $PD + DE = PD + DE' = PE' = AA' = 2AM$.

□



Note: One may also draw $AN \perp PD$ at N and show that N is the midpoint of PE. Refer to the diagram on the below. Since $AMDN$ is a parallelogram (and in fact, a rectangle), we have $AM = DN$. Now it suffices to show $PD + DE = 2DN$. Note that this is equivalent to $PD - DN = DN - DE$, or $PN = EN$.



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It is easy to see that N is the midpoint of PE because ΔAPE is an isosceles triangle where $AP = AE$. (Can you show it?)

Definition 1.4.5 A rectangle is a quadrilateral with four right angles.

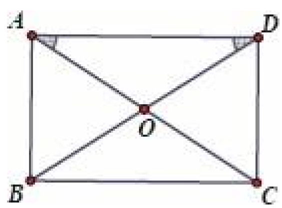
We give the following equivalent ways to define a rectangle.

(1) A rectangle is a parallelogram with a right angle.

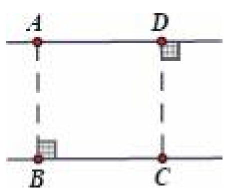
(2) A rectangle is a parallelogram with equal diagonals.

One may show that all these definitions are equivalent by the techniques of congruent triangles.

Note that (2) is an important property of rectangles. In particular, in a rectangle $ABCD$ where AC, BD intersect at O, we have $\angle OAD = \angle ODA$. Refer to the diagram on the below.



Given two parallel lines $\ell_1 \,//\, \ell_2$, the perpendicular distance from an arbitrary point on one line to the other line is a constant. Refer to the diagram on the below.



One could easily see that $ABCD$ is a rectangle and we always have $AB = CD$. This length is defined as the distance between ℓ_1 and ℓ_2.

Theorem 1.4.6 *In a right angled triangle ΔABC where $\angle A = 90°$ and M is the midpoint of BC, we have $AM = \frac{1}{2}BC$.*

Observe the fact that the right angled triangle is half of a rectangle. Refer to the diagram on the below. One may show the conclusion easily by congruent triangles.

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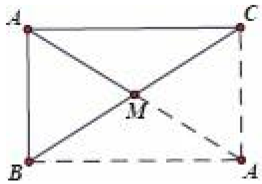


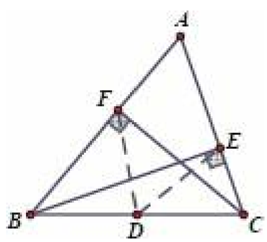
This is a simple but useful result. However, even the experienced contestants fail to recognize it occasionally, especially when the problem is complicated.

Note that Example 1.1.8 is the inverse of Theorem 1.4.6. In summary, given ΔABC where M is the midpoint of BC, $\angle A = 90°$ if and only if $AM = \frac{1}{2}BC$.

Example 1.4.7 In an acute angled triangle ΔABC, BE, CF are heights on AC, AB respectively. Let D be the midpoint of BC. Show that $DE = DF$.

***Proof*.** This is an immediate application of Theorem 1.4.6. In the right angled triangle ΔBEC, we have $DE = \frac{1}{2}BC$. Similarly, $DF = \frac{1}{2}BC$ The conclusion follows.

□



Example 1.4.8 In a right angled triangle ΔABC where $\angle A = 90°$ and $\angle C = 30°$, $AB = \frac{1}{2}BC$.

***Proof*.** Refer to the diagram on the below. Let M be the midpoint of BC. By Theorem 1.4.6, $AM = BM$ We see that ΔABM is an isosceles triangle where $\angle B = 60°$, and hence, an equilateral triangle.

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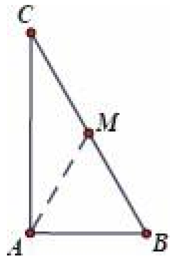
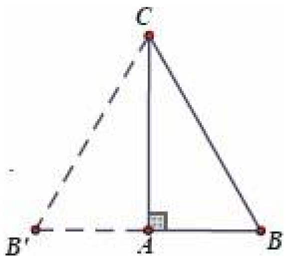


It follows that $AB = BM = \frac{1}{2}BC$.

□

Note:

(1) Refer to the diagram on the below. One may reflect ΔABC about the line AC and see that ΔABC is half of the equilateral triangle $\Delta BCB'$.



It is now clear that $AB = \frac{1}{2}BB' = \frac{1}{2}BC$.

(2) Notice that the inverse also holds: given ΔABC where $\angle A = 90°$, if $AB = \frac{1}{2}BC$, then $\angle C = 30°$. This is because $AM = BM = \frac{1}{2}BC = AB$ by Theorem 1.4.6, where M is the midpoint of BC. Hence, ΔABM is an equilateral triangle and $\angle B = 60°$.

Definition 1.4.9 A rhombus is a quadrilateral whose sides are of equal length.

We give the following equivalent ways to define a rectangle.

(1) A rhombus is a parallelogram with a pair of equal neighboring sides.
(2) A rhombus is a parallelogram whose diagonals are perpendicular to each other.

One may show that all these definitions are equivalent by the techniques of congruent triangles.

Example 1.4.10 Given a parallelogram $ABCD$ where $BC = 2AB$, E, F are on

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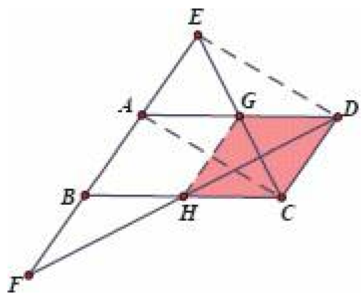
the line AB such that $AE = AB = BF$. Connect CE, DF. Show that $CE \perp DF$.

Insight. Refer to the diagram on the below. We are given a parallelogram $ABCD$ and $AE = AB = BF$. Hence, we can see more parallelograms, for example $ACDE$ (because $AE = CD$ and AE // CD).



It follows that AD and CE bisect each other. Now we can see that the condition $BC = 2AB$ is useful. Can you see a rhombus in the diagram?

Proof. Let AD, CE intersect at G and BC, DF intersect at H. Since $ABCD$ is a parallelogram, we have AE // CD and $AE = AB = CD$, which imply $ACDE$ is also a parallelogram. Hence, AD, CE bisect each other. Since $AD = 2AB$, $DG = AB = CD$. Similarly, $CH = CD$. It follows that $CDGH$ is a rhombus and hence, $CE \perp DF$.

□

Note: One may find an alternative solution using the technique of angle bisectors, parallel lines and isosceles triangle (Example 1.1.10):
Since $AB = BF$, we have $AF = 2AB = AD$, i.e., ΔAFD is an isosceles triangle. Now $\angle CDF = \angle AFD = \angle ADF$, i.e., DF bisects $\angle ADC$. Similarly, CE bisects $\angle BCD$. One sees $CE \perp DF$ because

$$\angle DCE + \angle CDF = \frac{1}{2}\angle BCD + \frac{1}{2} ADC = \frac{1}{2}\cdot 180^\circ = 90^\circ.$$

Notice that the last step is closely related to Example 1.1.9 that the angle bisectors of neighboring supplementary angles are perpendicular to each other.

Definition 1.4.11 A square is a rectangle whose sides are of equal length.

A square is a parallelogram which is both a rectangle and a rhombus. Hence, a square has all the properties of rectangles and squares, including equal sides, equal angles and diagonals of equal length which perpendicularly bisect each other. Of course, one may write down a lot of statements which are equivalent to the definition of a square.

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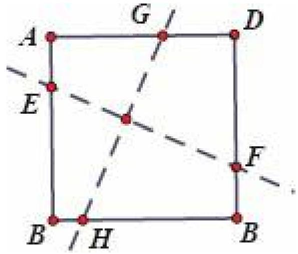
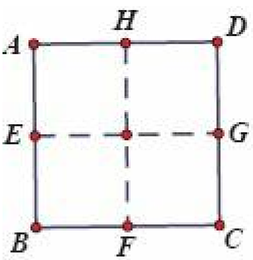
Example 1.4.12 Refer to the diagram on the below. *ABCD* is a square. Two lines, ℓ_1 and ℓ_2, intersect *ABCD* at *E*, *F* and *G*, *H* respectively. If $\ell_1 \perp \ell_2$, show that *EF* = *GH*.

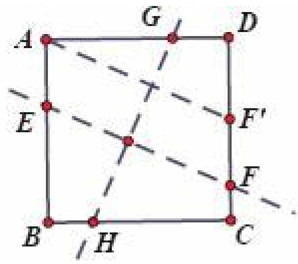


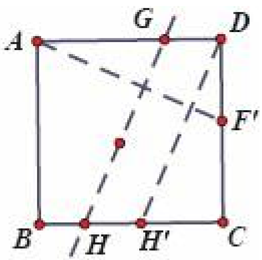
Insight. If ℓ_1, ℓ_2 are in the upright position, the conclusion is clear. Refer to the left diagram below.

Regrettably, we do not know the positions of ℓ_1 and ℓ_2 with respect to the square *ABCD*. Indeed, we are to show that for any $\ell_1 \perp \ell_2$, regardless of how they intersect *ABCD*, the conclusion holds.

Let us move ℓ_1, ℓ_2 around and observe. Refer to the middle diagram below. If we push *EF* upwards until *E* reaches *A*, we still have *EF* = *AF'* because *AEFF'* is a parallelogram. If we continue to push *GH* towards the right, we see that *GH* = *DH'*. Refer to the right diagram below. Hence, it suffices to show that *DH'* = *AF'*. This could be shown by congruent triangles.







Proof. Draw *AF'* // *EF*, intersecting *CD* at *F'* Draw *DH'* // *GH*, intersecting *BC* at *H'* Since *AEFF'* is a parallelogram, *EF* = *AF'*.

Similarly, *GH* = *DH'*. It suffices to show that *AF'* = *DH'*

Notice that ∠*DAF'* = 90° – ∠*ADH'* = ∠*CDH'*, ∠*ADC* = 90° = ∠*C* and *AD* = *CD*. Hence, Δ*ADF'* ≅ Δ*DCH'* (A.A.S.) and *AF'* = *DH'*

□

Example 1.4.13 Refer to the diagram on the below. *ABCD* is a square and *BDEF* is a rhombus such that *C*, *E*, *F* are collinear. Find ∠*CBF*.

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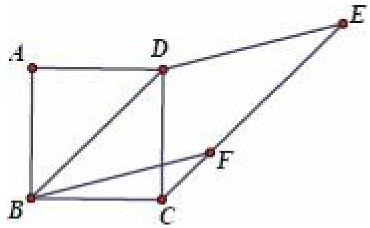
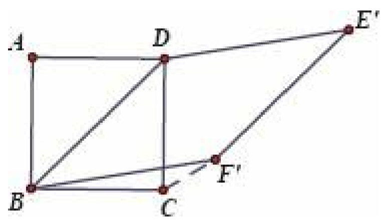
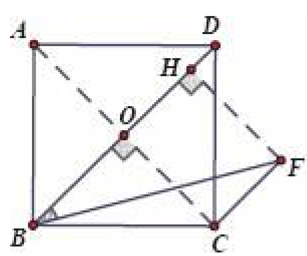


Insight. We have a square, a rhombus and the collinearity of *C*, *E*, *F*. One immediately sees that $\angle CBD = 45°$. Can we find $\angle DBF$? Notice that once $\angle DBF$ is known, the rhombus is uniquely determined. Which rhombus satisfies the conditions that *C*, *E*, *F* are collinear?
If we draw an arbitrary rhombus *BDE'F'* based on *BD*, as shown in the diagram on the below, we will still have *BD* // *E'F'*, but *C* will not lie on the line *E'F'*, i.e., we **must** use the fact *CE* // *BC* to show the conclusion.



One may also observe that if *F* is chosen, i.e., *CF* // *BD* and *BD* = *BF*, we do not need to draw *E* as it is not relevant to the problem anymore.

Hence, we may simplify the problem. Refer to the diagram on the below. Given *BD* // *CF* and *BD* = *BF*, what can we deduce about $\angle DBF$? We know ΔBDF is an isosceles triangle, but calculating $\angle BDF$ or $\angle BFD$ is not easy. How can we use *BD* = *BF* then? We know *AC* = *BD*. How is *AC* related to *BF*?



Given *BD* // *CF*, what is the distance between these two parallel lines? Can you see this distance is $\frac{1}{2}BD$? What if we introduce a perpendicular line to *BD* from *F* ?

Ans. Let *AC* and *BD* intersect at *O*. Draw $FH \perp BD$ at *H*. Since *CF* // *BD* and

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$AC \perp BD$ we have $FH = CO = \frac{1}{2}AC = \frac{1}{2}BD$. It follows that in the right angled triangle ΔBFH, $\angle FBH = 30°$ (Example 1.4.8).

Now $\angle CBF = \angle CBD - \angle FBD = 45° - 30° = 15°$.

□

Note: Since the distance between *BD* and *CF* is $\frac{1}{2}BD$, it is natural to think of $\angle FBH = 30°$ in a right angled triangle. In fact, one may even draw the diagram accurately and see that $\angle FBH = 30°$. Even though such a drawing will **NOT** be accepted as part of the solution, it gives us a clue. Now constructing a right angled triangle with $\angle FBH = 30°$, i.e., where one leg is half of the hypotenuse, becomes a natural strategy.

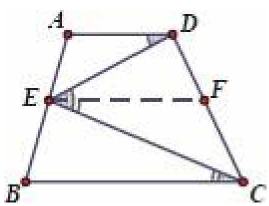
Definition 1.4.14 A trapezium is a quadrilateral with **exactly** one pair of parallel sides.

By definition, a trapezium cannot be a parallelogram.

Example 1.4.15 In a trapezium *ABCD* where *AD* // *BC*, *E* is a point on *AB*. Show that $\angle ADE + \angle BCE = \angle CED$.

Proof. Refer to the diagram on the below. Draw *EF* // *AD*, intersecting *CD* at *F*. Notice that $\angle ADE = \angle DEF$ and $\angle BCE = \angle CEF$. Hence, $\angle ADE + \angle BCE = \angle DEF + \angle CEF = \angle CDE$.

□



An isosceles trapezium is a trapezium whose unparalleled sides are of equal length. In fact, one obtains an isosceles triangle by extending the unparalleled sides. Refer to the diagram on the below where *ABCD* is an isosceles trapezium with *AB* = *CD*. It can be shown easily that $\angle B = \angle C$ and $AC = BD$.

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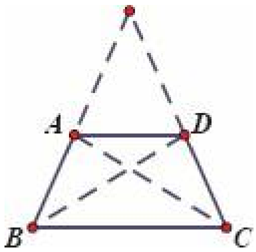
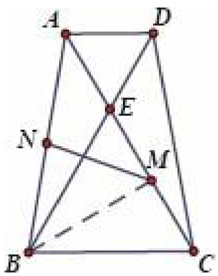


Example 1.4.16 *ABCD* is an isosceles trapezium where *AD* // *BC* and *AB* = *CD*. Its diagonals *AC*, *BD* intersect at *E* and $\angle AED = 60°$. Let *M*, *N* be the midpoints of *CE*, *AB* respectively. Show that $MN = \frac{1}{2}AB$.

Proof. Refer to the diagram on the below. Since *ABCD* is an isosceles trapezium with *AB* = *CD*, we must have $\angle ABC = \angle BCD$. Hence, $\Delta ABC \cong \angle DCB$ (S.A.S.), which implies $\angle BCE = \angle CBE$.



Since $\angle BEC = \angle AED = 60°$, ΔBCE must be an equilateral triangle. Since *M* is the midpoint of *CE*, we must have $BM \perp CE$.
Since *N* is the midpoint of *AB*, *MN* is the median on the hypotenuse of ΔAMB and hence, $MN = \frac{1}{2}AB$ (Theorem 1.4.6).

□

1.5 Exercises

1. In a right angled triangle ΔABC where $\angle A = 90°$, *P* is a point on *BC*. If *AP* = *BP*, show that *BP* = *CP*, i.e., *P* is the midpoint of *BC*.

2. Given ΔABC where $\angle B = 2\angle C$, *D* is a point on *BC* such that *AD* bisects $\angle A$. Show that *AC* = *AB* + *BD*.

3. Refer to the left diagram below. Given ΔABC, draw squares *ABDE* and *ACFG* outwards from *AB*, *AC* respectively. Show that *BG* = *CE* and $BG \perp CE$.

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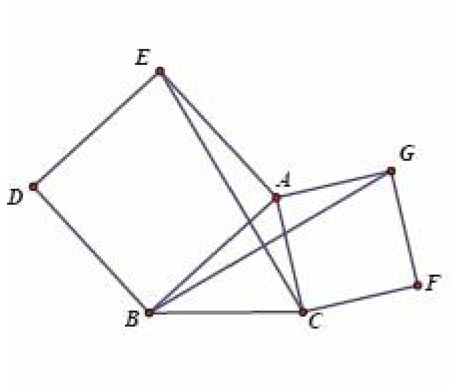
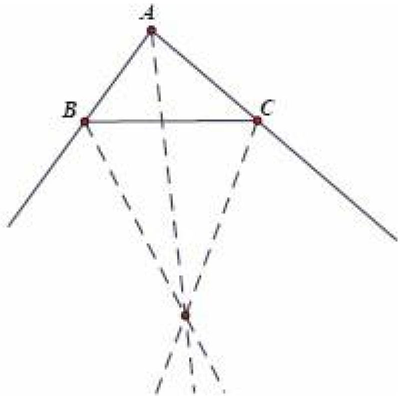




4. Refer to the right diagram above. Show that in ΔABC, the angle bisector of $\angle A$, the exterior angles bisectors of $\angle B$ and $\angle C$ are concurrent (i.e., they pass through the same point).

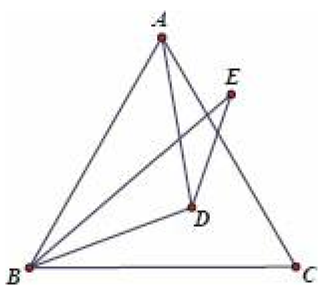
Note: This point is called the ex-center of ΔABC opposite A. One may see that each triangle has three ex-centers.

5. Given ΔABC, J_1 and J_2 are the ex-centers (refer to Exercise 1.4) opposite B and C respectively. Let I be the incenter of ΔABC. Show that $J_1J_2 \perp AI$.

6. Let $ABCD$ be a square. E, F are points on BC, CD respectively and $\angle EAF = 45°$. Show that $EF = BE + DF$.

7. In the acute angled triangle ΔABC, $BD \perp AC$ at D and $CE \perp AB$ at E. BD and CE intersect at Q. P is on BD extended such that $BP = AC$. If $CQ = AB$, find $\angle AQP$.

8. Refer to the diagram on the below. ΔABC is an equilateral triangle. D is a point inside ΔABC such that $AD = BD$. Choose E such that $BE = AB$ and BD bisects $\angle CBE$. Find $\angle BED$.



9. Let I be the incenter of ΔABC. AI extended intersects BC at D. Draw $IH \perp BC$ at H. Show that $\angle BID = \angle CIH$.

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10. Given a quadrilateral $ABCD$, the diagonal AC bisects both $\angle A$ and $\angle C$. If AB extended and DC extended intersect at E, and AD extended and BC extended intersect at F, show that for any point P on line AC, $PE = PF$.

11. In ΔABC, $AB = AC$ and D is a point on AB. Let O be the circumcenter of ΔBCD and I be the incenter of ΔACD. Show that A, I, O are collinear.

12. Given a quadrilateral $ABCD$ where BD bisects $\angle B$, P is a point on BC such that PD bisects $\angle APC$. Show that $\angle BDP + \angle PAD = 90°$.

13. $ABCD$ is a quadrilateral where $AD \parallel BC$. Show that if $BC - AB = AD - CD$, then $ABCD$ is a parallelogram.

14. Given a square $ABCD$, ℓ_1 is a straight line intersecting AB, AD at E, F respectively and ℓ_2 is a straight line intersecting BC, CD at G, H respectively. EH, FG intersect at I. If $\ell_1 \parallel \ell_2$ and the distance between ℓ_1, ℓ_2 is equal to AB, find $\angle GIH$.

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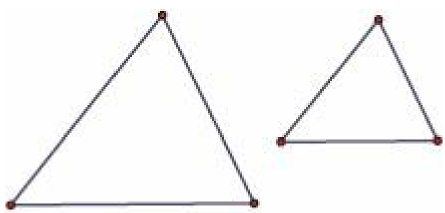
Chapter 2

Similar Triangles

Similar triangles are the natural extension of the study on congruent triangles. While congruent triangles describe a pair of triangles with identical shape and size (area), similar triangles focus on the shape. The diagram below gives an illustration.

Indeed, similar triangles are even more powerful tools than congruent triangles. Many interesting properties and important theorems in geometry could be proved by similar triangles.

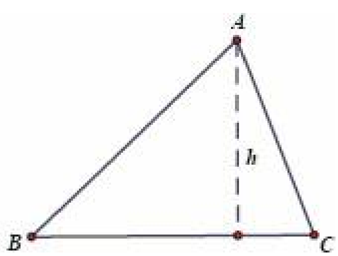


Two triangles of the same shape

One would see in this chapter that the Intercept Theorem plays a fundamental role in studying similar triangles, while the proof of this theorem is based on an even more fundamental concept: area.

2.1 Area of a Triangle

It is widely known that the area of ΔABC, denoted by $[\Delta ABC]$ or $S_{\Delta ABC}$, is given by $[\Delta ABC]=\frac{1}{2}BC\times h$, where h denotes the height on BC.



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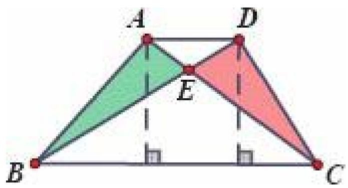
Of course, one may replace BC and h by any side of the triangle and the corresponding height on that side.

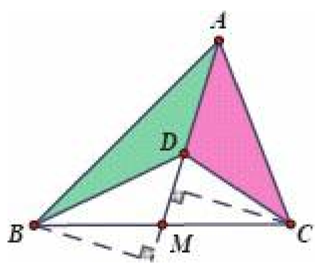
Notice that $[\Delta ABC]=\frac{1}{2}\text{base}\times\text{height}$ implies that if two triangles have equal bases and heights, they must have the same area. Even though this is a simple conclusion, it has a number of (important) variations:

- In a trapezium *ABCD* where $AD // BC$ and *AC,BD* intersect at *E*, we have [Δ*ABC*] = [Δ*DBC*] because both triangles have a common base and equal heights.



By substracting [Δ*BCE*] on both sides of the equation, we have [Δ*ABE*]= [Δ*CDE*]. Refer to the diagram above.

- In a triangle Δ*ABC* where *M* is the midpoint of *BC*, we must have [Δ*ABM* [Δ*ACM*]. Let *D* be any point on *AM*. We also have [Δ*BDM*]=[Δ*CDM*]. Refer to the diagram below.



It follows that [Δ*ABD*]=[Δ*ACD*]. Since Δ*ABD* and Δ*ACD* have a common base *AD*, we conclude that the perpendicular distance from *B,C* respectively to the line *AM* is the same.

Notice that the conclusion above still holds even if *D* is a point on *AM* extended. Refer to the diagram below.

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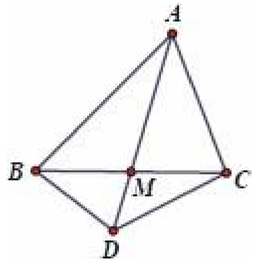
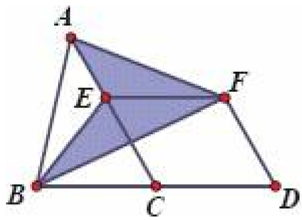


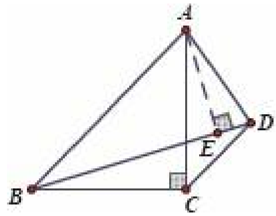
If M is the midpoint of BC, can you see [ΔABD]=[ΔACD]?

- Refer to the diagram below. Given a triangle ΔABC, extend BC to D such that $BC = CD$. E is a point on AC. Draw a parallelogram $CDFE$. Connect BE, BF and AF. One sees that the area of the shaded region is equal to the area of ΔABC.



This is because the shaded region consists of ΔAEF and ΔBEF, which have the same base EF. Hence, the heights of the triangles, called h_1 and h_2, are the distances from A and B to the line EF respectively. Now $[AEBF] = \frac{1}{2}EF \cdot h_1 + \frac{1}{2}EF \cdot h_2 = \frac{1}{2}EF \cdot (h_1 + h_2)$. Since $EF = CD = BC$ and $h_1 + h_2$ is equal to the distance from A to BC, we conclude that [$AEBF$]=[ΔABC].

- Given a right angled triangle ΔABC where $\angle C = 90°$, draw $CD // AB$. Refer the diagram below. Draw $AE \perp BD$ at E.



One sees that $AE \cdot BD = AC \cdot BC$, because $AE \cdot BD = 2$[ΔABD] and $AC \cdot BC = 2$[ΔABC]. We have [ΔABC]=[ΔABD] since both triangles have a common base AB and equal heights (because $AB // CD$). In fact, one may see this

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conclusion more clearly by recognizing the trapezium *ABCD*. (You may rotate the page and hence, look at the trapezium from the "upright" position.)

Note that using areas of equal triangles is an important technique to show equal products (or ratios, $\frac{AE}{AC}=\frac{BC}{BD}$ in this example) of line segments.

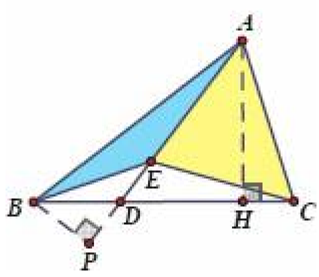
Moreover, $[\Delta ABC]=\frac{1}{2}BC\times h$ implies that if two triangles, say ΔABC and $\Delta A'B'C'$, have equal bases BC and $B'C'$, then $\frac{[\Delta ABC]}{[\Delta A'B'C']}=\frac{h}{h'}$, where h and h' are the respective heights. A similar conclusion could be drawn if two triangles have equal heights.

This is a very useful result because we may calculate the area of triangles indirectly by comparing its base and height with another triangle whose area is known.

Example 2.1.1 Given ΔABC, D is a point on BC such that $BC = 3BD$. E is a point on AD such that $AD = 4DE$. Show that:

(1) $[\Delta ACE] = 2[\Delta ABE]$

(2) $[\Delta ABC] = 4[\Delta BCE]$



Proof. Refer to the diagram below.

(1) Notice that ΔABD and ΔACD has the same height AH.

Hence, $\frac{[\Delta ABD]}{[\Delta ACD]}=\frac{BD}{CD}=\frac{1}{2}$, or $[\Delta ACD] = 2[\Delta ABD]$.

Similarly, $[\Delta CDE] = 2[\Delta BDE]$. Hence, we have:

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$[\Delta ACE] = [\Delta ACD] - [\Delta CDE] = 2([\Delta ABD] - [\Delta BDE]) = 2[\Delta ABE]$.

(2) Notice that ΔABD and ΔBDE have the same height BP.

Hence, $\frac{[\Delta ABD]}{[\Delta BDE]} = \frac{AD}{DE} = \frac{4}{1}$, or $[\Delta ABD] = 4[\Delta BDE]$.

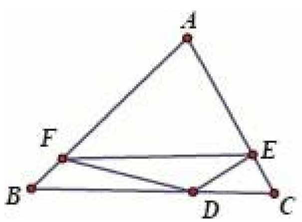
Similarly, $[\Delta ACD] = 4[\Delta CDE]$. Hence, we have:

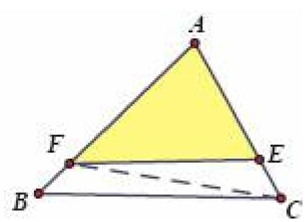
$[\Delta ABC] = [\Delta ABD] + [\Delta ACD] = 4([\Delta BDE] + [\Delta CDE]) = 4[\Delta BCE]$.

□

Note: One may see that similar arguments apply even if the ratios given (i.e., the positions of D and E) are different. Such an argument is commonly used in solving problems related to areas. In fact, experienced contestants in Mathematical Olympiads could see the conclusions almost instantaneously.

Example 2.1.2 Given ΔABC, D, E, F are points on BC, AC, AB respectively such that $BD = 2CD$, $AE = 3CE$ and $AF = 4BF$. If the area of ΔABC is 240cm^2, find the area of ΔDEF.





Insight. Refer to the left diagram above. Calculating ΔDEF directly will certainly be difficult because we do not know any of its bases or heights. We are given the area of ΔABC, but we do not know exactly how the areas of ΔDEF and ΔABC are related. However, we could obtain the area of ΔDEF by subtracting the areas of ΔAEF, ΔBDF and ΔCDE from ΔABC, where each of these triangles share a (part of) common side with ΔABC. Let us choose one of them, say ΔAEF. Refer to the right diagram above. Connect CF.

Observe that $\frac{[\Delta AEF]}{[\Delta ACF]} = \frac{AE}{AC} = \frac{3}{4}$. We also have $\frac{[\Delta ACF]}{[\Delta ABC]} = \frac{AF}{AB} = \frac{4}{5}$. It follows that $\frac{[\Delta AEF]}{[\Delta ABC]} = \frac{3}{5}$, or $[\Delta AEF] = \frac{3}{5}[\Delta ABC] = \frac{3}{5} \times 240 = 144$.

Similarly, $[\Delta BDF] = \frac{2}{3} \times \frac{1}{5} \times 240 = 32$ and $[\Delta CDE] = \frac{1}{4} \times \frac{1}{3} \times 240 = 20$.

Now $[\Delta DEF] = [\Delta ABC] - [\Delta AEF] - [\Delta BDF] - [\Delta CDE]$

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$= 240 - 144 - 32 - 20 = 44$ cm^2.

Note:

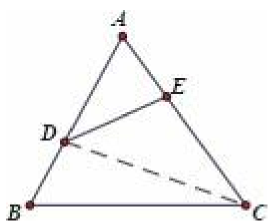
(1) One sees that $[\Delta DEF] = \frac{11}{60}[\Delta ABC]$ always holds regardless of the are and the shape of ΔABC. This is solely determined by the relative positions of D, E, F on BC, AC, AB respectively.

(2) In general, given ΔABC and D, E are on AB, AC respectively, we always have $\frac{[\Delta ADE]}{[\Delta ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC}$.

Refer to the diagram below.

One may see this conclusion by connecting CD and hence, relaying

$$\frac{[\Delta ADE]}{[\Delta ACD]} = \frac{AE}{AC} \text{ and } \frac{[\Delta ACD]}{[\Delta ABC]} = \frac{AD}{AB}.$$



Alternatively, one may apply Sine Rule, which we will discuss in Chapter 3.

(3) Using such a "relay" of area comparison is a useful technique because it links the unknown area to what is given. However, creating such a link literally requires a sequence of triangles, one after another which shares either a common side or a height. Of course, this may not be an easy task and one needs to draw one or more auxiliary lines wisely. Can you use this "relay" method to solve the following Example 2.1.3 and Example 2.1.4, without referring to the solution?

Example 2.1.3 Let $ABCD$ be a quadrilateral. E, F are on AB such that $AE = EF = BF = \frac{1}{3}AB$ and G, H are on CD such that $CG = GH = DH = \frac{1}{3}CD$. Show that $[EFGH] = \frac{1}{3}[ABCD]$.

Proof. Refer to the left diagram below. Since $AE = EF$, we must have $[\Delta EFH$

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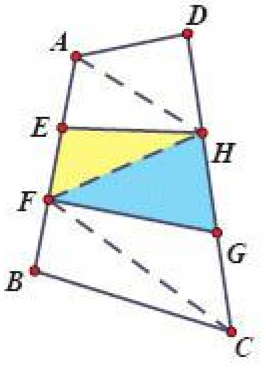
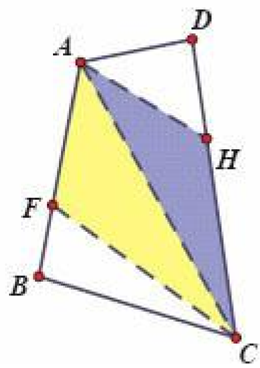
] = $\frac{1}{2}$[Δ*AFH*]. Similarly, [Δ*FGH*]= $\frac{1}{2}$[Δ*CFH*].





Hence, [*EFGH*] = [Δ*EFH*] + [Δ*FGH*] = $\frac{1}{2}$[Δ*AFH*] + $\frac{1}{2}$[Δ*CFH*]

= $\frac{1}{2}$[*AFCH*].

Now it suffices to show that [*AFCH*] = $\frac{2}{3}$[*ABCD*]. Refer to the previous right diagram. Since AF = 2*BF* and *CH* = 2*DH*, we have

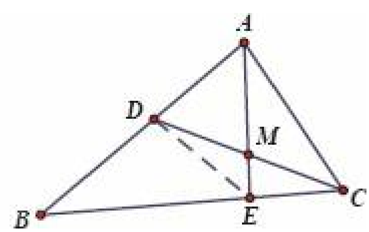
$$[EFGH]=[\Delta EFH]+[\Delta FGH]=\frac{1}{2}[\Delta AFH]+\frac{1}{2}[\Delta CFH]$$

□

Example 2.1.4 In Δ*ABC*, *D* is a point on *AB* and $\frac{AD}{AC}=\frac{AC}{AB}=\frac{2}{3}$. M is the midpoint of *CD* while *AM* extended intersects *BC* at *E*. Find $\frac{CE}{BE}$.

Ans. Refer to the diagram below. Connect *DE*. Since *CM* = *DM*, one sees that

[Δ*ACM*] = [Δ*ADM*] and [Δ*CEM*] = [Δ*DEM*].



It follows that [Δ*ADE*]= [Δ*ACE*].

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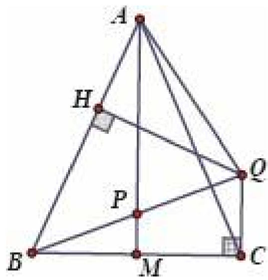
Since $\frac{[\Delta ADE]}{[\Delta ABE]}=\frac{AD}{AB}=\frac{AD}{AC}\cdot\frac{AC}{AB}=\frac{4}{9}$, we have $\frac{CE}{BE}=\frac{[\Delta ACE]}{[\Delta ABE]}=\frac{4}{9}$. ☐

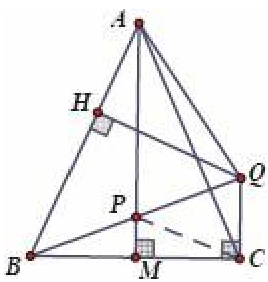
Note: We will see this example again in Section 3.2 and Section 3.4, where we will use two other methods (Intercept Theorem and Menelaus' Theorem) to solve it.

Example 2.1.5 Refer to the diagram below. In an acute angled triangle Δ*ABC* where *AB* = *AC*, *M* is the midpoint of *BC*. *P* is a point on *AM* and *Q* is a point on *BP* extended such that *QC* ⊥ *BC* at *C*. Draw *QH* ⊥ *AB* at *H*. Show that $\frac{AB}{AP}=\frac{BC}{HQ}$.



Insight. We are to show *AB* · *HQ* = *AP* · *BC*. Since *AB* · *HQ* = 2[Δ*ABQ*] and *AP* ⊥ *BC*, perhaps we can show the equality by area. Does *AP* · *BC* give the area of any triangle, or at least the area of a region in the diagram?

Proof. Refer to the diagram below. Connect *CP*. Since *M* is the midpoint of *BC* and *AB* = *AC*, *AM* must be the perpendicular bisector of *BC* (Theorem 1.2.2). It follows that *BP* = *CP* (Theorem 1.2.4).



Since ∠*BCQ* = 90°, we have *BP* = *PQ* (Exercise 1.1), i.e., *P* is the midpoint of *BQ*.

Notice that *AB* · *HQ* = 2[Δ*ABQ*] = 4[ΔABP] because *BQ* = 2*BP*.

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We also have $AP \cdot BC = 2AP \cdot BM = 2 \times 2[\Delta ABP] = 4[\Delta ABP]$.

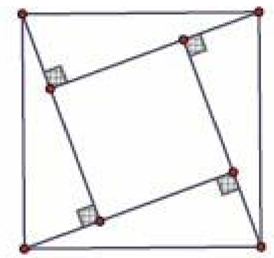
It follows that $AB \cdot HQ = AP \cdot BC$, or $\frac{AB}{AP} = \frac{BC}{HQ}$.

□

Note: Since $BM = CM$ and $MP // CQ$, one may obtain $BP = CQ$ easily by the Intercept Theorem. We will see this in the next section.

Pythagoras' Theorem

Pythagoras' Theorem is well known. Many of its popular proofs are based on the clever construction of a diagram. An example is given on the right. (We leave it to the reader to complete the proof based on this diagram.)



Proof of Pythagoras' Theorem

We shall introduce the classical proof of this theorem in Euclid's *Elements*. The proof is straightforward and is based on the area of triangles. It also illustrates a method applicable to many other problems related to areas of triangles.

Theorem 2.1.6 (Pythagoras' Theorem) *In* ΔABC *where* $\angle A = 90°$, $AB^2 + AC^2 = BC^2$.

Proof. Refer to the diagram below. We draw squares outwards from AB, AC, BC respectively. Since AB^2, AC^2, BC^2 represent the areas of squares, we are to show that the sum of the areas of the two small squares equals the area of the large square, i.e.,

$[ABDE] + [ACFG] = [BCHI]$. (1)

Notice that $[ACFG] = 2[\Delta ACF]$ and $[\Delta ACF] = \frac{1}{2} CF \cdot AC = [\Delta BCF]$.

Since $BCF \cong \Delta HCA$ (Exercise 1.3), we must have $[ACH] = [\Delta BCF]$

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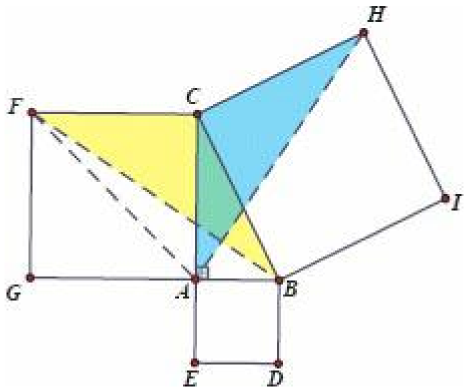
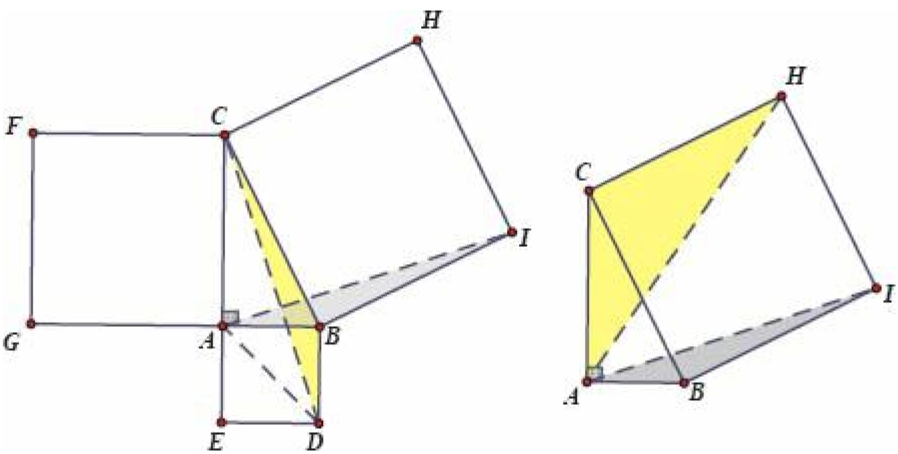
$= [\Delta ACF] = \frac{1}{2}[ACFG]$. (2)



Similarly, $[\Delta ABI] = [\Delta BCD] = [\Delta ABD] = \frac{1}{2}[ABDE]$. (3)

Refer to the left diagram below.



From (1), (2) and (3), it suffices to show $[\Delta ACH] + [\Delta ABI] = \frac{1}{2}[BCHI]$.

One sees that ΔACH and ΔABI have equal bases CH and BI with their respective heights added up to HI. Refer to the right diagram above. This completes the proof.

□

Example 2.1.7 *ABCD* is a trapezium such that $AD // BC$. If the two diagonals are perpendicular to each other, i.e., $AC \perp BD$, show that $AC^2 + BD^2 = (AD + BC)^2$.

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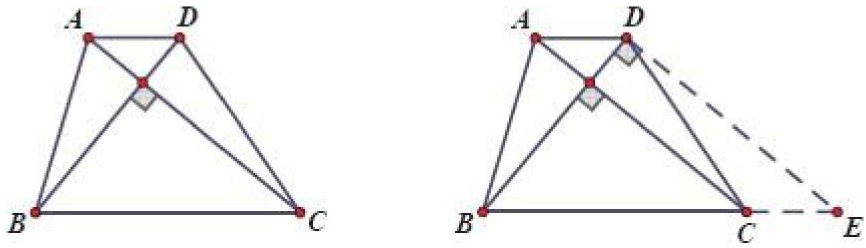
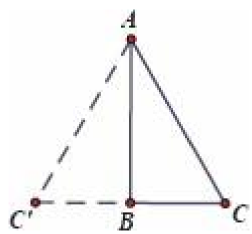
Insight. Refer to the left diagram below. Given $AC \perp BD$, we are asked about $AC^2 + BD^2$. Apparently, one should apply Pythagoras' Theorem. However, *AC*, *BD* are not intersecting at the endpoints. Can we bring them into a right angled triangle, say by moving the lines?



Proof. Draw $DE // AC$, intersecting *BC* extended at *E*. Refer to the previous right diagram. Clearly, *ACED* is a parallelogram and hence, $AC = DE$ and $AD = CE$. Now $AC^2 + BD^2 = DE^2 + BD^2 = BE^2$ because $DE \perp BD$. It is easy to see that $BE = AD + BC$ because $AD = CE$. This completes the proof.

□

We know that in a right angled triangle ΔABC where $\angle B = 90°$, if $\angle A = 30°$, then $AC = 2BC$ (Example 1.4.8). Hence, by Pythagoras' Theorem, $AB^2 = AC^2 - BC^2 = 3BC^2$, i.e., $AB = \sqrt{3}BC$.



Refer to the diagram above where $\Delta ACC'$ is an equilateral triangle with a side of length *a*, i.e., $BC = \frac{1}{2}a$. We have $AB = \frac{\sqrt{3}}{2}a$ and hence, the area of the equilateral triangle is $\frac{\sqrt{3}}{4}a^2$.

Similarly, in a right angled triangle ΔABC where $\angle B = 90°$, if $\angle A = 45°$, we must have $AB = BC$ and hence, $AC = \sqrt{2}AB$ by Pythagoras' Theorem.

Example 2.1.8 *ABCD* is an isosceles trapezium where $AD // BC$ and *AC*, *BD* intersect at *P*. If $BC = AC$ and $AC \perp BD$, show that $AD + BC = 2BP$.

Proof. Refer to the diagram below. Since *ABCD* is an isosceles trapezium and $AC \perp BD$, both ΔPAD and ΔPBC are right angled isosceles triangles.

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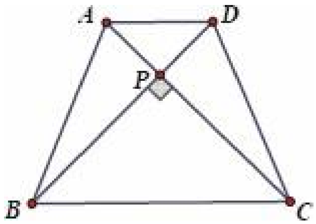


Let $AP = x$ and $CP = y$. We have $AD = \sqrt{2}x$ and $BC = \sqrt{2}y$.

Since $AC = BC$, we must have $x + y = \sqrt{2}y$. (1)

We are to show $AD + BC = 2BP$, i.e., $\sqrt{2}x + \sqrt{2}y = 2y$, but this can be obtained immediately from (1), by multiplying $\sqrt{2}$ on both sides.

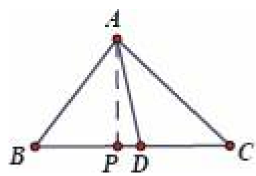
□

The inverse of Pythagoras' Theorem also holds, i.e., in ΔABC, if $AB^2 + AC^2 = BC^2$, then $\angle A = 90°$. This can be proved by contradiction.

The following result could be seen as an extension of the inverse of Pythagoras' Theorem.

Theorem 2.1.9 *Let A be a point outside the line BC and D is on the line BC. I* $AB^2 - BD^2 = AC^2 - CD^2$, *then* $AD \perp BC$.

Proof. Suppose otherwise. Refer to the diagram below. Draw $AP \perp BC$ at P. We may assume, without loss of generality, that $BD > BP$.



By Pythagoras' Theorem, $AP^2 = AB^2 - BP^2 = AC^2 - CP^2$.

Since $AB^2 - BD^2 = AC^2 - CD^2$, we have $BD^2 - CD^2 = AB^2 - AC^2 = BP^2 - CP^2$. This is impossible since $BD > BP$ and $CD < CP$, i.e., $BD^2 - CD^2 > BP^2 - CP^2$.

Note that the proof is not complete yet because one should also consider the cases where either D or P is outside the line segment BC. Refer to the following diagrams. Indeed, we have $BD^2 - CD^2 \neq BP^2 - CP^2$ in each case. We leave the details to the reader.

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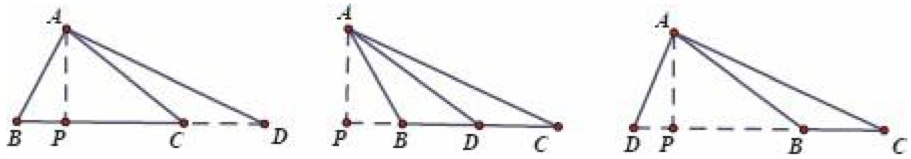


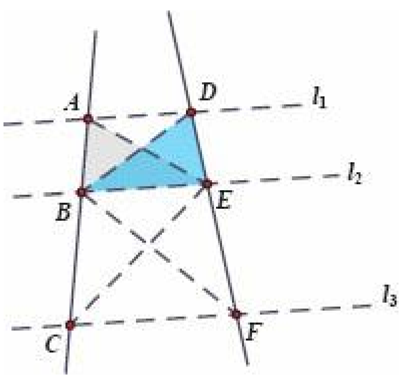
Indeed, Theorem 2.1.9 still holds even if *D* does not lie on the line *BC*. One may write down a similar proof by contradiction.

2.2 Intercept Theorem

Theorem 2.2.1 (Intercept Theorem) *Let ℓ_1, ℓ_2, ℓ_3 be a group of parallel lines which intersect two straight lines at A, B, C and D, E, F respectively. We have* $\frac{AB}{BC}=\frac{DE}{EF}$.

Proof. Refer to the diagram below.

Notice that $\frac{AB}{BC}=\frac{[\Delta ABE]}{[\Delta BCE]}$, since Δ*ABE* and Δ*BCE* share the same height from *E* to the line *AC*. Similarly, $\frac{DE}{EF}=\frac{[\Delta BDE]}{[\Delta BEF]}$.



Notice that [Δ*ABE*] = [Δ*BDE*] = $\frac{1}{2}BE\times h$, where *h* is the height on *BE*.

The two triangles have the same height as $\ell_1 \,//\, \ell_2$.

Similarly, [Δ*BCE*] = [Δ*BEF*].

It follows that $\frac{AB}{BC}=\frac{[\Delta ABE]}{[\Delta BCE]}=\frac{[\Delta BDE]}{[\Delta BEF]}=\frac{DE}{EF}$.

□

Note:

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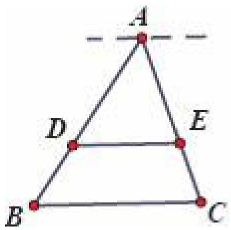
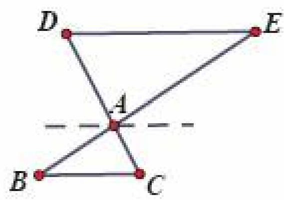
(1) One may easily see that the Intercept Theorem applies when more than three parallel lines intercept two straight lines: the corresponding line segments will still be in ratio.

(2) There are a few cases where the Intercept Theorem applies for only two parallel lines. Refer to the following diagrams.



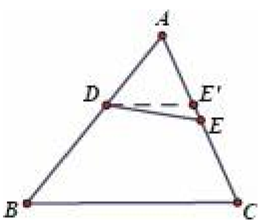


In these cases, we always have $\frac{AB}{AC}=\frac{AD}{AE}$.

Notice that one could always draw the third parallel line at *A* before applying the Intercept Theorem.

(3) Notice that the inverse of the Intercept Theorem holds as follows: In Δ*ABC* where *D*, *E* are on *AB*, *AC* respectively, if $\frac{AD}{AB}=\frac{AE}{AC}$, we must have *DE* // *BC*. This could be proved easily by contradiction:

Suppose otherwise. We draw *DE*' // *BC*, intersecting *AC* at *E*'. Refer to the diagram below.



We have $\frac{AD}{AB}=\frac{AE'}{AC}$ by the Intercept Theorem.

Since $\frac{AD}{AB}=\frac{AE}{AC}$, we must have AE = *AE*', i.e., *E* and *E*' coincide.

This completes the proof that *DE* // *BC*.

Corollary 2.2.2 *In* Δ *ABC, D, E are on AB, AC respectively such that DE // BC.*

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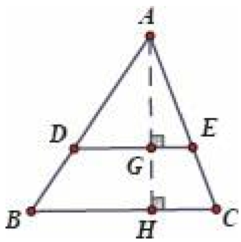
We have $\dfrac{AD}{AB}=\dfrac{AE}{AC}=\dfrac{DE}{BC}$.

Proof. Refer to the diagram below. Draw $AH \perp BC$ at H. Let AH intersect DE at G. Since $BC\ //\ DE$, $AG \perp DE$.



We have $\dfrac{AD}{AB}=\dfrac{AG}{AH}$ by the Intercept Theorem.

Let $\dfrac{AD}{AB}=\dfrac{AG}{AH}=k$, i.e., $AD = k \cdot AB$ and $AG = k \cdot AH$.

Pythagoras' Theorem gives $DG^2 = AD^2 - AG^2 = (k \cdot AB)^2 - (k \cdot AH)^2$ $= k^2\left(AB^2 - AH^2\right) = k^2 BH^2$, i.e., $\dfrac{DG}{BH}=k=\dfrac{AD}{AB}$.

Similarly, $\dfrac{EG}{CH}=k=\dfrac{AE}{AC}=\dfrac{AD}{AB}$.

Now $DE = DG + EG = k \cdot BH + k \cdot CH = k(BH + CH) = k \cdot BC$.

This implies $\dfrac{DE}{BC}=k=\dfrac{AD}{AB}$.

□

Note:

(1) The conclusion holds even if D, E lie on BA,CA extended respectively, i.e., when the lines BC, DE are on different sides of A. Refer to the diagrams in the remarks after Theorem 2.2.1.

(2) Refer to the diagram below where $BC\ //\ PQ$.

We have $\dfrac{PR}{QR}=\dfrac{BD}{CD}$, because $\dfrac{PR}{BD}=\dfrac{AP}{AB}=\dfrac{QR}{CD}$.

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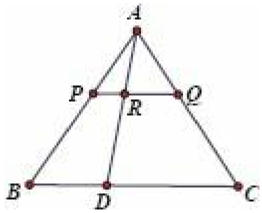


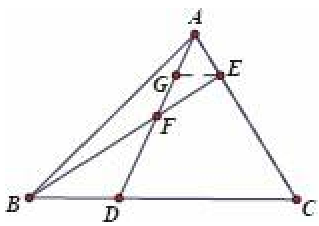
One familiar with similar triangles may see the conclusion almost immediately. We shall study similar triangles in the next section.

The Intercept Theorem and Corollary 2.2.2 are very useful in calculating the ratio of line segments.

Example 2.2.3 In ΔABC, D, E are on BC, AC respectively such that $BC = 3BD$ and $AC = 4AE$. If AD and BE intersect at F, find $\frac{BF}{EF}$.

Ans. Refer to the diagram below. Draw EG // BC, intersecting AD at G.

Since $\frac{EG}{CD} = \frac{AE}{AC} = \frac{1}{4}$ and $\frac{CD}{BD} = \frac{2}{1}$, we have $\frac{EF}{BF} = \frac{EG}{BD} = \frac{EG}{CD} \cdot \frac{CD}{BD} = \frac{1}{2}$, i.e., $\frac{BF}{EF} = 2$. □



Note: This solution shows a standard method solving this type of questions. Once the positions of D and E are known, one could always use this method to find $\frac{BF}{EF}$. Can you use the same technique to show that $AF = DF$? (**Hint**: Draw DP // AC, intersecting BE at P.)

Example 2.2.4 Given ΔABC, D, E, F are on AB, BC, CA respectively such that $AB = 3AF$, $BC = 3BD$ and $AC = 3CE$. Refer to the diagram below. Find $\frac{[\Delta PQR]}{[\Delta ABC]}$.

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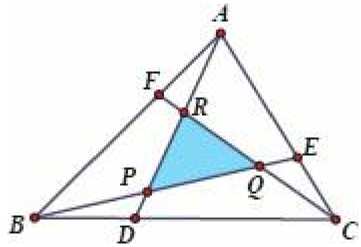
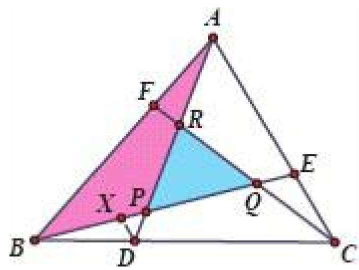


Insight. This is similar to Example 2.1.2. We can calculate [Δ*PQR*] by subtracting the unshaded areas from [Δ*ABC*]. In order to calculate the area of the unshaded region, we may divide it into a few triangles, say Δ*ABP*, Δ*BCQ* and Δ*CAR*. How can we calculate [Δ*ABP*]? We know $[\Delta ABD]=\frac{1}{3}[\Delta ABC]$ because $BD=\frac{1}{3}BC$. Notice that $\frac{[\Delta ABP]}{[\Delta ABD]}=\frac{AP}{AD}$. We can use the method illustrated in Example 2.2.3 to find $\frac{AP}{AD}$.

Proof. Refer to the diagram below. Draw *DX* // *AC*, intersecting *BE* at *X*.

We have $\frac{DX}{CE}=\frac{BD}{BC}=\frac{1}{3}$ and $\frac{CE}{AE}=\frac{1}{2}$.



Hence, $\frac{DX}{AE}=\frac{1}{6}=\frac{PD}{AP}$, i.e., $\frac{AP}{AD}=\frac{6}{7}$.

Since $\frac{[\Delta ABP]}{[\Delta ABD]}=\frac{AP}{AD}=\frac{6}{7}$ and $[\Delta ABD]=\frac{1}{3}[\Delta ABC]$, we must have

$\frac{[\Delta ABP]}{[\Delta ABC]}=\frac{2}{7}$, or $[\Delta ABP]=\frac{2}{7}[\Delta ABC]$.

Similarly, one sees that $[\Delta BCQ]=[\Delta CAR]=\frac{2}{7}[\Delta ABC]$.

It follows that $[\Delta PQR]=\frac{1}{7}[\Delta ABC]$, or $\frac{[\Delta PQR]}{[\Delta ABC]}=\frac{1}{7}$.

□

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Recall Example 2.1.4.

In ΔABC, D is a point on AB and $\frac{AD}{AC}=\frac{AC}{AB}=\frac{2}{3}$. *M is the midpoint of CD while AM extended intersects BC at E. Find* $\frac{CE}{BE}$.

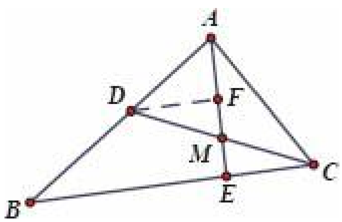
Can you solve it using the technique demonstrated above, drawing parallel lines and applying the Intercept Theorem?

Ans. Refer to the diagram below. Draw *DF* // *BC*, intersecting *AE* at *F*.

We have $\frac{DF}{CE}=\frac{DM}{CM}=1$ and $\frac{DF}{BE}=\frac{AD}{AB}=\frac{AD}{AC}\cdot\frac{AC}{AB}=\frac{4}{9}$.

It follows that $\frac{CE}{BE}=\frac{DF}{BE}=\frac{4}{9}$.

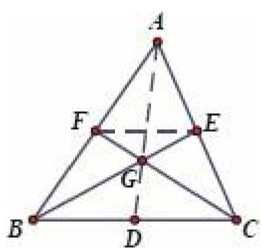
□



An important special case of Corollary 2.2.2 is the Midpoint Theorem.

Theorem 2.2.5 (Midpoint Theorem) *In Δ ABC, D, E, F are midpoints of BC, AC, AB respectively. We have EF // BC, EF =* $\frac{1}{2}$*BC and AD, BE, CF are concurrent*.

Proof. Since *E*, *F* are midpoints, *EF* // *BC* by the Intercept Theorem. Now Corollary 2.2.2 implies $\frac{EF}{BC}=\frac{AF}{AB}=\frac{1}{2}$. Refer to the diagram below.



Suppose *BE* and *CF* intersect at *G*. We have $\frac{BG}{GE}=\frac{BC}{EF}=2$, i.e., *CF* must

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intersect BE at the trisection point closer to E. Notice that this argument applies to AD as well, i.e., AD must also intersect BE at G where $\frac{BG}{BE}=\frac{2}{3}$. Indeed, AD, BE, CF are concurrent at G.

□

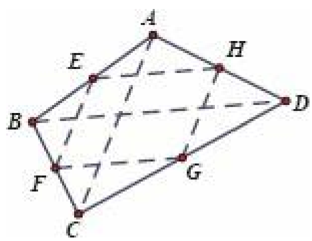
Note: One may derive the following important properties easily from the Midpoint Theorem.

(1) The medians of a triangle are concurrent (at the *centroid*) and the centroid is always at the lower one-third position of a median.

(2) A *midline*, i.e., a line segment connecting the midpoints of two legs, is always parallel to and has half of the length of the corresponding base of the triangle. Hence, drawing a midline is an important technique when solving problems related to midpoints as the line segments far apart could be brought together.

Example 2.2.6 Let $ABCD$ be a quadrilateral and E, F, G, H be the midpoints of AB, BC, CD, DA respectively. Show that $EFGH$ is a parallelogram.

Insight. This is a simple application of the Midpoint Theorem. Refer to the diagram below. One easily sees that EF // AC // GH and EH // BD // FG.



Example 2.2.7 Let D be a point inside ΔABC such that AD bisects $\angle A$ and $AD \perp BD$. Let M be the midpoint of BC.

(1) If AB = 11 and AC = 17, find MD.

(2) Show that M cannot lie on AD extended.

Insight.

(1) We are to find MD where M is the midpoint of BC. If D is the midpoint o another line segment, perhaps we could apply the Midpoint Theorem. Is there a line segment whose midpoint is D? Since AD is an angle bisector, it is a common technique to *reflect* ΔABD about AD. This technique is even more useful here because $AD \perp BD$. Refer to the diagram below. Can you see ΔABE is an isosceles triangle?

(2) Let E be the reflection of B about AD. If M lies on AD extended, can you see BM = CM = EM ? What does it imply?

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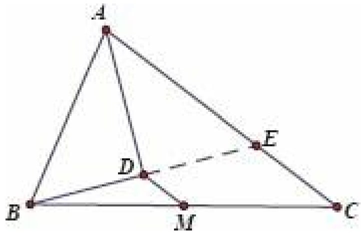


Ans.

(1) Let AD extended intersect AC at E. Since AD is the angle bisector and $AD \perp BE$, we have $\Delta ABD \cong \Delta AED$ (A.A.S.), which implies $BD = DE$ and $AE = AB$. Since $BM = CM$, we must have $MD = \frac{1}{2}CE$ by the Midpoint Theorem. It follows that $MD = \frac{1}{2}(AC - AE) = \frac{1}{2}(AC - AB) = 3$.

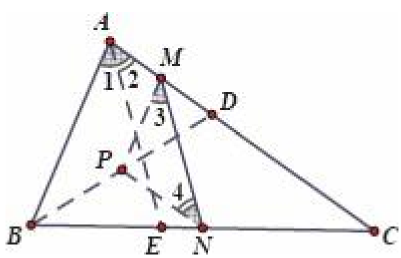
(2) Suppose otherwise that M lies on AD extended. It is easy to see that $\Delta ABM \cong \Delta AEM$ (S.A.S.), which implies $BM = EM$.
Now $BM = CM = EM$ implies $\angle BEC = 90°$ (Example 1.1.8). This is absurd because ΔABE is an isosceles triangle.

□

Example 2.2.8 Given ΔABC, D is a point on AC such that $AB = CD$. Let M, N be the midpoints of AD, BC respectively. Show that MN is parallel to the angle bisector of $\angle BAC$.

Insight. How can we apply $AB = CD$, where AB, CD are far apart? Since we are given the midpoints of AD, BC, if we connect BD and let P be the midpoint of BD, then $PM = \frac{1}{2}AB$ and $PN = \frac{1}{2}CD$.

Hence, $PM = PN$. Refer to the diagram below. Now ΔPMN is an isosceles triangle. Can we use the technique of the isosceles triangle and parallel line to obtain the angle bisector (Example 1.1.10)?



Proof. Let P be the midpoint of BD. Notice that $PM = \frac{1}{2}AB = \frac{1}{2}CD = PN$

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by the Midpoint Theorem. Hence, $\angle 3 = \angle 4$

Draw *AE* // *MN*, intersecting *BC* at *E*. Since *AB* // *PM* and *AE* // *MN*, one sees that $\angle 1 = \angle 3$ and similarly, $\angle 2 = \angle 4$. It follows that $\angle 1 = \angle 2$, i.e., *AE* bisects $\angle BAC$. This completes the proof.

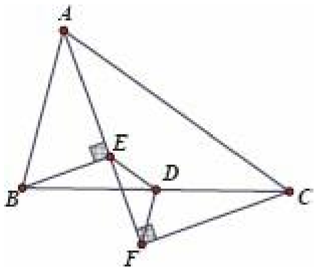
□

Note:

(1) One could see $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$ easily by recognizing corresponding angles, alternate angles and interior angles with respect to parallel lines.

(2) If one draws the angle bisector of $\angle BAC$ instead of *AE* // *MN*, the proof similar. One could show $\angle 3 + \angle 4 = \angle BAC$ (using parallel lines), which also leads to the conclusion.

Example 2.2.9 Refer to the diagram below. Given ΔABC, *D* is the midpoint of *BC* and *AF* bisects $\angle A$. Draw $BE \perp AF$ at *E* and $CF \perp AF$ at *F*. Show that $DE = DF$.



Insight. Considering the midlines (and medians) could be a wise strategy because we are given not only midpoints, but also right angled triangles. For example, say *P*, *Q* are the midpoint of *AB*, *AC* respectively, we have $QD = \frac{1}{2}AB$ by the Midpoint Theorem and $PE = \frac{1}{2}AB$ because *PE* is the median on the hypotenuse of the right angled triangle ΔABE. Hence, $QD = PE$. Can you see that $PD = QF$ as well?

Proof. Refer to the diagram below. Let *P*, *Q* be the midpoints of *AB*, *AC* respectively. We have $PE = \frac{1}{2}AB = QD$ and $PD = \frac{1}{2}AC = QF$.

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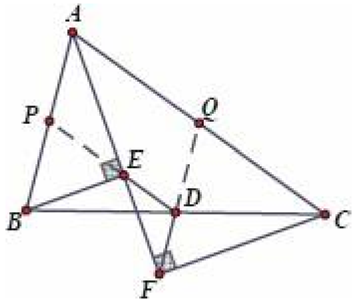
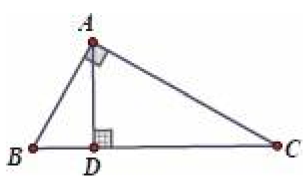


In the right angled triangle ΔABE, we have $AP = PE$ and hence, $\angle BPE = 2\angle BAE = \angle BAC$, which implies $PE \mathbin{//} AC$. Since $PD \mathbin{//} AC$, we must have P, D, E collinear. Similarly, D, F, Q are collinear. It follows that $DE = PD - PE = FQ - DQ = DF$.

Note: It seems from the diagram above that P, D, E are collinear, but one should **not** assume this without a proof. In fact, if an inaccurate diagram is casually drawn, one may even see $\Delta PDE \cong \Delta QFD$.

2.3 Similar Triangles

Congruent triangles are very useful in solving geometry problems, as a pair of congruent triangles are of not only the same size, but of identical *shape* as well. However, we may frequently encounter triangles which have identical shape, but differ in size. For example, a height on the hypotenuse of a right angled triangle gives three triangles of the same shape. Refer to the diagram below. Note that ΔABD, ΔCAD and ΔCBA show similarity in their shapes. We say two triangles ΔABC and $\Delta A'B'C'$ are similar if they have the same shape, or more precisely, if all the corresponding angles are the same and all the corresponding sides are of equal ratio, i.e., $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$ and $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$. We denote this by $\Delta ABC \sim \Delta A'B'C'$.



One may verify similar triangles by definition. However, this is often unnecessary. It is taught in most secondary schools that one can verify similar triangles by the following criteria, the proof of which is based on the Intercept Theorem:

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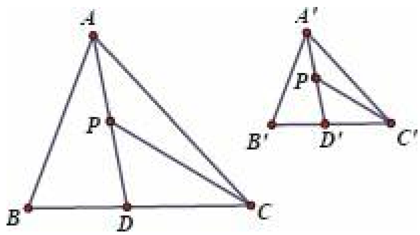
- If two pairs of corresponding angles are identical, then the two triangles are similar, i.e., if $\angle A = \angle A'$ and $\angle B = \angle B'$ (in which case one must have $\angle C = \angle C'$), then $\Delta ABC \sim \Delta A'B'C'$.

- If two pairs of corresponding sides are of equal ratio and the angles between them are identical, then the two triangles are similar, i.e., if $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ and $\angle A = \angle A'$, then $\Delta ABC \sim \Delta A'B'C'$.

- If all the corresponding sides are of equal ratio, then the two triangles a congruent, i.e., if $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$, then $\Delta ABC \sim \Delta A'B'C'$.

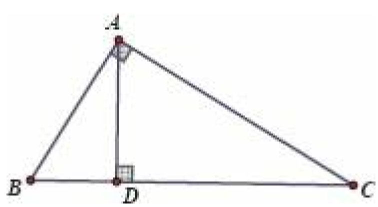
One may also determine a pair of similar right angled triangles by legs and hypotenuses. This is similar to determining congruent triangles using H.L. and it can be justified easily by Pythagoras' Theorem.

Notice that if $\Delta ABC \sim \Delta A'B'C'$, **all** the corresponding angles are the same and the corresponding line segments are of the same ratio. Refer to the diagram below for an example.



Given $\Delta ABC \sim \Delta A'B'C'$, let AD bisect $\angle A$ and $A'D'$ bisects $\angle A'$. If P, P' are the midpoints of AD, $A'D'$ respectively, we have $\frac{AB}{A'B'} = \frac{CP}{C'P'}$ and $\angle ACP = \angle A'C'P'$.

Now we can see that in a right angled triangle ΔABC where $\angle A = 90°$ and AD is a height, $\Delta ABC \sim \Delta ABD \sim \Delta ACD$. Refer to the diagram below. In particular, the following result is useful.



Example 2.3.1 ΔABC is a right angled triangle where $\angle A = 90°$ and AD is a

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height. We have $AB^2 = BD \cdot BC$, $AC^2 = CD \cdot BC$ and $AD^2 = BD \cdot CD$.

Proof. Since $\angle C = \angle BAD$, we have $\Delta ABC \sim \Delta DBA \sim \Delta DAC$, which gives $\frac{AB}{BD} = \frac{BC}{AB}$, $\frac{AC}{CD} = \frac{BC}{AC}$ and $\frac{AD}{BD} = \frac{CD}{AD}$.

It follows that $AB^2 = BD \cdot BC$, $AC^2 = CD \cdot BC$ and $AD^2 = BD \cdot CD$.

□

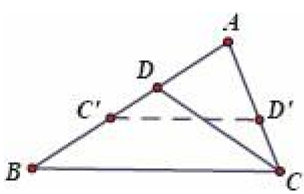
Note:

(1) Pythagoras' Theorem follows immediately from this example as $AB^2 + AC^2 = BD \cdot BC + CD \cdot BC = (BD + CD) \cdot BC = BC^2$.

(2) One sees from this example that $\frac{AB^2}{AC^2} = \frac{BD}{CD}$. This is a very useful conclusion. You may compare it with the Angle Bisector Theorem (Theorem 2.3.7).

Recognizing similar triangles is a very important technique because a pair of similar triangles gives equal angles and ratios of line segments. One may seek similar triangles via the following clues:

- Parallel lines
- Angle bisectors
- Opposite angles
- Refer to the diagram below. If $\angle ACD = \angle B$, then $\Delta ACD \sim \Delta ABC$.
  One may see this more clearly by reflecting ΔACD about the angle bisector of $\angle A$, which gives $\Delta AC'D'$. It is easy to show $BC \parallel C'D'$ and hence, $\Delta AC'D' \sim \Delta ABC$.
  Notice that Example 2.3.1 could be seen as a special case of this result, where $\angle ACB = 90°$.



Example 2.3.2 Given ΔABC where $\angle A = 120°$, D is a point of BC such that $BD = 15$, $CD = 5$ and $\angle ADB = 60°$. Find AC.

Ans. Refer to the diagram below. Since $\angle ADB = 60°$, we have $\angle ADC = 120° = \angle BAC$. It follows that $\Delta ABC \sim \Delta DAC$.

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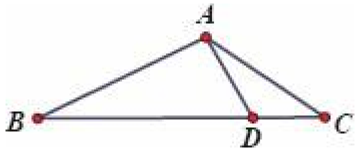
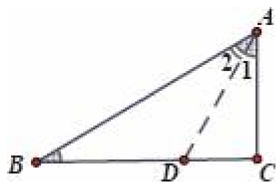


Now we have $\frac{CD}{AC}=\frac{AC}{BC}$, or $AC^2 = CD \cdot BC$. Since $CD = 5$ and $BC = BD + CD = 15 + 5 = 20$, we conclude that $AC = 10$.

□

Example 2.3.3 In ΔABC, $\angle A = 2\angle B$. Show that $BC^2 = AC \cdot (AB + AC)$.

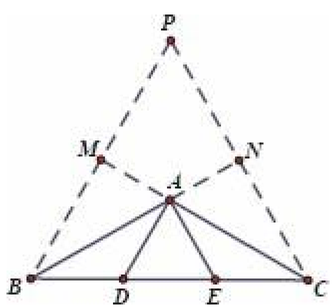


Insight. We are only given that $\angle A = 2\angle B$. Hence, it is natural to draw the angle bisector of $\angle A$ and we obtain equal angles $\angle B = \angle 1 = \angle 2$. Refer to the diagram above. Perhaps we shall seek similar triangles and set up the ratio.

Since $\angle 1 = \angle B$, $\Delta CAD \sim \Delta CBA$. We have $\frac{AC}{BC}=\frac{AD}{AB}=\frac{CD}{AC}$. Hence, $AC \cdot AB = BC \cdot AD$ and $AC^2 = BC \cdot CD$. Since $AD = BD$, we have $AC \cdot AB + AC^2 = BC \cdot BD + BC \cdot CD$, simplifying which gives the conclusion.

Example 2.3.4 In ΔABC, $\angle A = 120°$ and $AB = AC$. Let D, E be trisection points of BC, i.e., $BD = DE = CE$. Show that ΔADE is an equilateral triangle.

Proof. Refer to the diagram below. Draw an equilateral triangle ΔPBC from BC such that A is inside ΔPBC. It is easy to see that $\angle B = \angle C = 30°$, which implies that A is the incenter of ΔPBC. Clearly, A is also the centroid of ΔPBC.



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Now $\frac{AC}{CM} = \frac{2}{3} = \frac{CD}{BC}$, which implies AD // PB. Similarly, AE // PC.

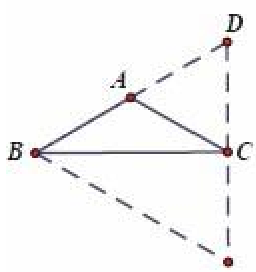
It follows that $\Delta ADE \sim \Delta PBC$ and hence the conclusion.

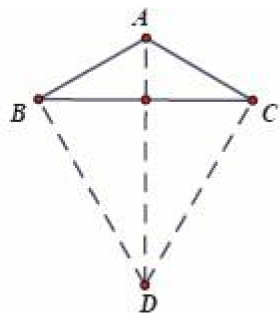
□

Note:

(1) ΔABD, ΔACE and ΔBCA are similar.

(2) An isosceles triangle with 120° at the vertex is closely related to equilateral triangles. Besides the example above, one may also double a leg. Refer to the left diagram below. Extend BA to D such that $AB = AD$. Notice that ΔACD is an equilateral triangle and $BC \perp CD$. Indeed, we are familiar with ΔBCD, which is half of a larger equilateral triangle.

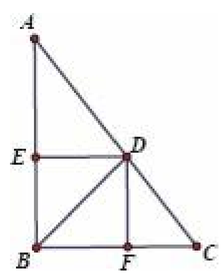




On the other hand, one may draw an equilateral triangle ΔBCD outwards. Refer to the right diagram above. Notice that both ΔABD and ΔACD are half of a larger equilateral triangle.

Example 2.3.5 In a right angled triangle ΔABC where $\angle B = 90°$, D is a point on AC such that BD bisects $\angle B$. Draw $DE \perp AB$ at E and $DF \perp BC$ at F. Show that $BD^2 = 2AE \cdot CF$.

Insight. Refer to the diagram below. It is given that BD bisects a right angle and DE, DF are perpendicular to AB, AC respectively. Can you see that $BEDF$ is a square!



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How is *BD* related to *AE* and *CF*? We know $BD = \sqrt{2}BF$ and it is easy to relate *AE*, *CF* and *BF* (or *DE*) together by similar triangles.

Proof. It is easy to see that *BFDE* is a rectangle because *DE* // *BF*, *BE* // *DF* and $\angle B = 90°$. We are given that $\angle ABD = \angle CBD = 45°$ and $\angle BED = 90°$. Hence, *BD* = *BE*, which implies that *BEDF* is a square. It follows that $BD = \sqrt{2}$ *BF*.

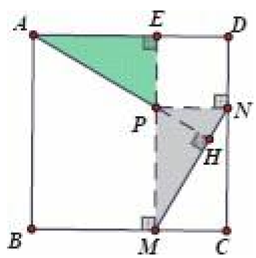
Clearly, $\Delta ADE \sim \Delta ACB$. Now $\frac{AE}{AB} = \frac{DE}{BC}$. Let $DE = BF = x$.

We have $\frac{AE}{AE+x} = \frac{x}{CF+x}$, simplifying which gives $x^2 = AE \cdot CF$.

It follows that $BD^2 = 2x^2 = 2AE \cdot CF$.

□

Example 2.3.6 Let *P* be a point inside the square *ABCD*. *M*, *N* are the feet of the perpendicular from *P* to *BC*, *CD* respectively. If $AP \perp MN$, show that either *AP* = *MN*, or $AP \perp BD$.



Insight. Refer to the diagram below. Notice that there are a lot of right angles. Clearly, *CMPN* is a rectangle and *MN* = *CP*. If *AP* = *MN*, we **should** have *AP* = *CP*, which implies *P* lies on *BD*. If $AP \perp BD$, then *P* lies on *AC*.

It seems from the diagram that $\Delta AEP \cong \Delta MPN$, which immediately gives *AP* = *MN*. However, this may **not** be true because it excludes the case for $AP \perp BD$. Nevertheless, we still have $\Delta AEP \sim \Delta MPN$ since $\angle PAE = \angle HPN = PMN$. Perhaps when $AP \neq MN$, we would have $AP \perp BD$. Notice that $AE + PN = PM + PE$ and $\frac{AE}{PE} = \frac{PM}{PN}$!

Proof. Let *AP* extended intersect *MN* at *H* and *MP* extended intersect *AD* at *E*. Since *PN* // *AD*, $\angle PAE = \angle HPN$. In the right angled triangle ΔPMN, we must have $\angle HPN = PMN$. Hence, $\angle PAE = PMN$, which implies $\Delta AEP \sim \Delta MPN$.

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Let $k = \frac{AE}{PE} = \frac{PM}{PN}$, i.e., $AE = k \cdot PE$ and $PM = k \cdot PN$. Since $AE + PN = PM + PE$, we have $k \cdot PE + PN = k \cdot PN + PE$, simplifying which gives $(k-1) \cdot PE = (k-1) \cdot PN$. Hence, either $k = 1$ or $PE = PN$.

If $k = 1$, we have $AE = PE$ and $PM = PN$. Now $AE = PE$ implies that $\angle PAE = 45°$, i.e., P lies on AC. $PM = PN$ implies $PMCN$ is a square and we must have $MN \parallel BD$. Hence, $AP \perp BD$.
If $PE = PN$, we have $\Delta AEP \cong \Delta MPN$ and hence, $AP = MN$.

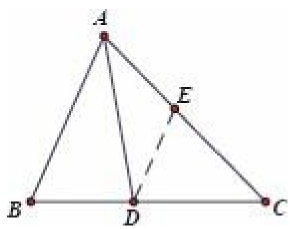
□

Similar triangles are even more frequently seen when circle properties are introduced, which we will discuss in Chapter 4.

The following is an important property of angle bisectors.

Theorem 2.3.7 (Angle Bisector Theorem) *In ΔABC, the angle bisector of $\angle A$ intersects BC at D. We have* $\frac{AB}{AC} = \frac{BD}{CD}$.

Proof. Refer to the diagram on the below. Draw $DE \parallel AB$, intersecting AC at E. We have $\angle BAD = \angle EDA$. Since AD bisects $\angle A$, $\angle EDA = \angle BAD = \angle EAD$. It follows that $AE = DE$.



Since $DE \parallel AB$, we have $\frac{BD}{CD} = \frac{AE}{CE} = \frac{DE}{CE}$. Notice that $\Delta ABC \sim \Delta EDC$.

Hence, $\frac{DE}{CE} = \frac{AB}{AC}$ and the proof is complete.

□

Note:

(1) We are still using the strategy of constructing an isosceles triangle with the angle bisector and parallel lines.

(2) One may easily see that the inverse of the Angle Bisector Theorem holc Given ΔABC where D is a point of BC, if $\frac{AB}{AC} = \frac{BD}{CD}$, then AD bisects $\angle A$. Otherwise, let AD' be the angle bisector and we have

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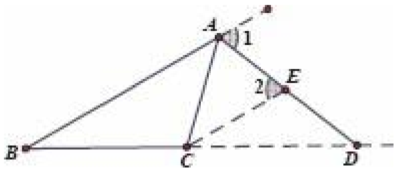
$\frac{BD'}{CD'} = \frac{AB}{AC} = \frac{BD}{CD}$, which implies *D* and *D*' coincide.

(3) Notice that the conclusion $\frac{AB}{AC} = \frac{BD}{CD}$ still holds even if *AD* is an exteri angle bisector, i.e., when *AD* bisects the supplementary angle of $\angle A$ Refer to the diagram below.



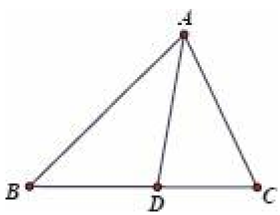
The proof is similar. Draw *CE* // *AB*, intersecting *AD* at *E*.

One sees that Δ*ACE* is an isosceles triangle where *AC* = *CE* (because $\angle 2 = \angle 1 = \angle CAE$).

Now $\frac{AB}{AC} = \frac{AB}{CE} = \frac{BD}{CD}$ by the Intercept Theorem.

Example 2.3.8 Let *AD* bisect $\angle A$ in Δ*ABC*, intersecting *BC* at *D*. Show that $BD = \frac{ac}{b+c}$, where *BC* = *a*, *AC* = *b* and *AB* = *c*.

***Proof*.** Refer to the diagram on the below. By the Angle Bisector Theorem, $\frac{BD}{CD} = \frac{AB}{AC} = \frac{c}{b}$.



Since $a = BC = BD + CD = \left(1 + \frac{b}{c}\right)BD$, we must have $BD = \frac{ac}{b+c}$.

□

Note: One may draw similar conclusions if *AD*, *BE*, *CF* are the angle bisectors of $\angle A$, $\angle B$, $\angle C$ respectively. This result is useful if angle bisectors are given and the ratios of sides are to be found.

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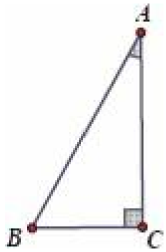
2.4 Introduction to Trigonometry

Since any two right angled triangles are similar if they have an equal pair of acute angles, a right angled triangle with a given acute angle, say $\angle A$, must have constant ratios between the legs and the hypotenuse.

Refer to the diagram on the below.



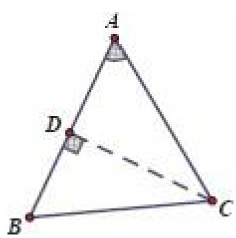
We define $\sin\angle A = \frac{BC}{AB}$, $\cos\angle A = \frac{AC}{AB}$ and $\tan\angle A = \frac{BC}{AC}$.

Trigonometry is taught in most secondary schools. The most important and commonly used properties are as follows, which one may see easily from the definition.

- $\sin\angle A = \cos(90° - \angle A)$
- $\tan\angle A = \frac{\sin\angle A}{\cos\angle A}$
- $(\sin\angle A)^2 + (\cos\angle A)^2 = 1$ by Pythagoras' Theorem.

Trigonometric methods are widely applicable in geometric calculations, which we do not emphasize in this book. Nevertheless, we still encounter simple trigonometry occasionally in problem-solving and hence, one should be very familiar with the basic properties.

One important application is about the area of triangles. Refer to the diagram on the below.



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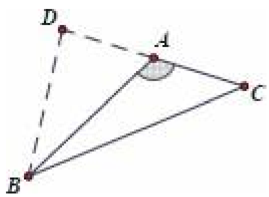
In ΔABC, given $CD \perp AB$ at D, we have $[\Delta ABC] = \frac{1}{2} AB \cdot CD$ and by definition,

$CD = AC \sin \angle A$. It follows that $[\Delta ABC] = \frac{1}{2} AB \cdot AC \sin \angle A$. Notice that heights are no longer involved in this formula.

If $\angle A > 90°$, we extend CA to D such that $AC = AD$. Refer to the diagram on the below.



Since $[\Delta ABC] = [\Delta ABD] = \frac{1}{2} AB \cdot AD \sin \angle BAD$ and $AC = AD$, if we define $\sin \angle A = \sin \angle BAD = \sin(180° - \angle A)$, we still have

$[\Delta ABC] = \frac{1}{2} AB \cdot AC \sin \angle A$. In particular, one sees that $\sin 90° = 1$.

Now $[\Delta ABC] = \frac{1}{2} AB \cdot AC \sin \angle A$ is consistent for any ΔABC.

Example 2.4.1 (HUN 10) Let $ABCD$ be a quadrilateral whose area is S. Show that if $(AB + CD)(AD + BC) = 4S$, then $ABCD$ is a rectangle.

Insight. We have $(AB + CD)(AD + BC) = AB \cdot AD + AB \cdot BC + CD \cdot AD + CD \cdot BC$. How are these related to S?

Proof. Notice that $S = [\Delta ABC] + [\Delta ACD] = \frac{1}{2} AB \cdot BC \sin B + \frac{1}{2} AD \cdot CD \sin D$. Refer to the diagram on the below. Similarly, we have $S = [\Delta ABD] + [\Delta BCD] = \frac{1}{2} AB \cdot AD \sin A + \frac{1}{2} BC \cdot CD \sin C$, i.e.,

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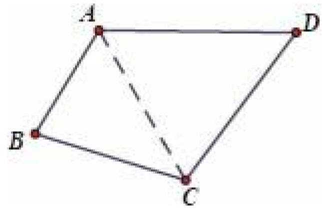


$4S = AB \cdot AD \sin A + AB \cdot BC \sin B + CD \cdot BC \sin C + CD \cdot AD \sin D.$

It is given that $4S = (AB + CD)(AD + BC)$. One sees that this is only possible when $\sin A = \sin B \sin C = \sin D = 1$. We must have $\angle A = \angle B = \angle C = \angle D = 90°$ and hence, $ABCD$ is a rectangle.

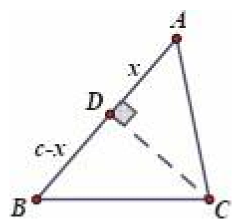
□

Cosine Rule is one of the most elementary and commonly used results in trigonometry. One may see it as an extension of Pythagoras' Theorem.

Theorem 2.4.2 (Cosine Rule) *In ΔABC where $BC = a$, $AC = b$ and $AB = c$, we have $a^2 = b^2 + c - 2bc \cos A$.*

Proof. We use Pythagoras' Theorem to prove Cosine Rule. Refer to the below diagram, where $\angle A$ is acute. Draw $CD \perp AB$ at D.
Let $AD = x$. We have $BD = c - x$.

Pythagoras' Theorem gives $AC^2 - AD^2 = CD^2 = BC^2 - BD^2$, i.e., $b^2 - x^2 = a^2 - (c - x)^2$. Simplifying the equation, we obtain

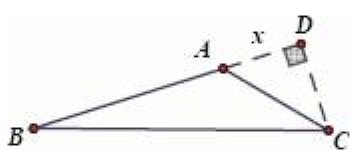


$b^2 = a^2 - c^2 + 2cx$, or $a^2 = b^2 + c^2 - 2cx$.

The conclusion follows as $x = b\cos A$.

A similar argument applies if $\angle A$ is obtuse. Refer to the diagram on the below. We draw $CD \perp AB$ intersecting BA extended at D.

Let $AD = x$. Pythagoras' Theorem gives



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$$AC^2 - AD^2 = CD^2 = BC^2 - BD^2, \text{ i.e., } b^2 - x^2 = a^2 - (c+x)^2.$$

Simplifying the equation, we obtain $a^2 = b^2 + c^2 + 2cx$, where $x = b\cos\angle CAD = b\cos(180° - \angle A)$. One sees that the conclusion holds if we define $\cos\theta = -\cos(180° - \theta)$ for $\theta \geq 90°$ and in particular, $\cos 90° = 0$.

Now $a^2 = b^2 + c^2 - 2bc\cos A$ is consistent for any triangle ΔABC.

□

Note:

(1) If $\angle A = 90°$, $a^2 = b^2 + c^2$ is exactly Pythagoras' Theorem.

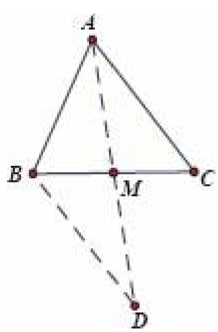
(2) One may perceive congruent triangles by Cosine Rule: Given a, b, c are the three sides of a triangle, we have $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Hence, one may calculate $\angle A$, and similarly $\angle B$ and $\angle C$. Now ΔABC is uniquely determined.
On the other hand, if b, c and $\angle A$ are given, one may calculate a using Cosine Rule. Hence, ΔABC is uniquely determined.
Notice that these are consistent with the criteria determining congruent triangles, S.S.S. and S.A.S. respectively.

One may apply Cosine Rule to calculate the length of a median in a given triangle.

Theorem 2.4.3 *In ΔABC where $BC = a$, $AC = b$, $AB = c$ and M is the midpoint of BC, we have* $AM^2 = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2$.

Proof. Refer to the diagram on the below. Extend AM to D such that $AM = MD$.



By Cosine Rule, $AD^2 = AB^2 + BD^2 - 2AB \cdot BD\cos\angle ABD$. Notice that AD and BC bisect each other, which implies $ABDC$ is a parallelogram. Hence, $BD = AC = b$

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and $\angle ABD = 180° - \angle A$.

We have $AD^2 = b^2 + c^2 - 2bc\cos(180° - \angle A) = b^2 + c^2 + 2bc\cos A$.

Since $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, we have

$$AD^2 = b^2 + c^2 + 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc} = 2b^2 + 2c^2 - a^2.$$

It follows that $AM^2 = \dfrac{1}{4}AD^2 = \dfrac{1}{2}b^2 + \dfrac{1}{2}c^2 - \dfrac{1}{4}a^2$.

□

Example 2.4.4 In ΔABC, $AB = 9$, $BC = 8$ and $AC = 7$. Let M be the midpoint of BC. Show that $AM = AC$.

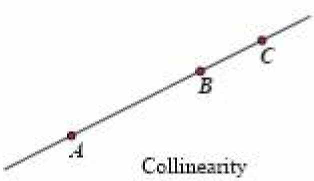
Proof. By Theorem 2.4.3, $AM^2 = \dfrac{1}{2}\left(AB^2 + AC^2\right) - \dfrac{1}{4}BC^2$

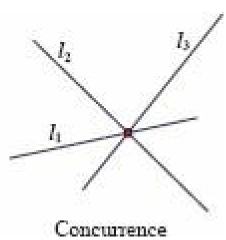
$= \dfrac{1}{2} \cdot \left(9^2 + 7^2\right) - \dfrac{1}{4} \cdot 8^2 = 49$, i.e., $AM = 7 = AC$.

□

2.5 Ceva's Theorem and Menelaus' Theorem

One important type of problems in geometry is on collinearity and concurrence. We know that any two points determine a unique straight line which passes through them. Hence, if we have three points say A, B, C, in general we can draw three lines AB, BC, CA, unless in the special case where A, B, C are collinear, i.e., they lie on the same line. Refer to the left diagram below.





Similarly, we know that any two distinct and non-parallel lines intersect at exactly one point. If we have three such straight lines say ℓ_1, ℓ_2, ℓ_3, in general we should have three points of intersection, unless in the special case where ℓ_1, ℓ_2, ℓ_3 are concurrent, i.e., they pass through the same point. Refer to the right diagram above.

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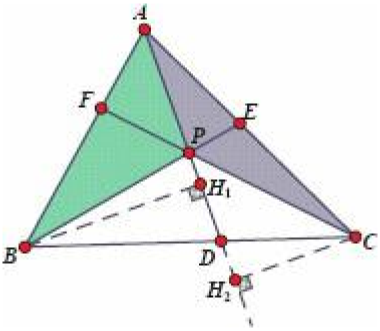
In many geometry questions, one may need to decide whether a given set of three points are collinear, or a given set of three lines are concurrent. For example, one may recall that we show in any triangles, the perpendicular bisectors of the three sides are concurrent (at the circumcenter). We have also shown the existence of the incenter, the ex-centers and the centroid of a triangle. We shall introduce Ceva's Theorem and Menelaus' Theorem, which provide more general criteria to determine concurrency and collinearity.

Theorem 2.5.1 (Ceva's Theorem) *In ΔABC, D, E, F are points on AB, AC, BC respectively such that AD, BE, CF are concurrent. We have* $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$.

Note: The conclusion is not difficult to remember. First, write down the three sides of the triangle *AB*, *BC*, *CA* in this manner: $\frac{A*}{B*}\cdot\frac{B*}{C*}\cdot\frac{C*}{A*}=1$. Notice that each letter appears in the numerator and denominator exactly once. Next, replace $*$ by the point which divides the respective side: $\frac{AF}{BF}$, $\frac{BD}{CD}$ and $\frac{CE}{AE}$. Notice that all the letters are "cancelled out"!

We use the area method to prove this theorem.

Proof. Refer to the diagram on the below. Let *AD*, *BE*, *CF* intersect at *P*. Draw $BH_1 \perp AP$ at H_1 and $CH_2 \perp AP$ at H_2.



We have $\frac{[\Delta ABP]}{[\Delta ACP]}=\frac{\frac{1}{2}\cdot AP\cdot BH_1}{\frac{1}{2}\cdot AP\cdot CH_2}=\frac{BH_1}{CH_2}=\frac{BD}{CD}$, because $BH_1 // CH_2$.

Similarly, $\frac{[\Delta CBP]}{[\Delta ABP]}=\frac{CE}{AE}$ and $\frac{[\Delta ACP]}{[\Delta BCP]}=\frac{AF}{BF}$.

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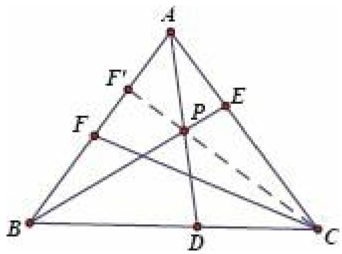
Now $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=\frac{[\Delta ACP]}{[\Delta BCP]}\cdot\frac{[\Delta ABP]}{[\Delta ACP]}\cdot\frac{[\Delta BCP]}{[\Delta ABP]}=1.$

□

Note:

(1) The inverse of Ceva's Theorem also holds: if *D*, *E*, *F* are points on *BC*, *AC*, *AB* respectively such that $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$, then *AD*, *BE*, *CF* are concurrent.

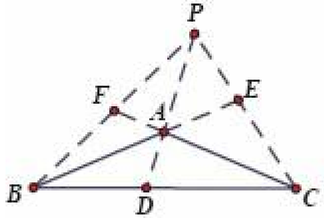
This can be proved easily by contradiction: Suppose otherwise that *AD*, *BE*, *CF* are not concurrent. Refer to the diagram on the below. Let *AD* and *BE* intersect at *P*. Suppose *CP* extended intersects *AB* at *F*'.
Now *AD*, *BE*, *CF*' are concurrent.

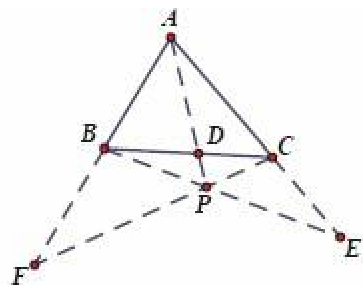


By Ceva's Theorem, one must have $\frac{AF'}{BF'}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$.

Since $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$, we must have $\frac{AF}{BF}=\frac{AF'}{BF'}$ which implies *F* and *F*' coincide.

(2) Ceva's Theorem also holds even if the points of division are on the extension of the sides of Δ*ABC*. Refer to the diagrams below where *AD*, *BE*, *CF* are concurrent at *P*.





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We still have $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$ in either case.

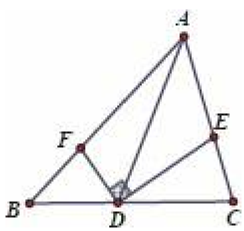
The proof is still by the area method. We leave the details to the reader. (**Hint**: Can you see that $\frac{AF}{BF}=\frac{[\Delta AFC]}{[\Delta BFC]}=\frac{[\Delta APC]}{[\Delta BPC]}$ in the diagram on the right? Notice that $\frac{[\Delta AFC]}{[\Delta APC]}=\frac{CF}{CP}=\frac{[\Delta BFC]}{[\Delta BPC]}$.)

Ceva's Theorem, especially its inverse, is very useful in showing concurrency. For example, the proof for the existence of the centroid of a triangle becomes trivial: if *D*, *E*, *F* are the midpoints of *BC*, *AC*, *AB* respectively, then $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1\times1\times1=1$. Hence, *AD*, *BE*, *CF* are concurrent.

One may also show the existence of the incenter using Ceva's Theorem (and the Angle Bisector Theorem). We leave it to the reader.

Example 2.5.2 In Δ*ABC*, *D* is on *BC*. *DE* bisects ∠*ADC*, intersecting *AC* at *E*. Draw *DF* ⊥ *DE*, intersecting *AB* at *F*. Show that *AD*, *BE*, *CF* are concurrent.

Insight. It suffices to show $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$. Since *DE* bisects ∠*ADC* and *DF* ⊥ *DE*, *DF* bisects ∠*ADB* (Example 1.1.9). Perhaps we should apply the Angle Bisector Theorem.



Proof. Since *DE* bisects ∠*ADC* and *DF* ⊥ *DE*, *DF* bisects ∠*ADB* (Example 1.1.9). By the Angle Bisector Theorem, we have $\frac{AF}{BF}=\frac{AD}{BD}$ and $\frac{AE}{CE}=\frac{AD}{CD}$. Now $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=\frac{AD}{BD}\cdot\frac{BD}{CD}\cdot\frac{CD}{AD}=1$. By Ceva's Theorem, *AD*, *BE*, *CF* are concurrent.

□

Example 2.5.3 Given a triangle Δ*ABC*, draw equilateral triangles Δ*ABF*, Δ*BCD*, Δ*ACE* outwards based on *AB*, *BC*, *AC* respectively. Show that *AD*, *BE*,

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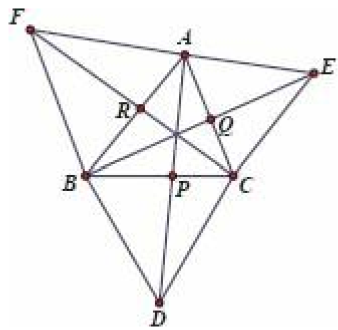
CF are concurrent.

Insight. Refer to the diagram on the below. It seems an application of Ceva's Theorem, i.e., say *AD* intersects *BC* at *P*, *BE* intersects *AC* at *Q* and *CF* intersects *AB* at *R*, we are to show $\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1$.



How could we express say $\frac{BP}{CP}$ in terms of what we are familiar with? We use areas of triangles when proving Ceva's Theorem, but we cannot use the same triangles once more because we do **not** know whether *AD*, *BE*, *CF* are collinear.

Notice that $\frac{BP}{CP}=\frac{[\Delta ABP]}{[\Delta ACP]}=\frac{[\Delta BDP]}{[\Delta CDP]}$. Hence, $\frac{BP}{CP}=\frac{[\Delta ABD]}{[\Delta ACD]}$

$$=\frac{\frac{1}{2}AB\cdot BD\sin\angle ABD}{\frac{1}{2}AC\cdot CD\sin\angle ACD}=\frac{AB\sin(\angle ABC+60^\circ)}{AC\sin(\angle ACB+60^\circ)}\ \text{since}\ BD=CD.$$

Similarly, $\frac{CQ}{AQ}=\frac{BC\sin(\angle ACB+60^\circ)}{AB\sin(\angle BAC+60^\circ)}$ and $\frac{AR}{BR}=\frac{AC\sin(\angle BAC+60^\circ)}{BC\sin(\angle ABC+60^\circ)}$.

It follows that $\frac{BP}{CP}\cdot\frac{CQ}{AQ}\cdot\frac{AR}{BR}=1$.

Example 2.5.4 In Δ*ABC*, *M* is the midpoint of *BC*. *AD* bisects ∠*A*, intersecting *BC* at *D*. Draw *BE* ⊥ *AD*, intersecting *AD* extended at *E*. If *AM* extended intersect *BE* at *P*, show that *AB* // *DP*.

Insight. Refer to the diagram on the below.

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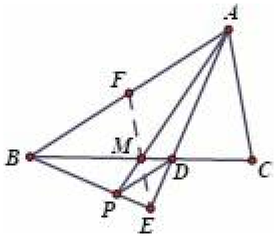
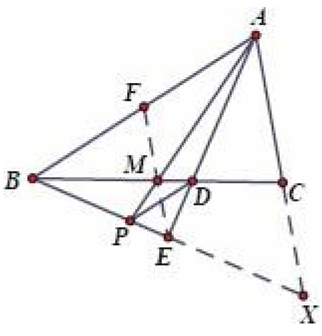


We **should** have $AB // DP$, i.e., $\frac{AD}{DE}=\frac{BP}{PE}$.

Hence, if *EM* extended intersects *AB* at *F*, we **should** have $\frac{AF}{BF}\cdot\frac{BP}{EP}\cdot\frac{ED}{AD}=1$ by Ceva's Theorem, which implies *AF* = *BF*.

Given that *M* is the midpoint of *BC*, we **should** have *MF* // *AC*, or equivalently, *EM* // *AC*. How can we show it?
We have not used the condition *AE* ⊥ *BE* and the angle bisector *AE*. It is a common technique to reflect Δ*ABE* about *AE* (Example 1.2.5) and obtain an isosceles triangle!

Proof. Refer to the diagram on the below, where *BE* extended intersect *AC* extended at *X*, and *EM* extended intersect *AB* at *F*. Since *AE* bisects ∠*BAX* and *AE* ⊥ *BX*, Δ*ABX* must be an isosceles triangle where *AB* = *AX*. It follows that *BE* = *XE*.



By the Midpoint Theorem, *ME* // *AX*, or equivalently, *FF* // *AX*. It follows from the Intercept Theorem that *F* is the midpoint of *AB*.

By Ceva's Theorem, $\frac{AF}{BF}\cdot\frac{BP}{EP}\cdot\frac{ED}{AD}=1$. Since $\frac{AF}{BF}=1$, we must have $\frac{BP}{EP}=\frac{AD}{ED}$, which implies *PD* // *AB* by the Intercept Theorem.

□

Note:

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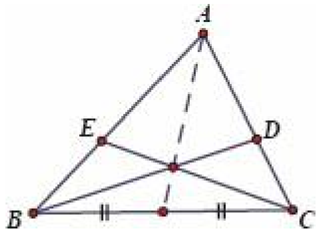
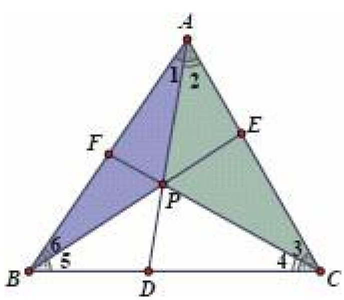
(1) One may easily show the following result by applying Ceva's Theorem. Refer to the diagram on the below. Given Δ*ABC* where *D*, *E* are on *AC*, *AB* respectively and *BD*, *CE* intersect at *P*, we have *DE* // *BC* if and only if *AP* extended passes through the midpoint of *BC*.



(2) Example 2.5.4 is not an easy problem. However, one may see the clues more clearly by dividing it into three sub-problems: reflecting Δ*ABE* about the angle bisector *AE* (Example 1.2.5), applying the Midpoint Theorem and the Intercept Theorem to the midline *EM*, and applying Ceva's Theorem with the median *EF*. Hence, one could understand how the auxiliary lines are constructed. (You may draw the diagrams separately for each sub-problem.)

Ceva's Theorem has a **trigonometric form**. Refer to the diagram below.

If AD, BE, CF are concurrent, then $\frac{\sin\angle 1}{\sin\angle 2}\cdot\frac{\sin\angle 3}{\sin\angle 4}\cdot\frac{\sin\angle 5}{\sin\angle 6}=1$.



Proof. We still use the area method. Recall the area formula of a triangle: $[\Delta ABC]=\frac{1}{2}bc\sin A$.

We have $\frac{[\Delta ABP]}{[\Delta ACP]}=\frac{\frac{1}{2}\cdot AP\cdot AB\sin\angle 1}{\frac{1}{2}\cdot AP\cdot AC\sin\angle 2}=\frac{AB\sin\angle 1}{AC\sin\angle 2}$.

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Similarly, $\frac{[\Delta ACP]}{[\Delta BCP]} = \frac{AC \sin\angle 3}{BC \sin\angle 4}$ and $\frac{[\Delta BCP]}{[\Delta BAP]} = \frac{BC \sin\angle 5}{AB \sin\angle 6}$.

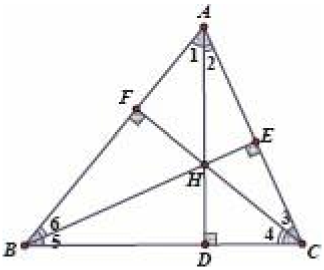
Multiply the three equations and we obtain:

$$\frac{[\Delta ABP]}{[\Delta ACP]} \cdot \frac{[\Delta ACP]}{[\Delta BCP]} \cdot \frac{[\Delta BCP]}{[\Delta BAP]} = 1 = \frac{AB \cdot AC \cdot BC}{AC \cdot BC \cdot AB} \cdot \frac{\sin\angle 1}{\sin\angle 2} \cdot \frac{\sin\angle 3}{\sin\angle 4} \cdot \frac{\sin\angle 5}{\sin\angle 6},$$

which leads to the conclusion.

□

Applying the trigonometric form of Ceva's Theorem, it is easy to show that the three heights of a triangle are concurrent. Refer to the diagram on the below for the case of an acute angled triangle.

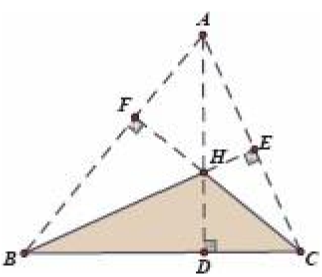


Notice that $\angle 1 = 90° - \angle AHF = \angle 4$. Similarly, $\angle 2 = \angle 5$ and $\angle 3 = \angle 6$.

It follows immediately that $\frac{\sin\angle 1}{\sin\angle 2} \cdot \frac{\sin\angle 3}{\sin\angle 4} \cdot \frac{\sin\angle 5}{\sin\angle 6} = 1$.

Hence, *AD*, *BE*, *CF* are concurrent, i.e., they pass through a common point *H*, which is called the **orthocenter** of Δ*ABC*.

A similar argument applies for obtuse angled triangles. Refer to the obtuse angled triangle Δ*HBC* in the diagram on the below. Its orthocenter is *A* (while *H* is the orthocenter of Δ*ABC*).



Example 2.5.5 Let *H* be the orthocenter of an acute angled triangle Δ*ABC*.

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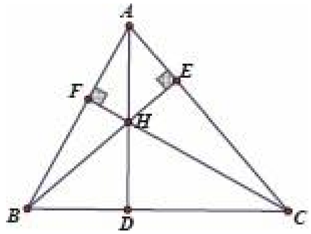
Show that $\angle BHC = 180° - \angle A$.

One easily sees the conclusion by considering the internal angles of the quadrilateral *AEHF*. Refer to the diagram on the below.



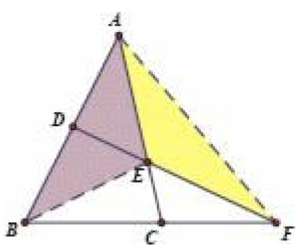
Note that there are a lot of pairs of equal angles in the diagram above. We will study more about the orthocenter of a triangle after we introduce the circle properties in Chapter 3.

Theorem 2.5.6 (Menelaus' Theorem) *Given ΔABC, a straight line intersects AB, AC and the extension of BC at D, E, F respectively. We have* $\frac{AD}{BD}\cdot\frac{BF}{CF}\cdot\frac{CE}{AE}=1$.

Note: The conclusion of Menelaus' Theorem is similar to that of Ceva's Theorem: it is also of the form $\frac{A*}{B*}\cdot\frac{B*}{C*}\cdot\frac{C*}{A*}=1$ where $*$ is to be replaced by the point which divides (internally or externally) the respective side of Δ*ABC*. Notice that all the letters are "cancelled out"!

We also use the area method to prove Menelaus' Theorem.

Proof. Connect *AF* and *BE*. We denote $S_1 = [\Delta ABE]$, $S_2 = [\Delta AEF]$ and $S_3 = [\Delta BEF]$. Refer to the diagram on the below.



Notice that $\frac{AD}{BD}=\frac{S_2}{S_3}$ because Δ*AEF* and Δ*BEF* share a common base *EF* and their heights on *EF* are of the ratio $\frac{AD}{BD}$. Similarly, $\frac{BC}{CF}=\frac{S_1}{S_2}$.

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Hence, $\frac{BF}{CF}=\frac{BC+CF}{CF}=\frac{BC}{CF}+1=\frac{S_1}{S_2}+1=\frac{S_1+S_2}{S_2}$.

We also have $\frac{CE}{AE}=\frac{[\Delta BCE]}{S_1}=\frac{[\Delta FCE]}{S_2}=\frac{[\Delta BCE]+[\Delta FCE]}{S_1+S_2}=\frac{S_3}{S_1+S_2}$.

Now $\frac{AD}{BD}\cdot\frac{BF}{CF}\cdot\frac{CE}{AE}=\frac{S_2}{S_3}\cdot\frac{S_1+S_2}{S_2}\cdot\frac{S_3}{S_1+S_2}=1$.

□

Note:

(1) The inverse of Menelaus' Theorem also holds: if *D*, *E*, *F* are points on *AB* *AC* and *BC* extended respectively and $\frac{AD}{BD}\cdot\frac{BF}{CF}\cdot\frac{CE}{AE}=1$, then *D*, *E*, *F* are collinear.

This can be proved easily by contradiction: Suppose otherwise, say *DE* extended intersects *BC* extended at *F*'.

By Menelaus' Theorem, we have $\frac{AD}{BD}\cdot\frac{BF'}{CF'}\cdot\frac{CE}{AE}=1$.

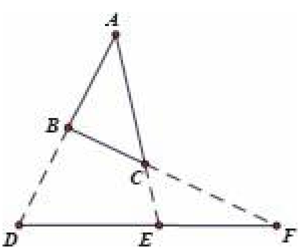
Hence, $\frac{BF'}{CF'}=\frac{BD}{AD}\cdot\frac{AE}{CE}=\frac{BF}{CF}$ by the condition given. We conclude that *F* and *F*' coincide.

Applying Menelaus' Theorem, especially its inverse, is an important method when showing collinearity.

(2) Menelaus' Theorem applies regardless of the relative positions of the division points, i.e., the division points can be on the extension of the sides of a triangle. Refer to the below diagram where the line *DE* does **not** intersect Δ*ABC*.

We still have $\frac{AD}{BD}\cdot\frac{BF}{CF}\cdot\frac{CE}{AE}=1$.

One may prove it by the similar area method. We leave the details to the reader.



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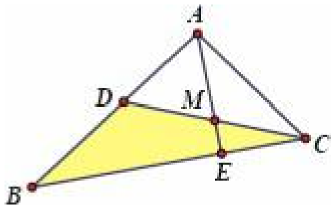
(3) Although the conclusions of Ceva's Theorem and Menelaus' Theorem ar highly similar, one may see their different geometric meanings easily from the diagrams.

One may apply Menelaus' Theorem and calculate the ratio of line segments very efficiently. Recall Example 2.1.4.

In ΔABC, D is a point on AB and $\frac{AD}{AC}=\frac{AC}{AB}=\frac{2}{3}$. *M is the midpoint of CD while AM extended intersects BC at E. Find* $\frac{CE}{BE}$.

Ans. Refer to the diagram on the below. Apply Menelaus' Theorem when the line *AE* intersects ΔBCD. We have $\frac{CE}{BE}\cdot\frac{BA}{DA}\cdot\frac{DM}{CM}=1$.

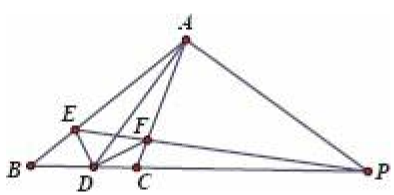


Since $\frac{BA}{DA}=\frac{BA}{AC}\cdot\frac{AC}{DA}=\left(\frac{3}{2}\right)^2=\frac{9}{4}$ and $\frac{DM}{CM}=1$, we have $\frac{CE}{BE}=\frac{4}{9}$. □

Note: Choosing an appropriate triangle and a line intersecting it is very important when applying Menelaus' Theorem. For example, if we choose the line *CD* intersecting Δ*ABE* in this example, we will not be able to obtain $\frac{CE}{BE}$.

Example 2.5.7 Given Δ*ABC*, *D* is a point on *BC* such that *AD* bisects ∠*A*. *E*, *F* are on *AB*, *AC* respectively such that *DE*, *DF* bisect ∠*ADB* and ∠*ADC* respectively. If *EF* extended intersects the line *BC* at *P*, show that *AP* ⊥ *AD*.

Insight. Refer to the diagram on the below. It seems we should consider the line *EF* intersecting Δ*ABC* and apply Menelaus' Theorem.



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Can we apply the Angle Bisector Theorem for $\frac{AE}{BE}$ and $\frac{CF}{AF}$?

On the other hand, since we are to show $AP \perp AD$, AP **should** be the exterior angle bisector of $\angle BAC$ (Example 1.1.9). Hence, we **should** have $\frac{BP}{CP}=\frac{AB}{AC}$ by the Angle Bisector Theorem.

Proof. By Menelaus' Theorem, $\frac{AE}{BE}\cdot\frac{BP}{CP}\cdot\frac{CF}{AF}=1$. (*)

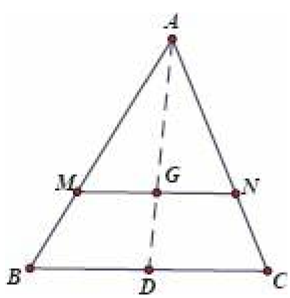
Since DE, DF are angle bisectors, we must have $\frac{AE}{BE}=\frac{AD}{BD}$ and $\frac{CF}{AF}=\frac{CD}{AD}$ by the Angle Bisector Theorem. Now $\frac{AE}{BE}\cdot\frac{CF}{AF}=\frac{CD}{BD}=\frac{AC}{AB}$ because AD bisects $\angle BAC$. It follows from (*) that $\frac{BP}{CP}=\frac{AB}{AC}$.

Hence, AP is the exterior angle bisector of $\angle BAC$. We conclude that $AP \perp AD$ (Example 1.1.9).

□

Example 2.5.8 In ΔABC, M, N are points on AB, AC respectively such that the centroid G of ΔABC lies on MN. Show that $AM \cdot CN + AN \cdot BM = AM \cdot AN$.

Insight. Let AG intersect BC at D. Notice that $\frac{AG}{AD}=\frac{2}{3}$. Since G lies on MN, if $MN \,//\, BC$, $\frac{AM}{AB}=\frac{AN}{AC}=\frac{2}{3}$ and $\frac{CN}{AC}=\frac{BM}{AB}=\frac{1}{3}$. Refer to the diagram on the below.



It follows that $AM\cdot CN+AN\cdot BM=\frac{2}{3}AB\cdot\frac{1}{3}AC+\frac{2}{3}AC\cdot\frac{1}{3}AB$

$=\frac{4}{9}AB\cdot AC=\frac{2}{3}AB\cdot\frac{2}{3}AC=AM\cdot AN.$

Otherwise, say MN extended intersects BC extended at P. Refer to the

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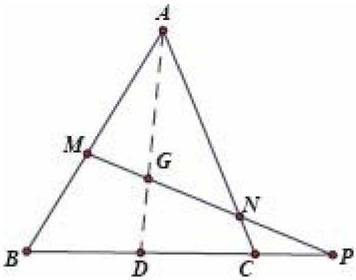
diagram on the below. We see that the line *MP* intersects several triangles. Moreover, we know $BD = CD$ and $AG = 2DG$. Hence, applying Menelaus' Theorem would probably help us to find the relationship among those line segments.



It is also noteworthy that the common factors *AM* and *AN* appear on both sides of the equation in the conclusion. Hence, we may consider dividing both sides by $AM \cdot AN$.

Proof. It is easy to show the conclusion when $MN \parallel BC$. Otherwise, say *MN* extended intersects *BC* extended at *P*. By dividing $AM \cdot AN$ on both sides of the equation, it suffices to show that $\frac{CN}{AN}+\frac{BM}{AM}=1$.

Apply Menelaus' Theorem when the line *MN* intersects ΔACD:

$$\frac{AG}{DG}\cdot\frac{DP}{CP}\cdot\frac{CN}{AN}=1\text{, i.e., }\frac{CN}{AN}=\frac{CP}{2DP}\text{ since }\frac{AG}{DG}=2.$$

Apply Menelaus' Theorem when the line *MN* intersects ΔABD:

$$\frac{AM}{BM}\cdot\frac{BP}{DP}\cdot\frac{DG}{AG}=1\text{, i.e., }\frac{BM}{AM}=\frac{BP}{2DP}.$$

Hence, we are to show $\frac{CP}{2DP}+\frac{BP}{2DP}=1$, or $CP+BP=2DP$.

This is clear because $CP + BP = CP + CP + BC = 2CP + 2DC = 2DP$.

□

Example 2.5.9 (USA 11) In a non-isosceles acute angled triangle ΔABC where *AD*, *BE*, *CF* are heights, *H* is the orthocenter. *AD* and *EF* intersect at *S*. Draw $AP \perp EF$ at *P* and $HQ \perp EF$ at *Q*. If the lines *DP* and *QH* intersect at *R*, show that $HQ = HR$.

Insight. Refer to the diagram on the below. Besides the feet of perpendicular *D*, *E*, *F* and the orthocenter *H*, the diagram is constructed by drawing perpendicular lines and we also have $AP \parallel QR$. In particular, for any

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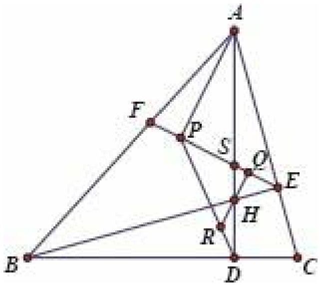
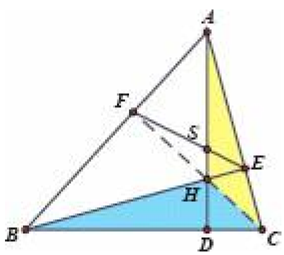
given ΔABC, Q and R are uniquely determined.



How could we show $\frac{HQ}{HR}=1$? Menelaus' Theorem could be very useful in such a diagram which is purely constructed by the intersection of straight lines.

Since AP // QR, we have $\frac{HQ}{AP}=\frac{HS}{AS}$ and $\frac{HR}{AP}=\frac{HD}{AD}$. It suffices to show that $\frac{HS}{AS}=\frac{HD}{AD}$. Which triangle (and the line intersecting it) should we apply Menelaus' Theorem to?

Proof. Refer to the diagram on the below. Apply Menelaus' Theorem to ΔAHC and EF.



We have $\frac{AS}{HS}\cdot\frac{HF}{CF}\cdot\frac{CE}{AE}=1$. (1)

Since $AP // QR$, we have $\frac{HQ}{AP}=\frac{HS}{AS}$ and $\frac{HR}{AP}=\frac{HD}{AD}$. We claim that $\frac{HS}{AS}=\frac{HD}{AD}$, which implies $\frac{HQ}{AP}=\frac{HR}{AP}$ and hence, $HQ=HR$.

By (1), it suffices to show $\frac{AD}{HD}\cdot\frac{HF}{CF}\cdot\frac{CE}{AE}=1$. (2)

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Let $S_1 = [\Delta ABH]$, $S_2 = [\Delta BCH]$ and $S_3 = [\Delta ACH]$.

We have $\frac{AD}{HD} = \frac{S_1 + S_2 + S_3}{S_2}$, $\frac{HF}{CF} = \frac{S_1}{S_1 + S_2 + S_3}$ and $\frac{CE}{AE} = \frac{S_2}{S_1}$.

Now it is easy to see that (2) holds. This completes the proof.

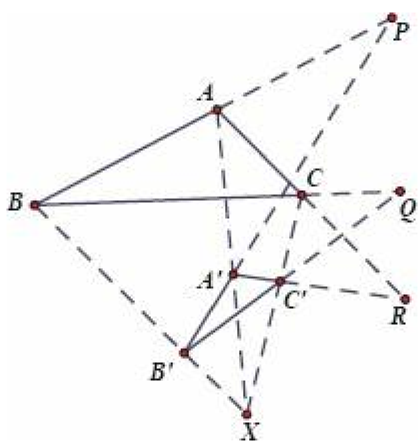
□

Note: One may perceive (2) as Ceva's Theorem applied to ΔAHC where lines AF, HE, CD are concurrent at B. Of course, beginners may find difficulties in recognizing Ceva's Theorem when the point of concurrency is outside the triangle. In such cases, one may always use the area method. We can see from the proof above that this is not difficult.

As an application of Menelaus' Theorem, we will show Desargues' Theorem, which is also an important result in showing collinearity and concurrency.

Theorem 2.5.10 (Desargues' Theorem) *Given ΔABC and $\Delta A'B'C'$ such that the lines AB, AB' intersect at P, the lines BC, $B'C'$ intersect at Q and the lines AC, $A'C'$ intersect at R, if the lines AA', BB', CC' are concurrent, then P, Q, R are collinear*.

Proof. Refer to the diagram on the below, where AA', BB', CC' are concurrent at X. Apply Menelaus' Theorem when $B'P$ intersects ΔXAB and we obtain:



$$\frac{XB'}{BB'} \cdot \frac{BP}{AP} \cdot \frac{AA'}{XA'} = 1. \quad (1)$$

Similarly, when $B'Q$ intersects ΔXBC, we have:

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$\frac{BB'}{XB'} \cdot \frac{CQ}{BQ} \cdot \frac{XC'}{CC'} = 1$. (2)

When $A'R$ intersects ΔXAC, we have $\frac{CC'}{XC'} \cdot \frac{XA'}{AA'} \cdot \frac{AR}{CR} = 1$. (3)

Multiplying (1), (2) and (3) gives $\frac{CQ}{BQ} \cdot \frac{BP}{AP} \cdot \frac{AR}{CR} = 1$, which implies P, Q, R are collinear by Menelaus' Theorem.

□

Note:

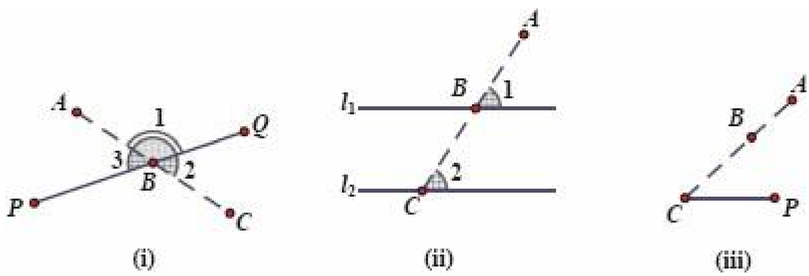
(1) We apply Menelaus' Theorem extensively in this proof, which does not depend on the relative positions of ΔABC and $\Delta A'B'C'$.

(2) The inverse of Desargues' Theorem also holds, i.e., if P, Q, R are collinear, then lines AA', BB', CC' are concurrent (or parallel to each other). One may follow a similar argument as above: given P, Q, R collinear and the lines AA', BB' intersect at X, show that C, C', X are collinear by Menelaus' Theorem.

Applying Desargues' Theorem changes the conclusion of concurrency to an equivalent one of collinearity, or vice versa. This may be a wise strategy when solving difficult problems, say if the conclusion to be shown seems unrelated to the conditions given. We will see examples in Chapter 6.

Ceva's Theorem and Menelaus' Theorem are very useful in showing concurrency and collinearity. However, we shall point out there are many other ways to show concurrency and collinearity.

- Collinearity: Showing equal or supplementary angles is the most fundamental and straightforward method. Refer to the diagrams below.



(i) PQ is a straight line where B lies. We have A, B, C collinear if $\angle 1 + \angle 2 = 180°$ or $\angle 2 = \angle 3$.

(ii) B, C are on ℓ_1, ℓ_2 respectively and $\ell_1 \parallel \ell_2$. We have A, B, C collinear if $\angle 1 = \angle 2$.

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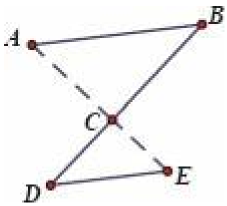
(iii) We have *A*, *B*, *C* collinear if $\angle ACP = \angle BCP$.

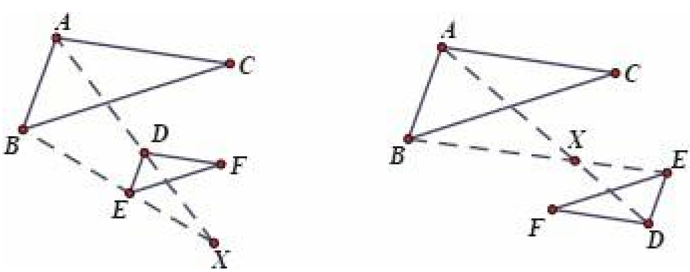
Another commonly used method is via the properties of similar triangles. Refer to the diagram on the below for an example, where *C* is a point on *BD* and *AB* // *DE*. Now *A*, *C*, *E* are collinear if $\frac{AC}{CE} = \frac{BC}{CD}$.



- Concurrency: One may suppose two lines meet at a point and show that the third line also passes through that point. We used this method to show the existence of the incenter, circumcenter, centroid and ex-centers (Exercise 1.4) of a triangle. Another commonly used method is via the properties of similar triangles, an example of which is given below.

Theorem 2.5.11 *Given ΔABC and ΔDEF such that AB// DE, BC // EF and AC // DF, then AD, BE, CF are either parallel or concurrent*.

Proof. Notice that there are two possible cases regarding the relative positions of Δ*ABC* and Δ*DEF*. Refer to the diagrams below.



It is easy to see that Δ*ABC* ~ Δ*DEF* because all the corresponding angles are equal. Suppose *AD* and *BE* intersect at *X*. It suffices to show that *CF* passes through *X* as well, i.e., *C*, *F*, *X* are collinear.

Connect *CX*, *FX*. Since *AB* // *DE*, we must have $\frac{AX}{DX} = \frac{AB}{DE} = \frac{AC}{DF}$.

Clearly, $\angle CAX = \angle FDX$. We conclude that Δ*ACX* ~ Δ*DFX* and hence, $\angle DXF = \angle AXC$. Now *C*, *X*, *F* are collinear.

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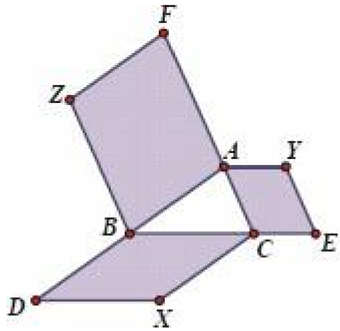
Notice that the proof does not depend on the diagram.

□

2.6 Exercises

1. Refer to the diagram on the below. Given ΔABC, extend AB to D such that $AB = BD$, extend BC to E such that $BC = 2CE$ and extend CA to F such that $AF = 2AC$. Draw parallelograms $BCXD$, $ACEY$ and $ABZF$. If the total area of these three parallelograms is 175cm^2, find the area of ΔABC in cm^2.



2. Given ΔABC, draw squares $ABDE$ and $ACFG$ outwards from AB, AC respectively. Let O_1, O_2 denote the centers of squares $ABDE$ and $ACFG$ respectively. If M, N are the midpoints of BC, EG respectively, show that MO_1NO_2 is a square.

3. In a quadrilateral $ABCD$, $AB \perp AD$ and $BC \perp CD$. F is a point on CD such that AF bisects $\angle BAD$. If BD and AF intersect at E and $AF // BC$, show that $AE < \frac{1}{2}CD$.

4. In a right angled triangle ΔABC, $\angle A = 90°$ and D, E are on AB, AC respectively. If M, N, P, Q are the midpoints of DE, BC, BE, CD respectively, show that $MN = PQ$.

5. Let $ABCD$ be a quadrilateral and E, F, G, H be the midpoints of AB, BC, CD, DA respectively. Let M be the midpoint of GH and P be a point on EM such that $FG = PG$. Show that $PF \perp EM$.

6. Given a square $ABCD$, E, F are the midpoints of AB, BC respectively. Let CE, DF intersect at P. Connect AP. Show that $AP = AB$.

7. Let G be the centroid of ΔABC. Show that if $BG \perp CG$, then $AB^2 + AC^2 = 5BC^2$.

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8. Given a triangle ΔABC, a line ℓ_1 // BC intersects AB, AC at D, D' respectively, a line ℓ_2 // AC intersects BC, AB at E, E' respectively and a line ℓ_3 // AB intersects AC, BC at F, F' respectively. Show that $[\Delta DEF][\Delta D'E'F']$.

9. Let ΔABC be an equilateral triangle and D is a point on BC. The perpendicular bisector of AD intersects AB, AC at E, F respectively. Show that $BD \cdot CD = BE \cdot CF$.

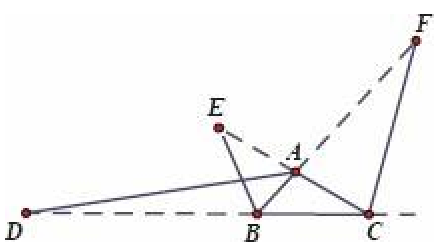
10. Given an acute angled triangle ΔABC where H is the orthocenter, show that $\frac{BC}{AH} = \tan \angle A$.

11. In a right angled triangle ΔABC where $\angle A$ = 90°, D, E are on BC such that $BD = DE = CE$. Show that $AD^2 + AE^2 = \frac{5}{9} BC^2$.

12. In ΔABC, M is the midpoint of AB and D is a point on AC. Draw CE // AB, intersecting BD extended at E. Show that lines AE, BC, MD are concurrent.

13. Given ΔABC, draw squares $ABDE$, $BCFG$ and $CAHI$ outwards based on AB, BC, AC respectively. Let P, Q, R be the midpoints of DE, FG, HI respectively. Show that AQ, BR, CP are concurrent.

14. Refer to the diagram on the below. ΔABC is a non-isosceles triangle. AD, BE, CF are the exterior angle bisectors of $\angle A$, $\angle B$, $\angle C$ respectively, intersecting the lines BC, AC, AB at D, E, F respectively. Show that D, E, F are collinear.



15. Given an isosceles triangle ΔABC where $AB = AC$, M is the midpoint of BC. A line ℓ passing through M intersects AB at D and intersects AC extended at E. Show that $\frac{1}{AD} + \frac{1}{AE} = \frac{2}{AB}$.

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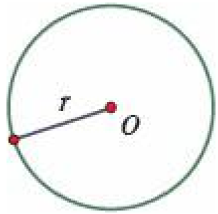
Chapter 3

Circles and Angles

A circle is uniquely determined by its center and radius, i.e., if two circles have the same center and radius, they must coincide. We use $\odot O$ to denote a circle centered at O.

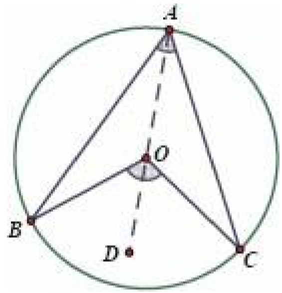
It is widely known that given a circle with radius r, its perimeter equals $2\pi r$ and the area of the disc is πr^2. Indeed, there are many more interesting properties about circles. In this chapter, we will focus on the properties of angles related to circles.



3.1 Angles inside a Circle

Theorem 3.1.1 *An angle at the center of a circle is twice of the angle at the circumference.*

Proof. Refer to the diagram below. We are to show $\angle BOC = 2\angle BAC$. Extend AO to D. Since O is the center of the circle, we have $AO = BO$. Now $\angle B = \angle OAB$ in ΔAOB, and the exterior angle $\angle BOD = \angle B + \angle OAB = 2\angle OAB$.



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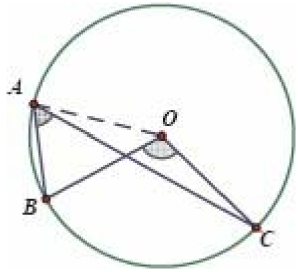
Similarly, $\angle COD = \angle C + \angle OAC = 2\angle OAC$.
Now $\angle BOC = \angle BOD + \angle COD = 2\angle OAB + 2\angle OAC = 2\angle BAC$.

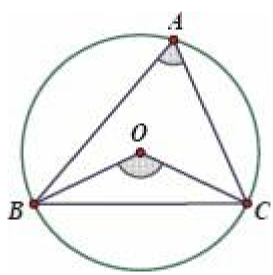
Notice that the proof is **not** completed yet: there is another possible situation as illustrated in the diagram on the right. Notice that the proof above does not apply in this situation, but an amended version following the same idea (using subtraction instead of addition) leads to the conclusion. We leave it to the reader.



Example 3.1.2 Let O be the circumcenter of ΔABC. We have:
(1) $\angle BOC = 2\angle A$
(2) $\angle OBC = 90° - \angle A$

Proof. (1) follows directly from Theorem 3.1.1.

(2) is because $\angle OBC = \frac{1}{2}(180° - \angle BOC)$



$= \frac{1}{2}(180° - 2\angle A) = 90° - \angle A$.

□

Theorem 3.1.1 has a few immediate corollaries which are very important in circle geometry.

Corollary 3.1.3 *Angles in the same arc are the same.*

Refer to the left diagram below. $\angle 1 = \angle 2$ because they are both equal to

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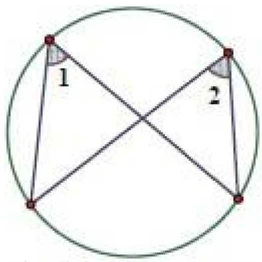
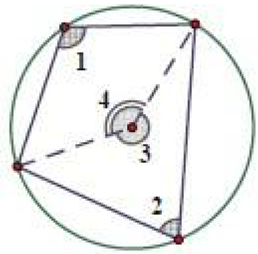
half of the angle at the center of the circle.



Angles in the same arc



Opposite angles in a cyclic quadrilateral

We call a quadrilateral **cyclic** if it is inscribed inside a circle.

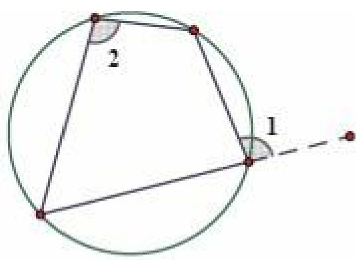
Corollary 3.1.4 *Opposite angles of a cyclic quadrilateral are supplementary, i.e., their sum is* 180°.

Refer to the previous right diagram. We have $\angle 1 + \angle 2$

$$= \frac{1}{2}\angle 3 + \frac{1}{2}\angle 4 = \frac{1}{2}\cdot 360^\circ = 180^\circ.$$

Notice that $\angle 3$ in the diagram is greater than 180°, but one can easily show that Theorem 3.1.1 still applies.

Corollary 3.1.5 *An exterior angle of a cyclic quadrilateral is equal to the corresponding opposite angle*.



Exterior angle of a cyclic quadrilateral

Refer to the diagram below where $\angle 1 = \angle 2$. This is immediately from Corollary 3.1.4.

In Section 2.5, we studied the relationship between points and lines, i.e., collinearity and concurrence. Similarly, we will study the relationship between points and circles in this chapter. First, one sees that any three non-collinear points uniquely determine a circle: for points A, B, C not collinear, there exists a unique circle passing through A, B, C. This is simply the circumcircle of ΔABC.

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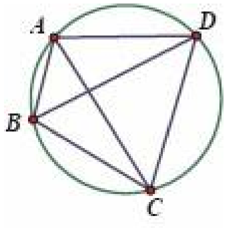
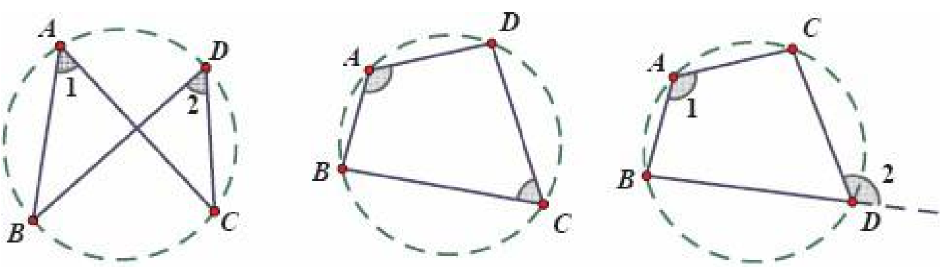
In general, four points do not lie on the same circle. Hence, it is noteworthy if the contrary happens, in which case we say the four points are concyclic. Refer to the diagram below for an example.



Concyclic

Showing concyclicity seems harder than collinearity or concurrence. For example, one may prove collinearity by showing the neighboring angles are supplementary, or prove concurrence by showing the intersection of two lines lies on the third. Are there any similar and *straightforward* techniques applicable to show concyclicity?

We have to accept that circles are not as *straight* as lines. Nevertheless, circle geometry has a rich structure which provides us abundant methods in showing concyclicity. For example, one sees that the inverse statements of Corollaries 3.1.3 to 3.1.5 also hold, which can be shown easily by contradiction. Now we have simple and effective criteria to determine concyclicity. Refer to the diagrams below. In any of these cases, *A*, *B*, *C*, *D* are concyclic.



$\angle 1 = \angle 2$ $\quad$ $\angle A + \angle C = 180°$ $\quad$ $\angle 1 = \angle 2$

Example 3.1.6 In an acute triangle ΔABC, *AD*, *BE*, *CF* are heights. Show that the line *AD* is the angle bisector of $\angle EDF$.

Proof. Refer to the diagram below. Since $\angle BFH = \angle BDH = 90°$, *B*, *D*, *H*, *F* are concyclic by the inverse of Corollary 3.1.4. Hence, $\angle 1 = \angle 3$. Similarly, *C*, *D*, *H*, *E* are concyclic and we have $\angle 2 = \angle 4$.
Since $\angle BFC = \angle BEC = 90°$, *B*, *C*, *E*, *F* are concyclic by the inverse of Corollary

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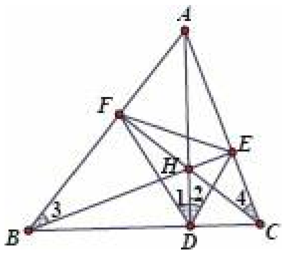
3.1.3. It follows that ∠1=∠3 = ∠4 = ∠2.

□



Note:

(1) Since $AD \perp BC$, ∠1=∠2 also implies ∠*BDF* = ∠*CDE*. Since *A*, *C*, *D*, *F* are concyclic, we also have ∠*BDF* = ∠*CDE* = ∠*BAC* (Corollary 3.1.5).

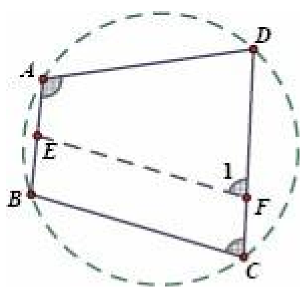
(2) One sees that a lot of concyclicity appear in this diagram. In fact, experienced contestants know this diagram very well and are able to recall those basic facts almost instantaneously.

(3) The conclusion implies that *H*, the orthocenter of Δ*ABC*, is the incenter of Δ*DEF*.

Example 3.1.7 Let *ABCD* be a cyclic quadrilateral. A line ℓ parallel to *BC* intersects *AB*, *CD* at *E*, *F* respectively. Show that *A*, *D*, *F*, *E* are concyclic.

Proof. Refer to the diagram below. Since *EF* // *BC*, ∠1=∠*C*. Notice that ∠*A* +∠*C* = 180° by Corollary 3.1.4. Hence, ∠*A*+∠1 = 180°, which implies *A*, *D*, *F*, *E* are concyclic.

□



Example 3.1.8 ⊙O_1 and ⊙O_2 intersect at *P* and *Q*. If O_1P extended intersects ⊙O_2 at *B* and O_2P extended intersects ⊙O_1 at *A*, show that O_1,O_2, *A*, *B*, *Q* are concyclic.

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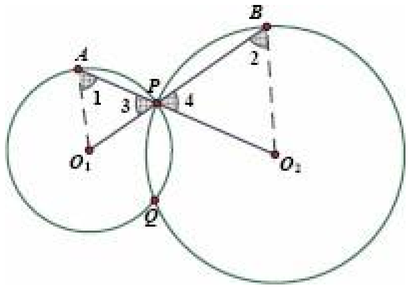
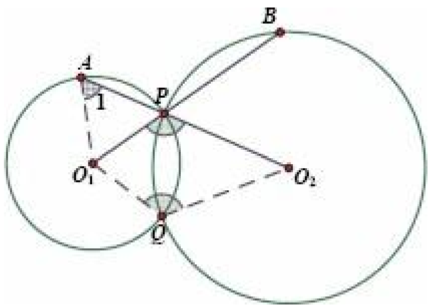


Insight. We are to show five points are concyclic. So many of them! Perhaps we can show four points are concyclic first, say O_1, O_2, A, B. Refer to the diagram below.

The simplest method is to show that $\angle 1=\angle 2$. Are there any equal angles in the diagram? Yes, say $\angle 1=\angle 3$ (because $O_1A = O_1P$) and similarly $\angle 2 = \angle 4$. We also have opposing angles $\angle 3 = \angle 4$. Job done!

Next, we may show that O_1, O_2, A, Q are concyclic. Let us draw the quadrilateral. Refer to the diagram below. Can we show $\angle 1+\angle O_1QO_2$ = 180° ? This seems not difficult.



Observe that $\angle O_1QO_2 = \angle O_1PO_2$ ($\Delta O_1PO_2 \cong \Delta O_1QO_2$), $\angle 1=\angle APO_1$ and $\angle APO_1 + \angle O_1PO_2 = 180°$. Job done!

In conclusion, both O_1,O_2, A, B and O_1, O_2, A, Q are concyclic, which means that B and Q lie on the circumcircle of ΔO_1AO_2. Indeed, O_1,O_2, A, B, Q are concyclic.

Note: One may show that O_1,O_2, A, Q are concyclic and hence, O_1,O_2, B, Q are concyclic by similar reasoning. This would also complete the proof.

Example 3.1.9 Refer to the diagram below. A, B, C are points on the circle. PC extended intersects the circle at D. Q is a point on CD such that $\angle DAQ = \angle PBC$. Show that $\angle DBQ = \angle PAC$.

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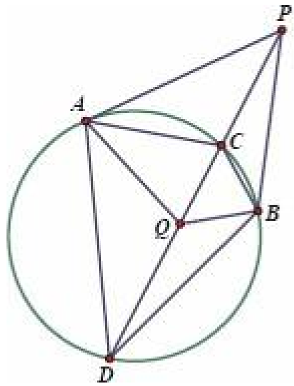
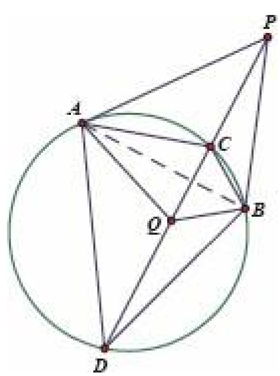


Insight. We are given a circle and a pair of equal angles. Could we find more pairs of equal angles? How are they related to our conclusion $\angle DBQ = \angle PAC$?

One may see the difficulty as $\angle PAC$ (and $\angle PBC$) are **not** extended by an arc. Perhaps we should relate $\angle PBC$ to another angle on the circumference besides $\angle DAQ$ and seek clues. How about $\angle PBC = \angle BCD - \angle BPD$? We may connect AB. Now $\angle BCD = \angle BAD$ is also related to $\angle DAQ$!

Proof. Refer to the diagram below. We have $\angle PBC = \angle BCD - \angle BPC$. Connect AB. Notice that $\angle BCD = \angle BAD$ (angles in the same arc). It is given that $\angle DAQ = \angle PBC$. Hence, $\angle DAQ = \angle BAD - \angle BPC$, or $\angle BPC = \angle BAD - \angle DAQ = \angle BAQ$. This implies P, A, Q, B are concyclic. Now $\angle DBQ = \angle PQB - \angle CDB = \angle PAB - \angle CAB = \angle PAC$.

□



Example 3.1.10 Given an equilateral ΔABC and its circumcircle, M is a point on the minor arc $\overset{\frown}{BC}$. Show that $MA = MB + MC$.

Insight. We are to show $MA = MB + MC$. Hence, it is a common technique to "cut" MB from MA and see whether the remaining portion equals to MC, i.e., we choose D on MA such that $MB = MD$ and attempt to show $MC = AD$.

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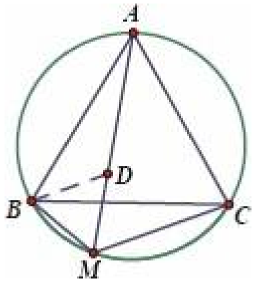
Refer to the diagram below. Notice that there are many equal sides and angles due to the equilateral triangle and the circle. Can you find congruent triangles?

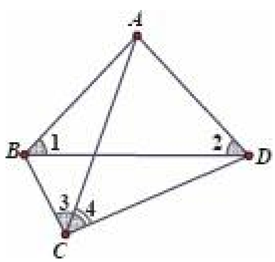


Proof. Choose D on MA such that $MB = MD$. It suffices to show that $AD = MC$. Notice that $\angle AMB = \angle ACB = 60°$ (angles in the same arc). Hence, ΔMBD is an isosceles triangle with the vertex angle 60°, i.e., an equilateral triangle. Now $BD = BM$ and $\angle DBM = 60°$, which implies $\angle CBM = 60° - \angle CBD = \angle DBA$. It follows that $\Delta CBM \cong \Delta ABD$ (S.A.S.). Hence, $AD = MC$ and the conclusion follows.

□

Example 3.1.11 In a quadrilateral $ABCD$, $AB = AD$ and $BC \neq CD$. If CA bisects $\angle BCD$, then A, B, C, D are concyclic.

Insight. Refer to the diagram below. If A, B, C, D are concyclic, we have $\angle 1 = \angle 4 = \angle 3 = \angle 2$. It seems exactly right! Perhaps we can show the conclusion by contradiction: what if A, B, C, D are not concyclic?



Proof. Suppose otherwise that A, B, C, D are not concyclic. Let the circumcircle of ΔABD intersect the line AC at P. Refer to the diagram below. Notice that $\angle 1 = \angle APD$ and $\angle 2 = \angle APB$ (angles in the same arc).

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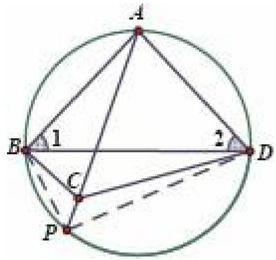


Since $AB = AD$, $\angle 1 = \angle 2$ and hence, $\angle APB = \angle APD$. We are also given that $\angle ACB = \angle ACD$. Hence, AP is the perpendicular bisector of BD (Example 1.2.10). This is impossible because $BC \neq CD$.

□

Note:

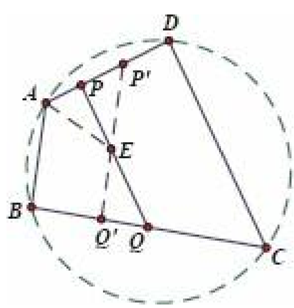
(1) This proof does not depend on the diagram, i.e., it still holds if C is outside the circle.

(2) One may also show $BC = CD$ by $\Delta PBC \cong \Delta PDC$ (A.A.S.).

Example 3.1.12 Let $ABCD$ be a cyclic quadrilateral where the angle bisectors of $\angle A$ and $\angle B$ intersect at E. Draw a line passing through E parallel to CD, intersecting AD, BC at P, Q respectively. Show that $PQ = PA + QB$.

Insight. Given angle bisectors and parallel lines, can we have isosceles triangles? Not exactly in this case because $PQ \,//\, CD$: if $PQ \,//\, AB$, we will obtain isosceles triangles. Hence, we may draw $P'Q' \,//\, AB$, intersecting AD, BC at P',Q' respectively. Refer to the diagram below.

Since AE bisects $\angle A$, we have $\angle P'AE = \angle BAE = \angle P'EA$, which implies $P'A = P'E$. Similarly, $Q'B = Q'E$. We have $P'Q' = P'A + Q'B$. How are PQ and $P'Q'$ related? If we randomly draw a line PQ passing through E, we shall **not** have $PQ = PA + QB$. Notice that we have not used the conditions $PQ \,//\, CD$ and A, B, C, D concyclic!



Proof. Draw $P'Q' \,//\, AB$, intersecting AD, BC at P',Q' respectively. Since $\angle P'AE = \angle BAE = \angle P'EA$, we have $P'A = P'E$ and similarly, $Q'B = Q'E$. Hence,

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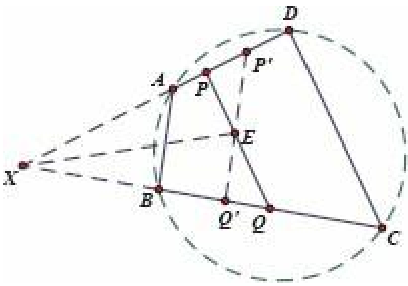
$P'Q' = P'A + Q'B$. (1)
Since $P'Q'$ // AB, PQ // CD and A, B, C, D are concyclic, we have $\angle PP'Q = 180° - \angle A = \angle C = \angle PQQ'$. Similarly, $\angle P'PQ = \angle P'Q'Q$.

Let the lines AD and BC intersect at X. Refer to the diagram below. Observe that E is the ex-center of ΔXAB opposite X.



Hence, XE bisects $\angle AXB$. One easily sees that $\Delta XP'E \cong \Delta XQE$ (A.A.S.) and $\Delta XPE \cong \Delta XQ'E$ (A.A.S.). It follows that $P'E = QE$, $PE = Q'E$ and $PP' = QQ'$. Now $PQ = PE + QE = P'E + Q'E = P'Q'$ (2) and $P'A + Q'B = PA + Q'B + PP' = PA + Q'B + QQ' = PA + QB$. (3) (1), (2) and (3) imply that $PQ = PA + QB$.

Note that the proof still holds if the lines AD and BC intersect at the other side of PQ, in which case E is the incenter of ΔXAB instead of the ex-center, and we still have XE bisects $\angle X$.

□

Note:
(1) Once it is shown that the corresponding angles in $\Delta PP'E$ and $\Delta Q'QE$ are the same, we should probably have $\Delta PP'E \cong \Delta Q'QE$ (which leads to the conclusion immediately). Hence, it is natural to consider the intersection of the lines AD and BC, which gives congruent triangles with common sides.
(2) Another strategy to solve the problem is via "cut and paste": since we a to show $PQ = PA + QB$, we choose F on PQ such that $BQ = FQ$ and we attempt to show $AP = FP$. Refer to the diagram below. Since PQ // CD, we have A, B, Q, P concyclic (Example 3.1.7).

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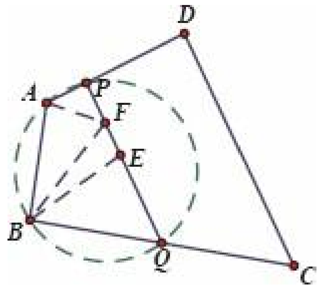
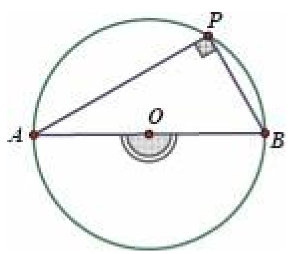


Since $BQ = FQ$, we have $\angle BFE = \angle FBQ = \frac{1}{2}\angle CQP = \frac{1}{2}\angle BAD = \angle BAE$, i.e., A, B, E, F are concyclic. We are to show that $\angle PAF = \angle PFA = \frac{1}{2}\angle DPQ$, while $\frac{1}{2}\angle DPQ = \frac{1}{2}\angle ABQ = \angle ABE$. Since A, B, E, F are concyclic, we must have $\angle PFA = \angle ABE$ (Corollary 3.1.5). This completes the proof.

One should also take note of another immediate corollary from Theorem 3.1.1 that the diameter of the circle always extends a right angle on the circumference. This is a common method in identifying right angles.

Corollary 3.1.13 *If AB is the diameter of* $\odot O$ *and P is a point on the circle, then* $\angle APB = 90°$.

***Proof*.** Refer to the diagram below. Notice that $\angle AOB = 180°$.



By Theorem 3.1.1, $\angle APB = \frac{1}{2}\angle AOB = 90°$

□

Note: The inverse of this corollary also holds, i.e., if a chord AB extends an angle of 90° on the circumference, then AB is the diameter (which passes through the center of the circle).

Example 3.1.14 Refer to the diagram below. Given a circle where AB is a diameter, C, D, E are on the circle such that C, E are on the same side of AB

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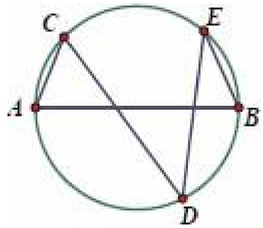
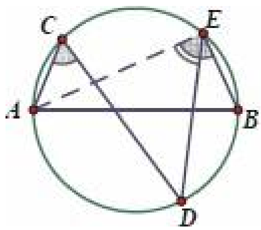
while D is on the other side. Show that $\angle C + \angle E = 90°$.

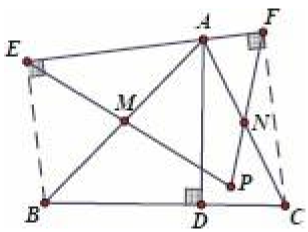


Proof. Refer to the diagram below. Connect AE. Since AB is a diameter, we have $\angle E = 90° - \angle AED$. Notice that $\angle AED = \angle C$ (angles in the same arc) and the conclusion follows.



Example 3.1.15 Given an acute angled ΔABC where $AD \perp BC$ at D, M, N are the midpoints of AB, AC respectively. Let ℓ be a line passing through A. Draw $BE \perp \ell$ at E and $CF \perp \ell$ at F. If the lines EM, FN intersect at P, show that D, E, F, P are concyclic.

Insight. Refer to the diagram below. E We could probably show the concyclicity by equal angles. Can you see A, D, B, E (and similarly A, D, C, F) are concyclic?



What do we know about P? P is obtained by intersecting EM and FN. Notice that EM, FN are medians on the hypotenuses of right angled triangles. This gives us more equal angles!

Proof. Since $\angle AEB = \angle ADB = 90°$, A, D, B, E are concyclic and in particular, M is the center of the circle. Clearly, $\angle AEM = \angle EAM$. Similarly, $\angle AFN =$

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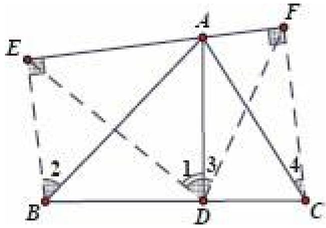
$\angle FAN$. Now $\angle P = 180° - (\angle AEM + \angle AFN) = 180° - (\angle EAM + \angle FAN) = \angle BAC$. On the other hand, we have $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (angles in the same arc). Refer to the diagram below.

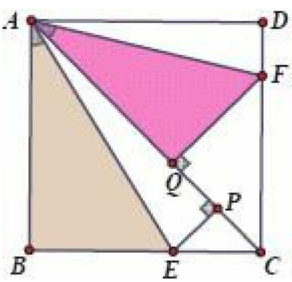


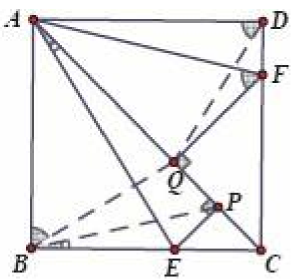
It follows that $\angle EDF = \angle 1 + \angle 3 = \angle 2 + \angle 4 = \angle BAC$, since $BCFE$ is a trapezium (Example 1.4.15). Now $\angle P = \angle EDF$, which implies D, E, F, P are concyclic.

□

Example 3.1.16 Let $ABCD$ be a square. E, F are points on BC, CD respectively and $\angle EAF = 45°$. Draw $EP \perp AC$ at P and $FQ \perp AC$ at Q (P, Q do not coincide). Show that the circumcenter of ΔBPQ lies on BC.
EAF ? One may recall

Insight. How shall we use the condition $\angle EAF = 45°$? One may recall Exercise 1.6. However, rotating ΔABE seems not useful this time.





Notice that $\angle BAE = 45° - \angle CAE = \angle CAF$. Refer to the left diagram above. It follows that $\Delta ABE \sim \Delta AQF$ and $\angle AEB = \angle AFQ$. In fact, one may find other pairs of equal angles due to symmetry. Refer to the right diagram above. We have $\angle ABQ = \angle ADQ = \angle AFQ$ (since A, D, F, Q are concyclic where $\angle ADF = \angle AQF = 90°$). Similarly, $\angle PAE = \angle PBE$ because A, B, P, E are concyclic.

Now we have $\angle ABQ = \angle AFQ = \angle AEB = 90° - \angle BAE$, which implies $\angle ABQ + \angle BAE = 90°$, i.e., $BQ \perp AE$.
We are to show the circumcenter of ΔBPQ lies on BC.
Let us draw the circumcircle. Refer to the diagram below. Let the

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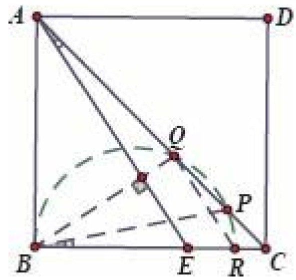
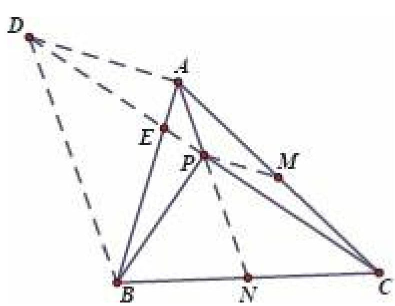
circumcircle of ΔBPQ intersect BC at R. Now it suffices to show that BR is a diameter, i.e., $\angle BQR = 90°$.
Note that this is equivalent to showing QR // AE. We have already shown $\angle CAE = \angle CBP$. Since $\angle CBP = \angle PQR$ (angles in the same arc), we have $\angle CAE = \angle PQR$ and AE // QR. This completes the proof.



Occasionally, one may need to apply circle properties to solve a problem, even though no circle is given explicitly.

Example 3.1.17 Let P be a point inside ΔABC such that $\angle BPC = 90°$ and $\angle BAP = \angle BCP$. Let M, N be the midpoints of AC, BC respectively. Show that if $BP = 2PM$, then A, P, N are collinear.

Insight. We are given a few conditions about the point P. However, neither $\angle BAP = \angle BCP$ nor $BP = 2PM$ seems helpful in determining the position of P. On the other hand, M, N are midpoints. If we can find a triangle where PM is a midline, the Midpoint Theorem will give a line segment equal to $2PM$!
Refer to the diagram above. If we extend CP to D such that $CP = DP$, then $AD = 2PM = BP$. Since A, P, N **should** be collinear, $ADBP$ **should** be an isosceles trapezium, i.e., A, D, B, P **should** be concyclic and we **should** have $\angle BAP = \angle BDP$. Now the condition $\angle BAP = \angle BCP$ seems useful and we may complete the proof by showing that ΔBCD is isosceles.



Proof. Extend CP to D such that $CP = DP$. Let CD intersect AB at E. Since M,

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N are the midpoints of *AC*, *BC* respectively, by the Midpoint Theorem, we have *AD* = 2*PM* = *BP* and *PN* // *BD*. (*)

Since ∠*BPC*= 90°, we have Δ*BCP* ≅ Δ*BDP* (S.A.S.). It follows that ∠*BDP* = ∠*BCP* = ∠*BAP* and hence, *A*, *D*, *B*, *P* are concyclic. Since *AD* = *BP*, one sees that Δ*ADE* ≅ Δ*PBE* (A.A.S.) and hence, *ADBP* is an isosceles trapezium where *BD* // *AP*. By (*), *A*, *P*, *N* are collinear.

□

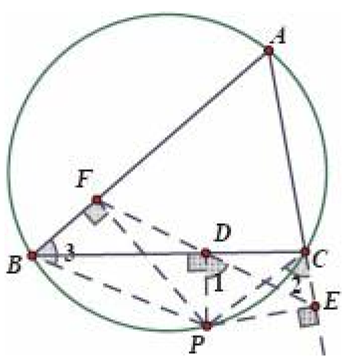
Note: An experienced contestant may write down an elegant proof starting with "Let the circumcircle of Δ*ABP* intersect *CP* extended at *D*. …" Of course, beginners may feel puzzled because the motivation of constructing the circumcircle of Δ*ABP* is not clear. Nevertheless, by showing *BC* = *BD* and *ADBP* is an isosceles trapezium, one sees that this is equivalent to the given proof.

As shown in the examples above, Corollary 3.1.3 to Corollary 3.1.5, including their inverse, are useful in showing equal angles and concyclicity. One may also use these simple results to show the following theorem.

Theorem 3.1.18 (Simson's Line) *Let P be a point on the circumcircle of ΔABC. Let D, E, F be the feet of the perpendiculars from P to the lines BC, AC, AB respectively. We have D, E, F collinear, called the Simson's line of ΔABC with respect to P.*

Proof. Refer to the diagram below. Notice that *P*, *D*, *C*, *E* are concyclic because ∠*PDC* = ∠*PEC* = 90°. Hence, we have ∠1=∠2 (Corollary 3.1.3). Notice that ∠2 = ∠3 (Corollary 3.1.5). Now ∠1=∠3 = 180°−∠*PDF* (Corollary 3.1.4.).

This implies ∠1+∠*PDF* = 180°, or *D*, *E*, *F* are collinear.



Note:

(1) The inverse of this theorem also holds, i.e., if *P* is a point such that the feet of its perpendicular to the sides of Δ*ABC* are collinear, then *P* lies

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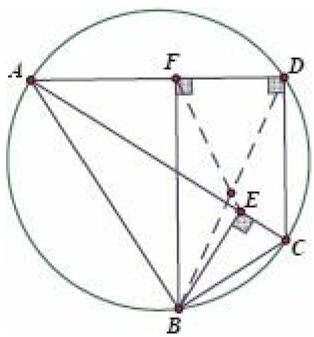
on the circumcircle of ΔABC. This can be shown by reversing the reasoning: if D, E, F are collinear, we have $\angle 1 + \angle PDF = 180°$. Hence, $\angle 3 = 180° - \angle PDF = \angle 1 = \angle 2$, which implies A, B, C, P are concyclic.

(2) Naturally, beginners may find it difficult to recognize pairs of equal angles, especially when the diagram is complicated. Such angle-chasing skills can only be enhanced via practice. For example, can you see $\angle 1 = \angle 2 = \angle 3$ from the diagram without referring to the proof? (**Hint**: One may occasionally *erase* extra lines and simplify the diagram.)

Example 3.1.19 A quadrilateral $ABCD$ is inscribed inside a circle and $AD \perp CD$. Draw $BE \perp AC$ at E and $BF \perp AD$ at F. Show that the line EF passes through the midpoint of the line segment BD.

Insight. From the first glance, it is not clear how EF is related to the midpoint of BD. Refer to the diagram below. What do we know about the midpoint of BD? One may easily see that BD is the hypotenuse of the right angled triangle ΔBDF. In fact, the only clues we have are the given right angles!

Can we show $\angle EFD = \angle BDF$? This may not be easy because $\angle EFD$ is neither an angle on the circumference nor closely related to other angles.



Perhaps the other right angles can help us. Since $\angle BFD = \angle CDF = 90°$, we see that BD is *almost* the diagonal of a rectangle, except that $BCDF$ is not a rectangle yet while one of the corners is cut. What if we fix it? Refer to the diagram below. We draw $BP \perp CD$ at P. If EF indeed passes through the midpoint of BD, EF should be part of the other diagonal of the rectangle $BPDF$. Indeed, that diagonal is PF and what we need to show is that P, E, F are collinear. Do you recognize a Simson's line?

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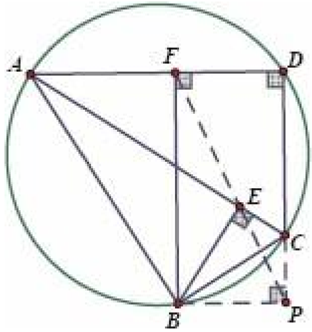


Proof. Draw $BP \perp CD$ at P. Since $AD \perp PD$ and $BF \perp AD$, we have AD // BP and BF // PD, i.e., $BPDF$ is a parallelogram (and a rectangle). Since P, E, F are the feet of the perpendiculars from B to the sides of ΔACD respectively, we must have P, E, F collinear (Simson's Line). Now the conclusion follows as the diagonals of a parallelogram bisect each other, i.e., EF passes through the midpoint of BD.

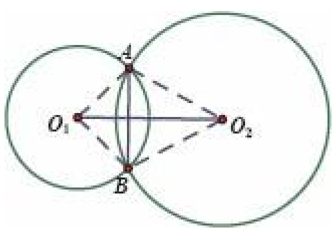
□

We mention the following elementary but very useful theorem as the end of this section. It is widely applicable when solving problems related to a few circles intersecting each other.

Theorem 3.1.20 *If $\odot O_1$ and $\odot O_2$ intersect at A, B, then O_1O_2 is the perpendicular bisector of AB.*

Proof. Refer to the diagram below. Notice that $\Delta O_1AO_2 \cong \Delta O_1BO_2$ (S.S.S.).

□



3.2 Tangent of a Circle

Definition 3.2.1 A line AB is tangent to (or touches) a circle $\odot O$ at A if $\angle OAB = 90°$ case, A is called the point of tangency.

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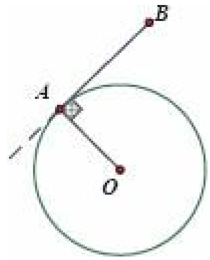
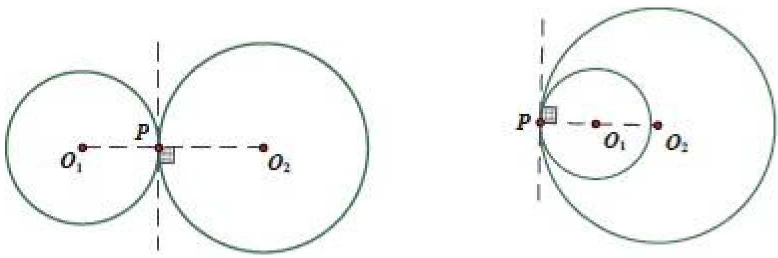


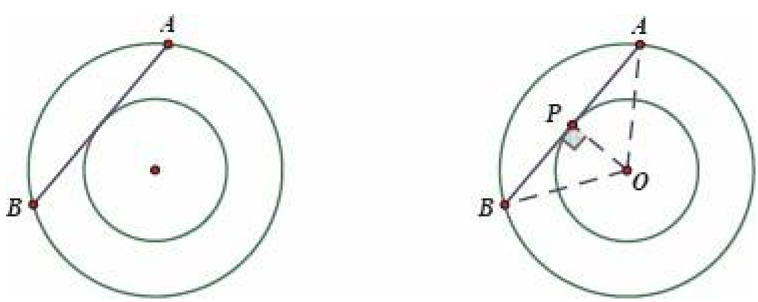
It is easy to see that a tangent line cannot intersect the circle more than once. Otherwise, we will have a triangle with two right angles!

Notice that $\odot O_1$ and $\odot O_2$ are tangent to each other (i.e., touch exactly once) at P if and only if P introduces a common tangent to both circles. Refer to the diagrams below.



Notice that O_1O_2 is perpendicular to the common tangent in either case. One may consider this as an extreme case of Theorem 3.1.20.

Example 3.2.2 Refer to the left diagram below. The area of the ring between two concentric circles is 16π cm^2. AB is a chord of the larger circle and is tangent to the smaller circle. Find AB.



Ans. Refer to the right diagram above. Let the center of the circles be O and the point of tangency be P. Since $OA = OB$ and $OP \perp AB$, one sees that $\Delta OAP \cong \Delta OBP$ (H.L.). Hence, $AB = 2AP$.

The area of the ring is the difference between the areas of two discs, i.e.,

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$\pi \cdot OA^2 - \pi \cdot OP^2 = 16\pi$. Hence, $16 = OA^2 - OP^2 = AP^2$ by Pythagoras' Theorem. It follows that $AP = 4$ cm and $AB = 8$ cm.

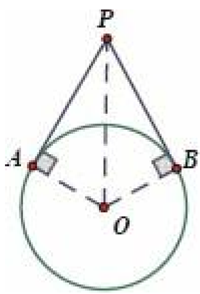
□

Note: If AB is a chord in ⊙O and M is the midpoint of AB, we always have $OM \perp AB$ because ΔOAB is an isosceles triangle.

Theorem 3.2.3 *Let P be a point outside a circle and PA, PB are tangent to the circle at A, B respectively. We have PA = PB (called equal tangent segments).*

Proof. Refer to the diagram below. Connect OA, OB, OP. Since $OA = OB$, one observes that $\Delta PAO \cong \Delta PBO$ (H.L.). The conclusion follows.

□



Note: An immediate corollary from the proof above is that $OP \perp AB$. In fact, OP is the perpendicular bisector of AB (Theorem 1.2.4).

We say a circle is *inscribed* inside a polygon if it touches (i.e., is tangent to) every side of the polygon. For example, every triangle has an inscribed circle, called the *incircle* of the triangle, centered at the incenter of the triangle (where angle bisectors meet). Refer to the proof of Theorem 1.3.2.

Example 3.2.4 $ABCD$ is a quadrilateral with an inscribed circle. Show that $AB + CD = AD + BC$.

Proof. Refer to the diagram below. Let E, F, G, H be the points of tangency. Note that $AE = AH$ (equal tangent segments). Similarly, $BE = BF$, $CF = CG$, $DG = DH$. Now $AB + CD = AE + BE + CG + DG = AH + BF + CF + DH = BC + AD$.

□

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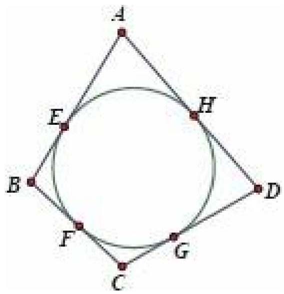
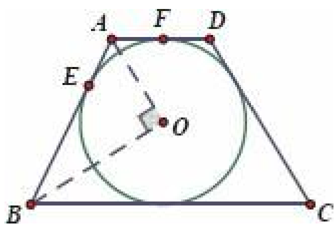


Note: This is called Pitot's Theorem. However, as the result is simple and well-known, the name of the theorem is seldom mentioned.

Example 3.2.5 *ABCD* is a trapezium with $AD \,//\, BC$ and $\odot O$ is inscribed inside *ABCD*. Show that $AO \perp BO$.

Proof. Refer to the diagram below. Let $\odot O$ touch *AB*, *AD* at *E*, *F* respectively. It is easy to see that $\Delta AOE \cong \Delta AOF$ (H.L.) and hence, *AO* bisects $\angle BAD$. Similarly, *BO* bisects $\angle ABC$.



Since $AD \,//\, BC$, $\angle BAD + \angle ABC = 180°$. It follows that

$$\angle BAO + \angle ABO = \frac{1}{2}\angle BAD + \frac{1}{2}\angle ABC = 90°\text{, i.e., } AO \perp BO.$$

□

Example 3.2.6 A circle is inscribed inside ΔABC and it touches the three sides *BC*, *AC*, *AB* at *D*, *E*, *F* respectively. Show that the lines *AD*, *BE*, *CF* are concurrent.

Insight. By Ceva's Theorem, we only need to show show $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = 1$.

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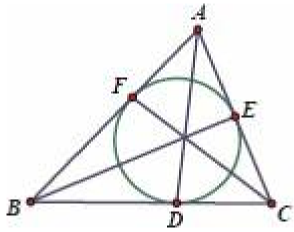
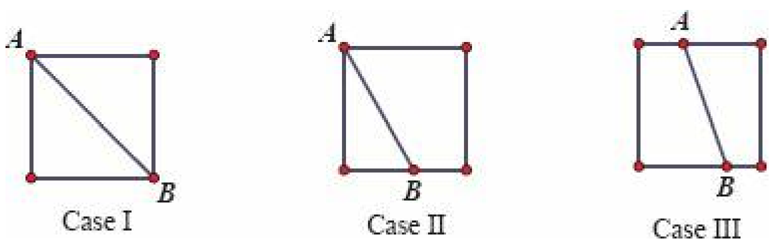


This is true because $AF = AE$, $CE = CD$ and $BD = BF$ (equal tangent segments).

Example 3.2.7 (IWYMIC 10) A straight line divides a square into two polygons, each of which has an inscribed circle. One of the circles has a radius of 6 cm while the other has an even longer radius. If the line intersects the square at A and B, find the difference, in cm, between the side length of the square and twice the length of the line segment .

Ans. There are a few cases when a line intersects a square.
Case I: Both A, B are vertices of the square.
One obtains two equal triangles and the radii of the inscribed circles must be the same. This contradicts the conditions given.



Case II: Only A is a vertex of the square.
One obtains a triangle and a quadrilateral. Notice that the quadrilateral cannot have an inscribed circle as the two pairs of opposite sides do not have equal sums (Example 3.2.4).

Case III: A, B lie on opposite sides of the square.
Similarly, the quadrilaterals obtained cannot have inscribed circles.

Case IV: A, B lie on neighboring sides of the square.
One obtains a triangle and a pentagon. Notice that the circle inscribed inside the pentagon is exactly the incircle of the square.
Refer to the right diagram below. We focus on the bottom right quarter of the square.

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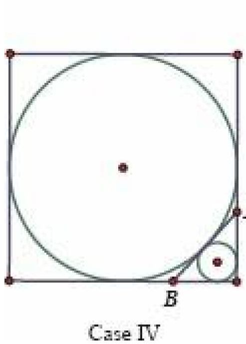
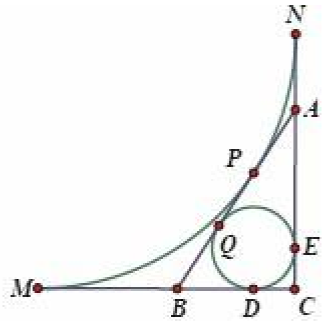


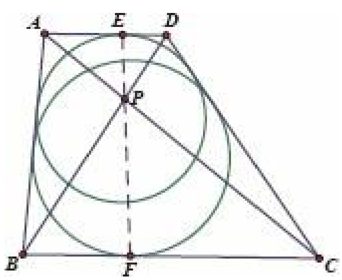


The square has a side length $2CM = CM + CN$.
Now $CM + CN - 2AB = CM + CN - (AP + BP) - AB$
$= CM + CN - (AN + BM) - AB = (CM - BM) + (CN - AN) - AB$
$= BC + AC - AB = BC + AC - (AQ + BQ)$
$= BC + AC - (AE + BD) = CD + CE = 12.$
Note that we applied equal tangent segments repeatedly.

□

Example 3.2.8 (CGMO 13) In a trapezium *ABCD*, *AD* // *BC*. Γ_1 is a circle inside the trapezium and is tangent to *AB*, *AD*, *CD*, touching *AD* at *E*. Γ_2 is a circle inside the trapezium and is tangent to *AB*, *BC*, *CD*, touching *BC* at *F*. Show that the lines *AC*, *BD*, *EF* are concurrent.

Insight. Refer to the diagram below. We know that Ceva's Theorem is useful in showing concurrency, but those three lines given are not inside a triangle. Perhaps we should use another method.



Notice that *ABCD* is an ordinary trapezium with no special properties. Hence, we shall show that *E*, *F*, *P* are collinear. Can we show that $\frac{AE}{CF} = \frac{DE}{BF}$? Notice that *AE*, *DE*, *BF*, *CF* are tangent segments of the circles and they could be expressed by the radii of the circles and the related angles.

Proof. Refer to the diagram below. Let Γ_1 be centered at O_1 with the

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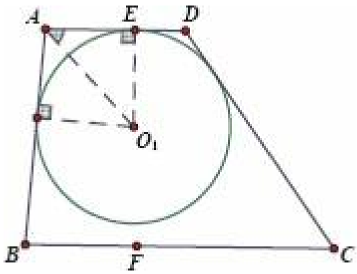
radius $O_1E = R_1$ Let $\angle BAD = 2\alpha$. We have $AE = R_1 \tan \angle O_1 AD = R_1 \tan \alpha$. Let $\angle CDA = 2\beta$. $DE = R_1 \tan \beta$.



Similarly, if Γ_2 has a radius R_2, we have

$$BF = R_2 \tan\left(\frac{1}{2}\angle ABC\right), \text{ where } \frac{1}{2}\angle ABC = \frac{1}{2}(180° - 2\alpha) = 90° - \alpha.$$

Hence, $BF = R_2 \tan(90°-\alpha)$. Similarly, $CF = R_2 \tan(90°-\beta)$.

Notice that $\tan \alpha \tan(90°-\alpha) = 1$ by definition. Hence, we have $AE \cdot BF = R_1R_2 \tan \alpha \tan(90°-\alpha) = R_1R_2$. Similarly, $DE \cdot CF = R_1R_2$. It follows that $\frac{AE}{CF} = \frac{DE}{BF}$, which implies AC, BD, EF are concurrent.

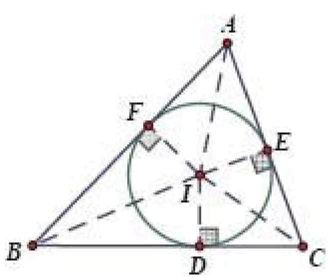
□

The following theorem describes the properties of the points of tangency and the radius of the incircle of a triangle.

Theorem 3.2.9 *Let I be the incenter of ΔABC where $AB = c$, $AC = b$ and $BC = a$. Let the incircle of ΔABC touch BC, AC, AB at D, E, F respectively. We have:*

(1) $BD = \frac{1}{2}(a - b + c)$

(2) $DI = \frac{2S}{a+b+c}$, *where* $S = [\Delta ABC]$.



Proof. Refer to the diagram below.

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(1) By equal tangent segments, $AE = AF = x$ say.
Similarly, let $BD = BF = y$ and $CD = CE = z$.
Notice that $a + b + c = 2(x + y + z)$ and $AE + CE = x + z = b$.

Hence, $y = \frac{1}{2}(a+b+c) - b = \frac{1}{2}(a-b+c)$.

(2) Let $DI = EI = FI = r$. Notice that DI, EI, FI are heights of ΔBCI, ΔACI and ΔAB respectively.

Hence, $S = [\Delta BCI] + [\Delta ACI] + [\Delta ABI] = \frac{1}{2}r \cdot a + \frac{1}{2}r \cdot b + \frac{1}{2}r \cdot c$

$= \frac{1}{2}r \cdot (a+b+c)$. It follows that $r = \frac{2S}{a+b+c}$.

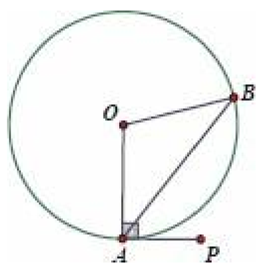
□

The following is another important circle property. It says the angle between the tangent and chord equals the angle in the alternate segment.

Theorem 3.2.10 *Let AP touch ⊙O at A. B is a point on the circle such that B, P are on the same side of the line OA. Then* $\angle BAP = \frac{1}{2}\angle AOB$.

Proof. Refer to the diagram below. Since AP is tangent to ⊙O, we have $OA \perp AP$. Now $\angle BAP = 90° - \angle OAB$. Since $OA = OB$, $\angle AOB = 180° - 2\angle OAB$. It follows that $\angle BAP = \frac{1}{2}\angle AOB$.

□



Note: By Theorem 3.1.1, we must have $\angle BAP = \angle ACB$ for any point C on the major arc $\overparen{AB}$. Refer to the diagram below. This is another commonly used result to show equal angles besides Corollaries 3.1.3 to 3.1.5.

It is easy to see that the inverse of this statement is also true, i.e., if $\angle BAP = \angle ACB$, then AP is tangent to the circle.

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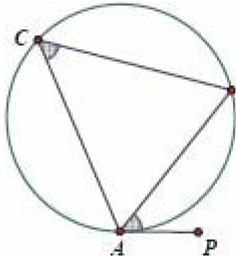
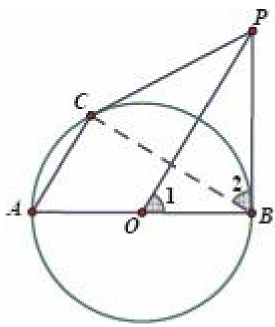


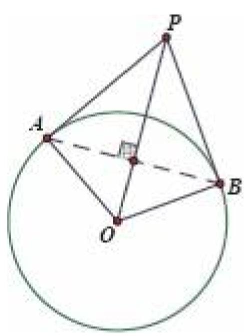
Example 3.2.11 Let AB be a diameter of $\odot O$. P is a point outside $\odot O$ such that PB, PC touch $\odot O$ at B and C respectively. Show that $AC // OP$.

***Proof*.** Refer to the diagram below. It suffices to show $\angle A = \angle 1$. Connect BC. Since PB is tangent to $\odot O$, we have $\angle A = \angle 2$ (Theorem 3.2.10). Since $AB \perp PB$ and $OP \perp BC$ (Theorem 3.2.3), we have $\angle 1 = 90° - \angle OPB = \angle 2$. It follows that $\angle A = \angle 1$.

□



Note: It is a common technique to connect AB if PA, PB are tangent to $\odot O$. Refer to the diagram below. By connecting OA, OB, one obtains right angled triangles with the heights on the hypotenuses. Moreover, we also see angles at the center of the circle, tangent lines and equal tangent segments, which, together with other conditions, may help us in finding equal angles.



Example 3.2.12 Refer to the diagram below. AB is a diameter of $\odot O$ and

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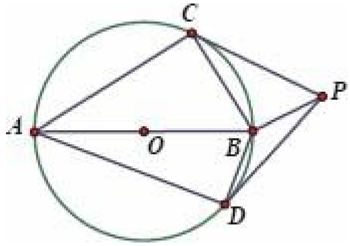
C, D are two points on the circle. P is a point outside the circle such that PC, PD touch $\odot O$ at C, D respectively.
Show that $\angle CPD = 180° - 2\angle CAD$.



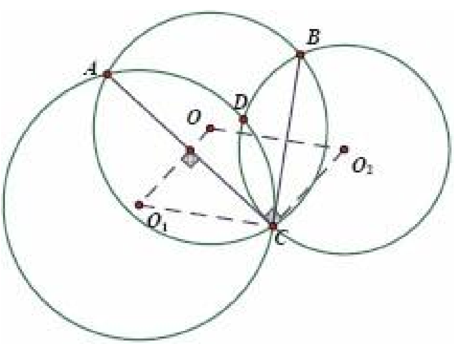
Proof. Since the sum of the interior angles of the quadrilateral $CODP$ is 360° and $\angle OCP = \angle ODP = 90°$, we have $\angle CPD = 180° - \angle COD$. The conclusion follows as $\angle COD = 2\angle CAD$ (Theorem 3.1.1).

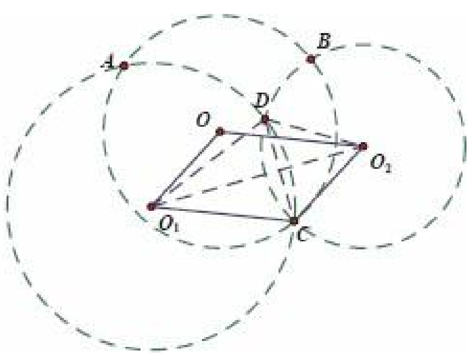
□

Note: One sees that the diameter AB is not useful. In particular, the point B complicates the diagram unnecessarily and should be deleted. One may also connect CD and see that $\angle PCD = \angle PDC = \angle CAD$ (Theorem 3.2.10), which also leads to the conclusion.

Example 3.2.13 Given $\odot O$ with radius R, A, B are two points on $\odot O$ and AB is NOT the diameter. C is a point on $\odot O$ distinct from A and B. $\odot O_1$ passes through A and is tangent to the line BC at C. $\odot O_2$ passes through B and is tangent to the line AC at C. If $\odot O_1$ and $\odot O_2$ intersect at C and D, show that $CD \le R$.

Insight. Refer to the left diagram below. It may not be easy to see the relationship between CD and R immediately. Notice that $OO_1 \perp AC$ and $OO_2 \perp BC$ (Theorem 3.1.20). Given that BC, AC are tangent to $\odot O_1$, $\odot O_2$ respectively, it is easy to see that OO_1CO_2 is a parallelogram!





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Let us focus on this parallelogram. Refer to the right diagram above. We are to show $CD \leq R = CO$. Can you see that CD is *vertical* and CO is *oblique* with respect to O_1O_2 ? Can you see that $\angle ODC = 90°$?

Proof. One sees that $OO_1 \perp AC$ and $O_2C \perp AC$. Hence, we have $OO_1 // O_2C$. Similarly, we have $OO_2 // O_1C$, which implies that OO_1CO_2 is a parallelogram.
It is easy to see that $\Delta O_1DO_2 \cong \Delta O_1CO_2 \cong \Delta O_1OO_2$, which implies OO_1O_2D is an isosceles trapezium. Hence, we have $OD // O_1O_2$, which implies $OD \perp CD$. It follows that $CD \leq CO = R$.

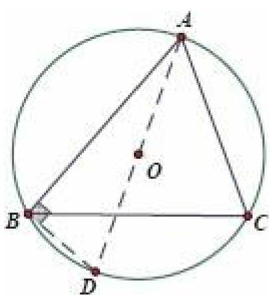
□

3.3 Sine Rule

Theorem 3.3.1 (Sine Rule) *In ΔABC, we have* $\frac{AB}{\sin\angle C} = \frac{BC}{\sin\angle A} = \frac{AC}{\sin\angle B} = 2R$, *where R is the circumradius of ΔABC.*

Proof. First, we show that $\frac{AB}{\sin\angle C} = 2R$.

Let O be the circumcenter of ΔABC. Refer to the diagram on the below. Let AD be a diameter of the circumcircle of ΔABC. Connect BD.



Clearly, $AD = 2R$ and we have $\angle ABD = 90°$.

By definition, $\frac{AB}{2R} = \sin\angle D$. Since $\angle C = \angle D$ (angles in the same arc), we have

we have $\frac{AB}{\sin\angle C} = 2R$. Similarly, $\frac{BC}{\sin\angle A} = 2R$ and $\frac{AC}{\sin\angle B} = 2R$.

□

Note: Sine Rule is taught in most secondary schools. However, the last equality, which links it to the circumradius (i.e., the radius of the circumcircle) of the triangle, is usually not included.

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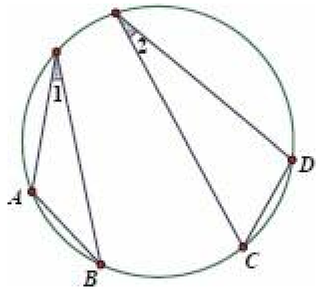
Corollary 3.3.2 *Let AB, CD be two chords in a circle. If AB, CD extend the same angle at the circumference, then AB = CD.*

Proof. Let the radius of the circle be R. Refer to the diagram below.



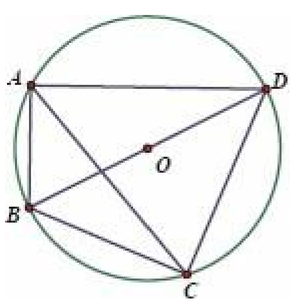
By Sine Rule, we have $\frac{AB}{\sin\angle 1} = 2R$, i.e., $AB = 2R \sin\angle 1$. Similarly, $CD = 2R \sin\angle 2$. The conclusion follows as $\angle 1 = \angle 2$.

□

Note:

(1) One sees that the corollary still holds if two chords extend the same angle at the center: Apply Theorem 3.1.1, or simply show that $\Delta AOB \cong \Delta COD$.

(2) The corollary still holds if we are given equal minor arcs $\overset{\frown}{AB} = \overset{\frown}{CD}$. Thi is because the arc length is proportional to the angle extended at the center (or on the circumference). Refer to the diagram below, which illustrates a variation of Corollary 3.3.2. *ABCD* is a quadrilateral inscribed in $\odot O$ where *BD* is a diameter. We have $AC = 2R \sin\angle D$. Notice that $2R = BD$. Hence, $AC = BD \sin\angle D = BD \sin\angle B$. This is a useful fact. One shall see this conclusion even if $\odot O$ is not shown explicitly, say if we are only given $AB \perp AD$ and $BC \perp CD$.



Corollary 3.3.3 *Given* ΔABC *and its circumcircle, show that the angle bisector of* $\angle A$ *passes through the midpoint of the minor arc* $\overset{\frown}{BC}$.

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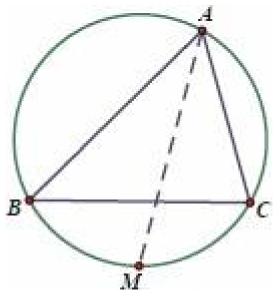
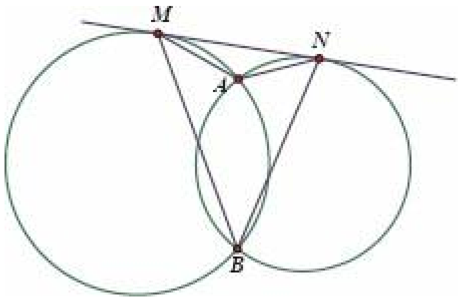
Proof. This follows immediately from the remarks above. Refer to the diagram below where AM bisects $\angle A$. One sees that $\overset{\frown}{BM} = \overset{\frown}{CM}$ because they extend equal angles on the circumference, i.e., $\angle BAM$ $\angle CAM$.



Example 3.3.4 Refer to the diagram below. Two circles intersect at A and B. A common tangent line touches the two circles at M, N respectively. Show that ΔMAN and ΔMBN have the same circumradius.



Insight. The two triangles have a common side MN. Sine Rule states that $\frac{MN}{\sin\angle MAN} = 2R$ where R is the circumradius of ΔAMN. Can you see that it suffices to show $\sin\angle MAN = \sin\angle MBN$?

Clearly, $\angle MAN \neq \angle MBN$ as one is acute and the other obtuse. How about $\angle MAN + \angle MBN = 180°$? Perhaps the tangent line would give us equal angles.

Proof. Refer to the diagram below. We have $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ (Theorem 3.2.10).
Since $\angle 1 + \angle 3 + \angle MAN = 180°$, we must have $\angle 2 + \angle 4 + \angle MAN = 180°$, i.e., $\angle MBN + \angle MAN = 180°$.

Hence, $\sin\angle MAN = \sin\angle MBN$. Let R_1, R_2 denote the circumradii of the two triangles. By Sine Rule, $\frac{MN}{\sin\angle MAN} = 2R_1$ and $\frac{MN}{\sin\angle MBN} = 2R_2$. It follows that $R_1 = R_2$.

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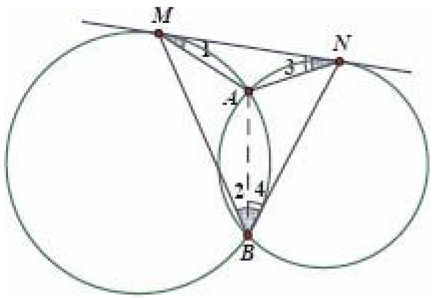
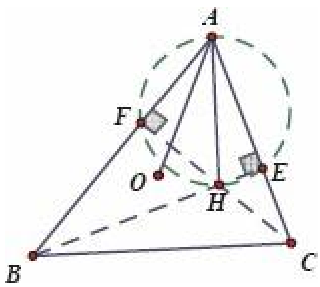
□



Example 3.3.5 Given an acute angled triangle ΔABC where $\angle A = 60°$, O and H are the circumcenter and orthocenter of ΔABC respectively. Show that $AO = AH$.



Insight. Refer to the diagram below. Of course, the most straightforward method is to show that $\angle AOH = \angle AHO$, but this is not easy because we do not know much about the line OH.

Let $BE \perp AC$ at E and $CF \perp AB$ at F. We know that A, E, H, F are concyclic. In particular, AH is the diameter of this circle (because $\angle AEH = 90°$, Corollary 3.1.13). Now it suffices to show that the radius of the circumcircle of ΔABC is twice of the radius of the circumcircle of ΔAEH. We may show this by Sine Rule. Notice that the right angled triangle with an internal angle of 60° gives sides of ratio 1:2.

Proof. Refer to the right diagram below where BE, CF are the heights in ΔABC. Since $\angle AEH = \angle AFH = \angle 90°$, A, E, H, F are concyclic. We denote R and r as the radii of the circumcircles of ΔABC and ΔAEH respectively.

By Sine Rule, $\dfrac{AB}{\sin\angle ACB} = 2R$ and $\dfrac{AE}{\sin\angle AFE} = 2r$.

Hence, $\dfrac{R}{r} = \dfrac{AB}{AE} \cdot \dfrac{\sin\angle AFE}{\sin\angle ACB}$.

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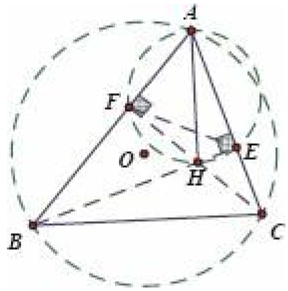
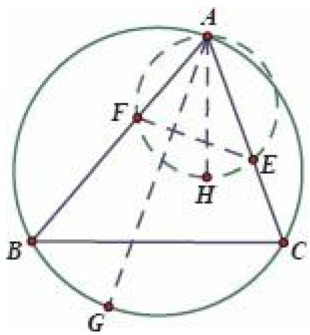


Notice that B, C, E, F are concyclic, which implies that $\angle ACB = \angle AFE$ (Corollary 3.1.5). We also have $AB = 2AE$ in the right angled triangle ΔABE since $\angle A = 60°$ (Example 1.4.8).

Hence, $\frac{R}{r} = 2$, or $R = 2r$, which implies the radius of the circumcircle of ΔABC equals the diameter of the circumcircle of ΔAEH. Since AH is the diameter of the circumcircle of ΔAEH, we have $AO = AH$.

□



Note: Refer to the diagram on the below. Let BE, CF be the heights of ΔABC. One sees that $\Delta ABC \sim \Delta AEF$. If BE, CF intersect at H and AG is a diameter of the circumcircle of ΔABC, we must have $\frac{AB}{AE} = \frac{AG}{AH}$ because these are corresponding line segments with respect to the similar triangles.

In Chapter 1, we learnt the criteria determining congruent triangles, among which S.A.S. requires two pairs of equal sides and one pair of equal angles **between** the sides. Otherwise, we cannot apply S.A.S. Nevertheless, given ΔABC and $\Delta A'B'C'$, if $AB = A'B'$, $AC = A'C'$ and $\angle B = \angle B'$, we have either $\angle C = \angle C'$ (which implies $\Delta ABC \cong \Delta A'B'C'$) or $\angle C = 180° - \angle C'$. This is because Sine Rule gives $\frac{AB}{\sin\angle C} = \frac{AC}{\sin\angle B} = \frac{A'C'}{\sin\angle B'} = \frac{A'B'}{\sin\angle C'}$ and hence, $\sin\angle C = \sin\angle C'$, which implies either $\angle C$ $\angle C'$ or $\angle C = 180° - \angle C$.

Example 3.3.6 (CGMO 03) In a non-isosceles triangle ΔABC, AD, BE, CF are angle bisectors of $\angle A$, $\angle B$, $\angle C$ respectively, intersecting BC, AC, AB at D, E,

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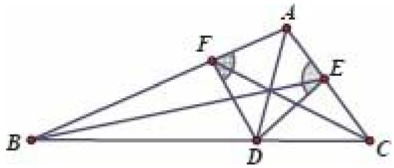
F respectively. Show that if *DE* = *DF,* then $\frac{a}{b+c}=\frac{b}{a+c}+\frac{c}{a+b}$, where $BC=a$, $AC=b$ and $AB=c$.



Insight. Refer to the diagram below. It is given that *AD* bisects ∠*A* and *DE* = *DF*. Consider ∠*ADE* and ∠*ADF*. We have either ∠*AED* = ∠*AFD* or ∠*AED* = 180°– ∠*AFD*.
If ∠*AED* = ∠*AFD,* we have Δ*ADE* ≅ Δ*ADF* and it seems the diagram is symmetric about *AD,* probably contradicting the fact that Δ*ABC* is non-isosceles. (Show it!) Perhaps we should work with the condition that ∠*AED* = 180° –∠*AFD*.

Notice that the conclusion is about the ratio $\frac{a}{b+c}$, $\frac{b}{a+c}$ and $\frac{c}{a+b}$. Is it reminiscent of the Angle Bisector Theorem? For example, $CE=\frac{ab}{a+c}$ and $BF=\frac{ac}{a+b}$ (Example 2.3.8). In fact, *CE* and *BF* are the only choices related to $\frac{b}{a+c}$ and $\frac{c}{a+b}$. Perhaps we can show that *CE* + *BF* equals to a length of $\frac{a^2}{b+c}$.

However, *CE* and *BF* are far apart. Can we put them together? Since *DE* = *DF* and ∠*AED* = 180° – ∠*AFD,* we may rotate Δ*BDF* so that *BF* and *CE* are on the same line.

Proof. By Sine Rule, $\frac{\sin\angle AFD}{\sin\angle DAF}=\frac{AD}{DF}=\frac{AD}{DE}=\frac{\sin\angle AED}{\sin\angle DAE}$ because *DE* = *DF*. Since ∠*DAE* = ∠*DAF,* we have sin ∠*AFD* = sin ∠*AED*, i.e., either ∠*AFD* = ∠*AED* or ∠*AFD* **+** ∠*AED* = 180°.

If ∠*AFD* = ∠*AED*, we immediately have Δ*ADF* ≅ Δ*ADE* (A.A.S.) and hence, *AE* = *AF*. Notice that $AE=\frac{bc}{a+c}$ and $AF=\frac{bc}{a+b}$. It follows that *b* = *c,* or *AC* = *AB,* contradicting the fact that Δ*ABC* is non-isosceles. Hence, ∠*AFD* ≠ ∠*AED*. We have ∠*AFD* + ∠*AED* = 180°.

Now ∠*CED* = ∠*AFD* and hence, we may choose *P* on *CE* extended such that ∠*CPD* = ∠*ABC*. Refer to the diagram below. It is easy to see that Δ*DEP* ≅

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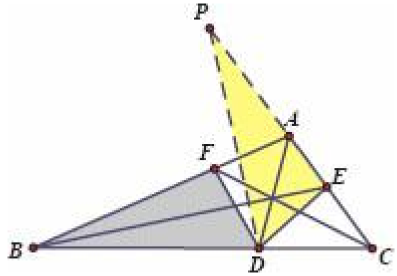
ΔDFB (A.A.S.).



Hence, $PE = BF = \frac{ac}{a+b}$.

Since $CE = \frac{ab}{a+c}$, it suffices to show that $PE + CE = PC = \frac{a^2}{b+c}$.

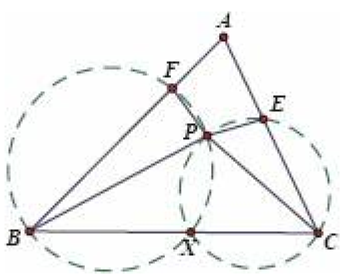
Since $\angle CPD = \angle ABC$, we have $\Delta PCD \sim \Delta BCA$.

Hence, $\frac{PC}{BC} = \frac{CD}{AC}$, which gives $PC = BC \cdot \frac{CD}{AC} = a \cdot \frac{\frac{ab}{b+c}}{b} = \frac{a^2}{b+c}$.

This completes the proof.

□

Example 3.3.7 In a non-isosceles acute angled triangle ΔABC, BE, $CF$are heights on AC, AB respectively. Let D be the midpoint of BC. The angle bisectors of $\angle BAC$ and $\angle EDF$ intersect at P. Show that the circumcircles of ΔBFP and ΔCEP has an intersection on BC.



Insight. Refer to the (simplified) diagram on the right. How can we show the concurrency of two circles and a line? Perhaps we can show that X, the intersection of the two circles, lie on BC, i.e., B, C, X are collinear. Thus, it suffices to show
$\angle BXP + \angle CXP = 180°$.
We do not know many properties of X, but given the circles, we know $\angle BXP = \angle AFP$ and $\angle CXP = \angle AEP$, where $\angle AEP$ and $\angle AFP$ are inside the quadrilateral $AEDF$ and the angle bisectors may give useful properties of

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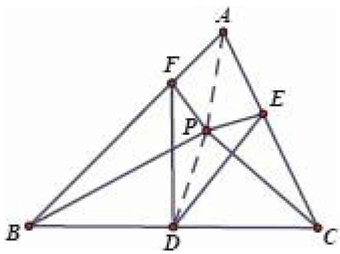
those angles. Now we are to show $\angle AFP + \angle AEP = 180°$. One may attempt to show *A, E, P, F* are concyclic, but it could be difficult (*) because we do not know much about the angles except for *AP* bisecting $\angle EAF$. How about considering ΔAEP and ΔAFP? The angle bisector *AP* could be useful if we apply Sine Rule, which gives

$$\frac{AP}{\sin\angle AFP}=\frac{PF}{\sin\angle FAP} \text{ and } \frac{AP}{\sin\angle AEP}=\frac{PE}{\sin\angle EAP}.$$

Since $\angle EAP = \angle FAP$ and we **should** have $\sin\angle AFP = \sin\angle AEP$, it seems we are to show $PE = PF$. Notice that $DE = DF$ (Example 1.4.7) and *P* is on the angle bisector of $\angle EDF$.

(*) One familiar with commonly used facts in circle geometry could see that if we are to show *A, E, P, F* are concyclic, it suffices to show $PE = PF$ (Example 3.1.11).



***Proof*.** In the right angled triangle ΔBCE, $DE=\frac{1}{2}BC$. Similarly, $DF=\frac{1}{2}BC=DE$.

Refer to the diagram below. Since *DP* bisects $\angle EDF$, we have $\Delta DPE \cong \Delta DPF$ (S.A.S.) and hence, $PE=PF$. Apply Sine Rule to ΔAFP and ΔAEP :

$\frac{AP}{\sin\angle AFP}=\frac{PF}{\sin\angle FAP}$ and $\frac{AP}{\sin\angle AEP}=\frac{PE}{\sin\angle EAP}$. Since *AP* is the angle bisector of $\angle EAF$, we must have $\sin\angle AFP = \sin\angle AEP$.

Case I: $\angle AFP = \angle AEP$

We have $\Delta AFP \cong \Delta AEP$ (A.A.S.) and hence, $AE = AF$. This implies $\Delta ABE \cong \Delta ACF$ (A.A.S.) and hence, $AB = AC$. This contradicts the fact that ΔABC is non-isosceles.

Case II: $\angle AFP = 180° - \angle AEP$

Let the circumcircle of $\angle BFP$ intersect *BC* at *X*. We must have $\angle BXP = \angle AFP = 180° - \angle AEP = \angle CEP$. Hence, *C, E, P, X* are concyclic, i.e., *X* lies on the circumcircle of $\angle CEP$. This completes the proof.

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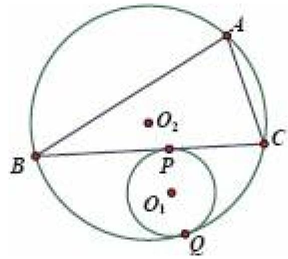
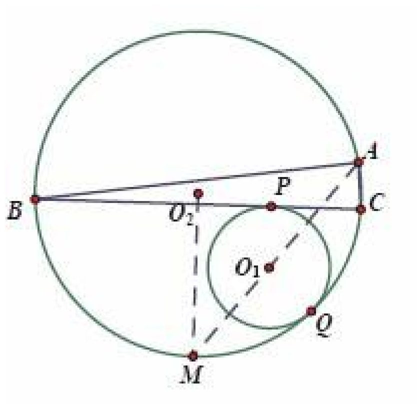
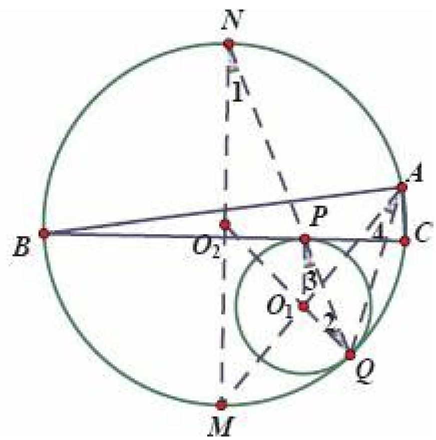


Example 3.3.8 Refer to the diagram below. ⊙O_1 touches ⊙O_2 at Q. BC is tangent to ⊙O_1 at P. Show that if $\angle BAO_1 = \angle CAO_1$, then $\angle PAO_1 = \angle QAO_1$.

Insight. Refer to the left diagram below. Let AO_1 extended intersect ⊙O_2 at M. Since $\angle BAO_1 = \angle CAO_1$, M is the midpoint of $\overset{\frown}{BC}$. Hence, $O_2M \perp BC$. We are to show $\angle PAO_1 = \angle QAO_1$. Notice that $O_1P = O_1Q$. We **should** have A, P, O_1,Q concyclic (Example 3.1.11). How can we show this? Notice that $O_1P \perp BC$, i.e., $O_1P//O_2M$. Perhaps the concyclicity and the parallel lines could give us equal angles.





Proof. Refer to the previous right diagram. Let AO_1 extended intersect ⊙O_2 at M Since $\angle BAO_1 = \angle CAO_1$, M is the midpoint of $\overset{\frown}{BC}$ and hence, $O_2M \perp BC$.(*)

Let MN be a diameter of ⊙O_2. We are given that ⊙O_1 touches ⊙O_2 at Q. Hence, O_2 lies on QO_1 extended. Since $O_1P \perp BC$, we must have $O_1P//MN$. Now the isosceles triangles ΔO_1PQ and ΔO_2NQ are similar and we must have $\angle 1 = \angle 2 = \angle 3$ where P, Q, N are collinear. Notice that $\angle 1 = \angle 4$ (angles in the same arc). We have $\angle 3 = \angle 4$ and hence, A, P, O_1,Q are concyclic. The conclusion follows as $O_1P = O_1Q$ (Corollary 3.3.2).

□

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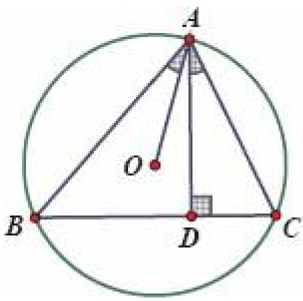
Note: (*) Since $O_2B = O_2C$ and $BM = CM$, O_2M is the perpendicular bisector of BC (Theorem 1.2.4). It is a simple but useful technique to introduce a perpendicular bisector of a chord, which passes through the center of the circle. We will illustrate this technique more in Chapter 5.

3.4 Circumcenter, Incenter and Orthocenter

We have learned the basic properties of the circumcenter, incenter and orthocenter of a triangle. In this section, we will study a few results related to these special points of a triangle using circle geometry techniques. These results are important and frequently referred to as lemmas in various competitions.

Example 3.4.1 Let O be the circumcenter of an acute angled triangle ΔABC. If $AD \perp BC$ at D, show that $\angle CAD = \angle BAO$.

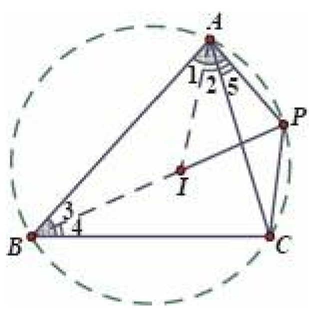


Proof. Refer to the diagram below. Recall that $\angle BAO = 90° - \angle C$ (Example 3.1.2). Clearly $\angle CAD = 90° - \angle C$. The conclusion follows.

□

Note: We also have $\angle CAO = \angle BAD$.

Example 3.4.2 Let I be the incenter of ΔABC. If BI extended intersects the circumcircle of ΔABC at P, show that $AP = CP = PI$.



Insight. One sees that $AP = CP$ follows directly from Corollary 3.3.3. To show $AP = PI$, we may consider showing $\angle AIP = \angle PAI$, as there are many equal angles in the diagram due to the incenter (i.e., angle bisectors) and

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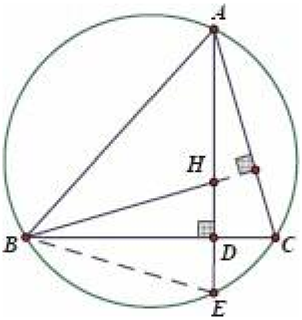
the circumcircle.

Proof. Refer to the diagram above. Since BP bisects $\angle B$, we have $AP = CP$ by Corollary 3.3.3. In ΔABI, we have the exterior angle $\angle AIP$ $\angle 1 + \angle 3$. On the other hand, $\angle PAI = \angle 2 + \angle 5$ where $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ $\angle 5$ (angles in the same arc). Now $\angle AIP = \angle 1 + \angle 3 = \angle 2 + \angle 5 = \angle PAI$. Hence, $AP = PI$ This completes the proof.

□

Example 3.4.3 Let H be the orthocenter of an acute angled triangle $\angle ABC$. Let D be the foot of the perpendicular from A to BC. If AD extended intersects the circumcircle of ΔABC at E, show that $DH = DE$.



Insight. Refer to the diagram below. Given that $BD \perp AE$, since we are to show $DH = DE$, we **should** have $\angle BEH$ isosceles, i.e., $BE = BH$ Both the circumcircle and the orthocenter give equal angles. Hence, one may show that $\angle CBH = \angle CBE$.

Proof. Notice that $\angle CBH = 90° - \angle BHD = \angle CAE$. Since $\angle CAE = \angle CBE$ (angles in the same arc), $\angle CBH = \angle CBE$.
The conclusion follows as $\Delta DBH \cong \Delta DBE$ (A.A.S.).

□

Example 3.4.4 Let H be the orthocenter of an acute angled triangle $\angle ABC$. Let M be the midpoint of BC. If HM extended intersects the circumcircle of ΔABC at A', show that:

(1) $HBA'C$ is a parallelogram
(2) AA' is a diameter of the circumcircle of ΔABC.

Insight. (1) follows from Example 2.5.5 and Example 1.4.3.
(2): It suffices to show that either $\angle ABA'$ or $\angle ACA'$ is $90°$.

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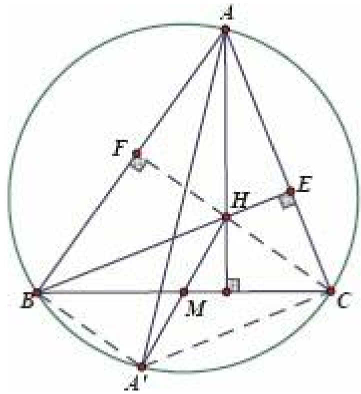


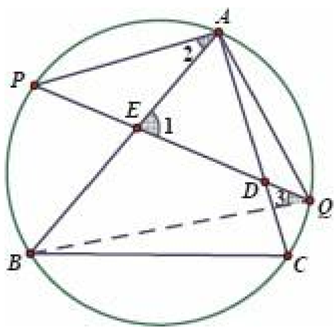
Proof. Refer to the diagram below.

(1) Since H is the orthocenter of ΔABC, we must have $\angle BHC = 180° - \angle BAC$ (Example 2.5.5).
Hence, $\angle BHC = \angle BA'C$ (Corollary 3.1.4). Consider the quadrilateral $A'BHC$. Since $BM = CM$, we conclude that $A'BHC$ is a parallelogram (Example 1.4.3).

(2) Since $CH \perp A'B$ and by (1), $CH \,//\, A'B$, we must have $A'B\ AB$, i.e., $\angle ABA' = 90°$. The conclusion follows.

□

Example 3.4.5 In an acute angled triangle ΔABC, BD, CE are heights. If the line DE intersects the circumcircle of ΔABC at P, Q respectively, show that $AP = AQ$.



Insight. One may show $\angle APQ = \angle AQP$ since there are many equal angles due to the circles. Notice that B, C, D, E are concyclic.

Proof. Refer to the diagram below. Since $\angle BDC = \angle BEC = 90°$, we must have B, C, D, E concyclic. Hence, $\angle 1 = \angle C$.
Now $\angle APQ = \angle 1 - \angle 2 = \angle C - \angle 3$ because $\angle 2 = \angle 3$ (angles in the same arc). On the other hand, $\angle AQP = \angle AQB - \angle 3$. Since $\angle C = \angle AQB$, we conclude that $\angle APQ = \angle AQP$ and hence, $AP = AQ$.

□

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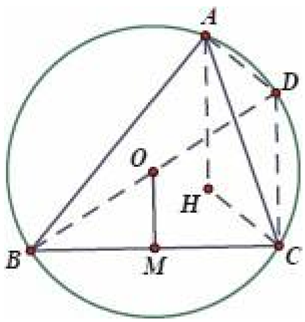
Notice that the argument still applies whenever *B, C, D, E* are concyclic: it is not necessary that *BD, CE* are heights of ΔABC.

Example 3.4.1 to Example 3.4.5 are very useful results. One familiar with these results may find it much easier to see the insight when solving geometry problems related to the circumcenter, incenter and orthocenter of a triangle.

Example 3.4.6 Let *O* and *H* be the circumcenter and orthocenter of an acute angled triangle ΔABC respectively. Let *M* be the midpoint of *BC*. Show that $AH = 2OM$.



Insight. Refer to the diagram below. We do not know much about the properties of *AH* or how it is related to *OM*. For example, it is not easy to find a line segment with length $\frac{1}{2}AH$.

However, one sees that *OM* is related to $\frac{1}{2}$: *M* is the midpoint of *BC* and *O* is the midpoint of a diameter of the circle. If we draw the diameter *BD*, we immediately have $OM = \frac{1}{2}CD$. Now it suffices to show that $AH = CD$.

Recall that Example 3.4.4 states that *ADCH* is a parallelogram, which completes the proof. (Beginners may spend a while to see how Example 3.4.4 is applied in the diagram above.) We leave the details to the reader.

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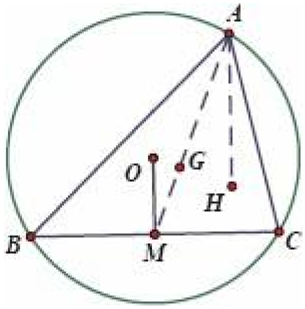


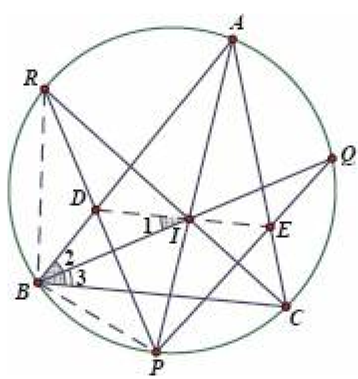
Note: Since M is the midpoint of BC, AM is a median of ΔABC and G, the centroid of ΔABC, lies on AM.

We have $\frac{AH}{OM}=\frac{AG}{GM}=\frac{2}{1}$ (Midpoint Theorem).

It follows that O, G, H are collinear (because $\Delta AGH \sim \Delta MGO$), which is called the Euler Line of ΔABC.

Example 3.4.7 Given ΔABC and it circumcircle, P, Q, R are midpoints of minor arcs $\overset{\frown}{BC}$, $\overset{\frown}{AC}$ and $\overset{\frown}{AB}$ respectively. If PR intersects AB at D and PQ intersects AC at E, show that $DE//BC$

Insight. It is easy to see that AP, BQ, CR are angle bisectors of ΔABC Recall Example 3.4.2 which is about angle bisectors intersecting the circumcircle. Can you see $BP = PI$? (AI extended intersects the circumcircle at P) Similarly, $BR = RI$. Hence, PR must be the perpendicular bisector of BI
Refer to the diagram below.



This implies $\angle 1 = \angle 2$. It is given that $\angle 2 = \angle 3$. Hence, $\angle 1 = \angle 3$, i.e., DI // BC. A similar argument gives EI // BC, which implies D, I, E collinear and $DE//BC$.

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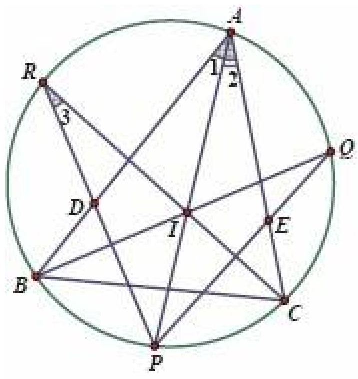
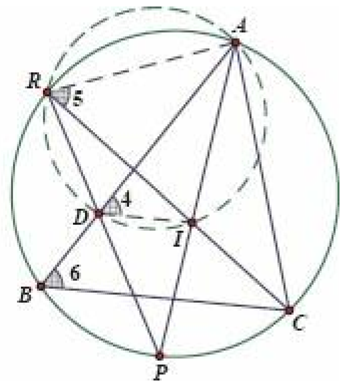
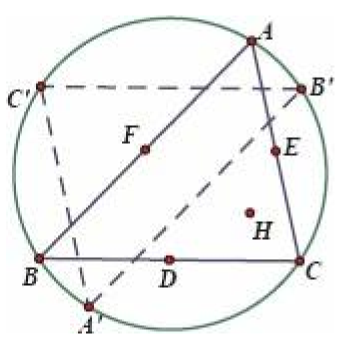
Alternatively, one may solve the problem without applying Example 3.4.2. Notice that angle bisectors in a circle give a lot of equal angles. Refer to the left diagram below. One sees that $\angle 1 = \angle 2 = \angle 3$ (angles in the same arc). This implies *A, I, D, R* are concyclic. Refer to the right diagram below. Now $\angle 4 = \angle 5 = \angle 6$ (angles in the same arc), which implies *DI // BC* A similar argument gives *EI // BC*. The conclusion follows.





Note:
(1) The first method is also an illustration of the relationship among the angle bisector, parallel lines and the isosceles triangle.
(2) It is important to draw the diagram properly. One may see the incenter appears between *D* and *E,* giving an inspiration that *D, I, E* might be collinear.

Example 3.4.8 (TUR 09) In an acute angled triangle ΔABC, *D, E, F* are the midpoints of *BC, CA, AB* respectively. Let *H* and *O* be the orthocenter and the circumcenter of ΔABC respectively. Extend *HD, HE, HF* to intersect the circumcircle of ΔABC at A', B', C' respectively. Let H' be the orthocenter of $\Delta A'B'C'$. Show that *O, H* and H' are collinear.



Insight. A well-constructed diagram is important. Refer to the diagram below. One may see that ΔABC and $\Delta A'B'C'$ are highly symmetric by a

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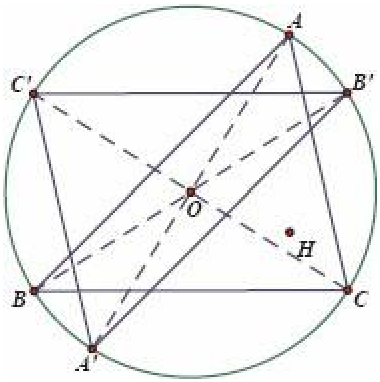
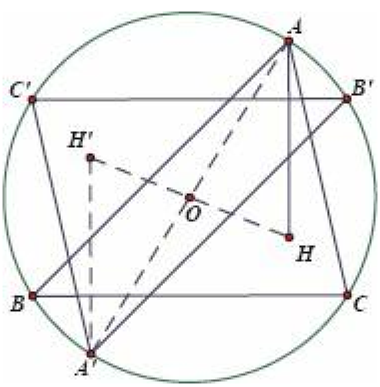
rotation of 180°. If we can show this is true, it is not far away from the conclusion.

On the other hand, the orthocenter and the midpoints remind us of Example 3.4.4, which states that $\Delta AA'$ is a diameter of the circumcircle. Similarly, BB' and CC' are also diameters. Now it is not difficult to show that ΔABC and $\Delta A'B'C'$ are symmetric about O, the center of the circumcircle.



Proof. By Example 3.4.4, we conclude that AA', BB', CC' are the diameters of $\odot O$, the circumcircle of ΔABC. Refer to the diagram below. Since AA' and BB' bisect each other, we conclude that $ABA'B'$ is a parallelogram (and in fact, a rectangle). Hence, $AB = A'B'$ and $AB//A'B'$.
Similarly, we have $BC = B'C'$, $BC//B'C'$ and $AC = A'C'$. It follows that $\Delta ABC \cong \Delta A'B'C$ (S.S.S.).
Refer to the diagram below. We claim that $AHA'H'$ is a parallelogram. Since $\Delta ABC \cong \Delta A'B'C'$, we must have $AH = A'H'$ because H and H' are corresponding points in ΔABC and $\Delta A'B'C'$ respectively. Since AH and $A'H'$ are heights and $BC//B'C,'$ we have $AH//A'H'$. Hence, $AHA'H'$ is a parallelogram and HH' must pass through O, the midpoint of AA'.

□



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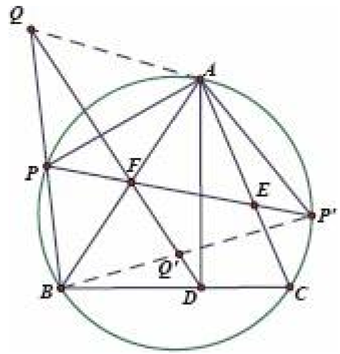
Example 3.4.9 (IMO 10) In an acute angled triangle ΔABC, AD, BE, CF are heights. EF extended intersects the circumcircle of ΔABC at P. BP extended and DF extended intersect at Q. Show that $AP = AQ$.



Proof. Let the line EF intersect the circumcircle of ΔABC at P, P'. Refer to the diagram below. By Example 3.4.5, $AP = AP'$. It suffices to show that $AP' = AQ$. Notice that $\angle ABP = \angle AP'P = \angle APP' = \angle ABP'$, i.e., BA is the angle bisector of $\angle P'BQ$. We also have $\angle BFP = \angle AFE = \angle BFD$ (Example 3.1.6).
It follows that $\Delta FPB \cong \Delta FQ'B$ (A.A.S.), where Q' is the intersection of BP' and DF. We conclude that $AP' = AQ$ (Exercise 1.10, or simply by congruent triangles). This completes the proof.

□

Note:

(1) Notice that $\angle ABP = \angle ABP'$ and $\angle BFP = \angle BFD$ imply P and Q' are symmetric about the line AB, and so are P' and Q. Hence, $AP' = AQ$. Such an argument based on symmetry is acceptable in competitions. However, beginners are recommended to write down a complete argument via congruent triangles.

(2) Notice that A, F, P, Q are concyclic since $\angle APQ = \angle ACB = \angle AFP' = \angle AFQ$. Hence, one may show the conclusion by applying Sine Rule to ΔAFQ and ΔAFP. (Can you show it?)

3.5 Nine-point Circle

First, we shall attempt the following examples.

Example 3.5.1 Let AB be the diameter of the semicircle centered at O. P is a point outside the semicircle and PC, PD are tangent to the semicircle at C, D respectively. If the chords AC, BD intersect at E, show that $PE \perp AB$.

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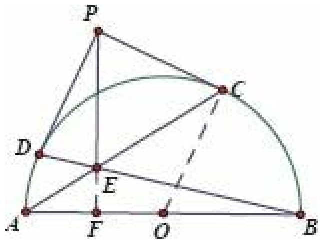

Insight. Of course, the most straightforward method is to show that $\angle A$ + $\angle AEF$ = 90°. Refer to the diagram below, where PE extended intersects AB at F.
Since $OA = OC$ and $OC \perp PC$, we have $\angle A = \angle OCA = 90° - \angle PCE$. On the other hand, $\angle AEF = \angle PEC$. Hence, we **should** have $PE = PC$.
Similarly, we **should** have $PD = PE$, i.e., $PC = PD = PE$ This implies that P **should** be the circumcenter of ΔCDE. Can we show it?

If P is the circumcenter of ΔCDE, Theorem 3.1.1 and Corollary 3.1.4 imply that $\angle P = 2\cdot(180° - \angle CED)$. Can we show this, or equivalently, $\angle CED+\frac{1}{2}\angle P=180°$? Notice that in the isosceles triangle ΔPCD, $180°-\frac{1}{2}\angle P=90°+\angle PCD$.

Proof. We claim that P is the circumcenter of ΔCDE. Notice that $\angle CED$ = $\angle BCE + \angle CBE$, where $\angle BCE$ = 90° (AB is the diameter) and $\angle CBE = \angle PCD$ (Theorem 3.2.10). Hence, $\angle CED = 90° + \angle PCD$. (1)

In the isosceles triangle ΔPCD, $\angle PCD=90°-\frac{1}{2}\angle P$. (2)

(1) and (2) give $\angle CED+\frac{1}{2}\angle P=180°$, or $\angle P=2\cdot(180°-\angle CED)$.

Since $\angle P$ is twice the supplementary angle of $\angle CDE$ and $PC = PD$, we claim that P is the circumcenter of ΔCDE. Otherwise, say O is the circumcenter of ΔCDE, we must have $\angle O = 2\cdot(180° - \angle CED) = \angle P$. Notice that O and P both lie on the perpendicular bisector of CD, and they are on the same side of CD because $\angle CED$ is obtuse. This is impossible.
In conclusion, P is the circumcenter of ΔCDE and hence, $PC = PD = PE$ It follows that $\angle A + \angle AEF = \angle ACO + \angle PEC = \angle ACO + \angle PCE$ = 90°, i.e., $PE \perp AB$.

□

Example 3.5.2 (CWMO 10) Let AB be the diameter of the semicircle centered at O. P is a point outside the semicircle and PC, PD are tangent to

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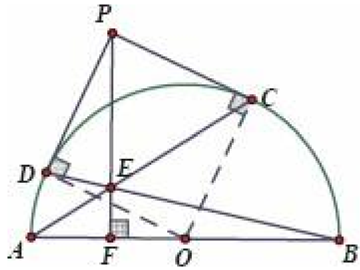
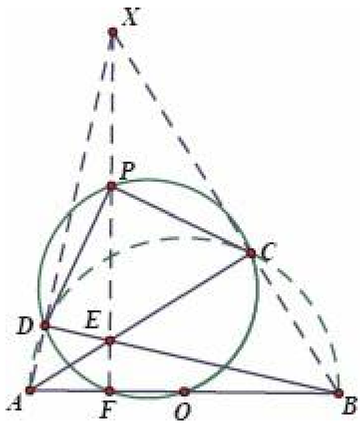
the semicircle at *C, D* respectively. If the chords *AC, BD* intersect at *E,* and *PE* extended intersects *AB* at *F,* show that *P, C, F, D* are concyclic.

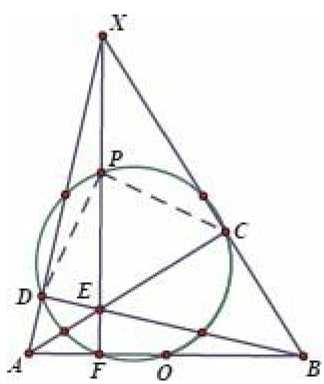


Proof. Refer to the diagram below. Clearly, *P, C, O, D* are concyclic because *OC* ⊥ *PC* and *OD* ⊥ *PD*. We also have *P, D, F, O* concyclic since *PF* ⊥ *AB* (Example 3.5.1). Now *P, D, F, O, C are* concyclic.

□



We shall review the diagrams in Example 3.5.1 and Example 3.5.2. Suppose *AD* extended and *BC* extended intersect at *X*. Since *BD* ⊥ *AX* and *AC* ⊥ *BX, E* is indeed the orthocenter of Δ*ABX,* i.e., *XE* ⊥ *AB*. Since *PE* ⊥ *AB*, *X, P, E, F* are collinear. Refer to the diagram on the left.



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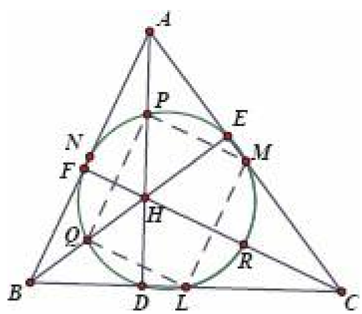
Example 3.5.2 states that *P, D, F, O, C* are concyclic. In fact, we may remove the semicircle centered at *O* and focus on Δ*ABX*. Refer to the diagram below. *C, D, F* are the feet of the altitudes in Δ*ABX* and the circumcircle of Δ*CDF* passes through *O,* the midpoint of *AB*. Similarly, this circle should pass through the midpoints of *AX, BX* as well.
On the other hand, since *PC = PD = PE,* one can show that *P* is the midpoint of *XE*. (**Hint**: Consider the right angled triangle Δ*XDE*. Apply Exercise 1.1.) By similar arguments, we see that the circumcircle of Δ*ABC* must pass through the midpoints of *AE, BE* as well. This circle is called the nine-point circle of Δ*ABC*.

Theorem 3.5.3 (Nine-point Circle) *In any triangle, the following nine points are concyclic: the midpoints of the three sides, the feet of the three altitudes and the midpoints of the line segments connecting each vertex to the orthocenter of the triangle*.

As shown above, one may derive this result from Example 3.5.2. The following is an alternative proof.



Proof. Refer to the diagram on the below. Let *D, E, F* be the feet of the altitudes on *BC, AC, AB* respectively, *L, M, N* be the midpoints of *BC, AC, AB* respectively and *P, Q, R* be the midpoints of *AH, BH, CH* respectively, where *H* is the orthocenter of Δ*ABC*.

Notice that *PM* is a midline in Δ*AHC,* i.e., $PM = \frac{1}{2}CH$ and $PM \,//\, CH$.

Similarly, *QL* is a midline in Δ*BCH* : $QL = \frac{1}{2}CH$ and $QL \,//\, CH$.

Hence, *PMLQ* is a parallelogram.
We also notice that *PQ* is a midline in Δ*ABH* and *PQ*//*AB*. Since *CH* ⊥ *AB* and *CH*//*PM*, we have *PM* ⊥ *PQ*. This implies that *PMLQ* is a rectangle and hence, *P, M, L, Q* are concyclic.
Similarly, *PRLN* is a rectangle and we have ∠*PNL* = 90° = ∠*PQL*. Hence, *N* lies on the circumcircle of Δ*PQL*. By similar arguments, we conclude that *P,*

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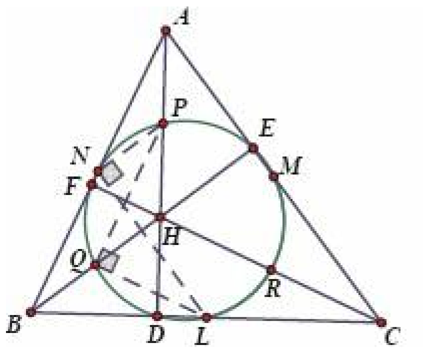
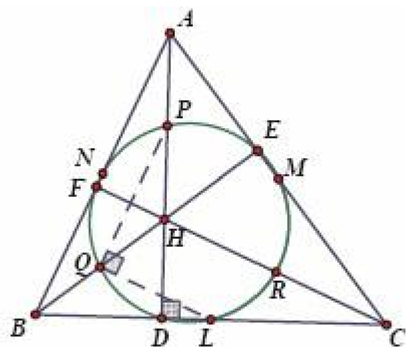
M, R, L, Q, N are concyclic. Refer to the left diagram below.





On the other hand, $\angle PDL = 90° = \angle PQL$, which implies *D* lies on the circumcircle of ΔPQL. Similarly, *E, F* also lie on the circumcircle of ΔPQL. Refer to the right diagram above.
In conclusion, *P, M, R, L, Q, N, D, E, F* are concyclic.

□

Note: Since $\angle PDL = 90°$, *PL* is a diameter of the nine-point circle. Hence, the midpoint of *PL* is the center of the nine-point circle. In particular, the lines *PL, QM, RN* are concurrent (since they all pass through the center of the nine-point circle).

Notice that the nine-point circle of a triangle could be determined by any three of the nine points, among which the most commonly seen ones are midpoints and feet of altitudes. Recall Example 3.1.15. Can you see that *P* lies on the nine-point circle of ΔABC ? (**Hint**: Show that $\angle P = \angle BAC = \angle MDN$. Now *P* lies on the circumcircle of ΔDMN, which is indeed the nine-point circle of ΔABC.)

3.6 Exercises

1. (a) Given a parallelogram *ABCD,* show that *ABCD* is cyclic if and only if is a rectangle.
(b) Given a trapezium *ABCD,* show that *ABCD* is cyclic if and only if it is an isosceles trapezium.

2. Let *ABCD* be a trapezium with *AD*//*BC*. Let *E, F* be on *AB, CD* respectively such that $\angle BAF = \angle CDE$. Show that $\angle BFA = \angle CED$.

3. In ΔABC, *I* is the incenter and *J* is the ex-center opposite *B*. Show that *A, I, C, J* are concyclic.

4. Let *AB* be the diameter of a semicircle. Let the chords *AC, BD* intersect

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at P. Draw $PE \perp AB$ at E. Show that P is the incenter of ΔCDE.

5. Let P be a point outside $\odot O$ and PA, PB are tangent to $\odot O$ at A, B respectively. Show that the incenter of ΔPAB is the midpoint of $\overset{\frown}{AB}$.

6. Let ΔABC be an acute angled triangle, where O, H are the circumcenter and the orthocenter respectively.

(a) If B, C, O, H are concyclic, find $\angle A$.

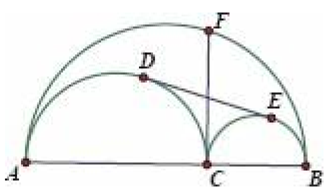
(b) Show that the circumcircles of ΔABC and ΔBCH have the same radius.

7. Given ΔABC and its circumcircle $\odot O$, D is the midpoint of BC and DO extended intersects AB at M. P is a point outside $\odot O$ such that PA,PB are tangent to $\odot O$ at A, B respectively. Show that $PM \mathbin{//} BC$.

8. (CGMO 07) Let D be a point inside ΔABC such that $\angle DAC = \angle DCA = 30°$ and $\angle DBA = 60°$. Let E be the midpoint of BC and F be a point on AC such that $AF = 2FC$, show that $DE \perp EF$.

9. Given ΔABC where $\angle A > 90°$, its circumcenter and orthocenter are O and H respectively. Draw $\odot O_1$ where CH is a diameter. $\odot O_1$ and $\odot O$ intersect at C and D. If HD extended intersects AB at M, show that $AM = BM$.

10. Refer to the diagram on the below. Let AB be the diameter of a semicircle and C be a point on AB. Draw two semicircles with diameters AC, BC respectively. Let D, E be points on these two semicircles respectively such that DE is a common tangent. Draw $CF \perp AB$, intersecting the large semicircle at F. Show that $CDFE$ is a rectangle.



11. Given ΔABC where $\angle B = 2\angle C$, D is a point on BC such that AD bisects $\angle A$. Let I be the incenter of ΔABC, show that the circumcenter of ΔCDI lies on AC.

12. (CZE-SVK 10) In a right angled triangle ΔABC where $\angle A = 90°$, P,Q, R are on the side BC such that $BP = PQ = QR = RC = \frac{1}{4}BC$. The circumcircles of ΔABP and ΔACR intersect at A and M. Show that A, M, Q are

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collinear.

13. In an acute angled triangle ΔABC, AD, BE are the heights. Let A' be the reflection of A about the perpendicular bisector of BC and B' be the reflection of B about the perpendicular bisector of AC. Show that $A'B' // DE$.

14. Let I be the incenter of ΔABC. Show that the circumcenter of ΔBIC lies on the circumcircle of ΔABC.

15. Given ΔABC, its incenter I and ex-centers J_1, J_2, J_3, show that the midpoints of the line segments IJ_1, IJ_2, IJ_3, JJ_1, JJ_2, JJ_3 all lie on the circumcircle of ΔABC.

16. Let $AXYZB$ be a convex pentagon inscribed in a semicircle centered at O with the diameter AB. Let P, Q, R and S denote the feet of the perpendiculars from point Y to the lines AX, BX, AZ and BZ respectively. Let PQ and RS intersect at C. Show that $\angle PCS = \frac{1}{2}\angle XOZ$.

17. (CHN 06) Let $ABCD$ be a trapezium such that $AD // BC$. Γ_1 is a circle tangent to the lines AB, CD, AD and Γ_2 is a circle tangent to the lines AB, BC, CD. Let ℓ_1 be the tangent line from A to Γ_2 (different from AB) and ℓ_2 be the tangent line from C to Γ_1 (different from CD). Show that $\ell_1 // \ell_2$.

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Chapter 4

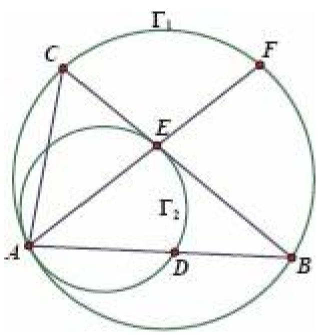
Circles and Lines

In Chapter 3, we learnt various properties about angles in circles. Indeed, one may also find important properties about line segments when straight lines intersect (or touch) a circle, or when triangles and quadrilaterals are inscribed in circles. We will study these properties in this chapter.

4.1 Circles and Similar Triangles

We have seen in Chapter 3 that straight lines intersecting a circle give equal angles. Hence, similar triangles could be constructed via circles. We will see a number of examples of circles and similar triangles in this section. Notice that one needs to be familiar with both circle and similar triangle properties in order to solve such problems.

Example 4.1.1 Refer to the diagram below. Γ_1 and Γ_2 are two circles touching each other at *A*. *AB* is a chord in Γ_1, intersecting Γ_2 at *D*. *BC* is a chord in Γ_1 which is tangent to Γ_2 at *E*. *AE* extended intersects Γ_1 at *F*.



Show that $AB \cdot AC = AE \cdot AF$.

Insight. Given two circles and two tangent lines (including a common tangent of the two circles), one should be able to see many pairs of equal angles. Since the conclusion is equivalent to $\frac{AB}{AF} = \frac{AE}{AC}$, we may show it by similar triangles, for example, $\Delta ABE \sim \Delta AFC$.

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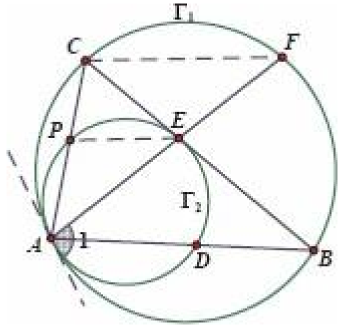
It is easy to see that $\angle ABE = \angle AFC$. Hence, we **should** have $\Delta ABE \sim \Delta AFC$. Can we show it by finding another pair of equal angles?

Proof. Refer to the diagram below.



Let AC intersect Γ_2 at P. Connect PE, CF and draw a common tangent of Γ_1 and Γ_2 at A. Since BC is tangent to Γ_2 at E, we have $\angle AEB = \angle APE = \angle 1 = \angle ACF$ by applying Theorem 3.2.10 repeatedly.

Since $\angle B = \angle F$ (angles in the same arc), we have $\Delta ABE \sim \Delta AFC$.

It follows that $\frac{AB}{AF} = \frac{AE}{AC}$ and hence the conclusion.

□

Note: One may also see $\angle AEB = \angle 1$ by equal tangent segments. Notice that the tangent line at A and the line BC are symmetric about the perpendicular bisector of AE.

Example 4.1.2 Let O be the circumcenter of an acute angled triangle ΔABC and AO extended intersects BC at D · BE, CF are heights of ΔABC. Let O_1 be the circumcenter of ΔAEF AO_1 intersects EF at P. Show that $AP \cdot BC = AD \cdot EF$.

Insight. Refer to the diagram below. Since $\angle BEC = \angle BFC = 90°$, B, C, E, F are concyclic and hence, $\angle ABC = \angle AEF$ (Corollary 3.1.5). We have $\Delta ABC \sim \Delta AEF$ · This is a standard result which an experienced contestant would recall instantaneously.

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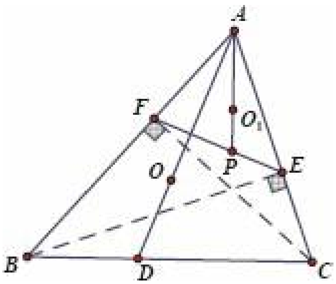


We are to show $AP \cdot BC = AD \cdot EF$. Since $AD \perp EF$ and, $AP \perp BC$, one may think of using the area method. However, $AP \cdot BC$ seems not the area of any existing triangle. Notice that *AP*, *BC*, *AD*, *EF* are in the similar triangles Δ*ABC* and Δ*AEF*. Can we show $\frac{AP}{AD} = \frac{EF}{BC}$ by the properties of similar triangles?

Proof. It is easy to see that *B, C, E, F* are concyclic, which implies ∠*ABC* = ∠*AEF* and hence, Δ*ABC* ~ Δ*AEF*. Notice that *AP* and *AD* are corresponding line segments in Δ*AEF* and Δ*ABC*. It follows that $\frac{AP}{AD} = \frac{EF}{BC}$ and hence the conclusion.

□

Note:

(1) Using the fact that the corresponding line segments are also in ratio as the corresponding sides in similar triangles is an effective technique. Beginners who are not familiar with this technique may also show $\frac{AP}{AD} = \frac{EF}{BC}$ as follows: First, we have $\Delta AOB \sim \Delta AO_1E$ because both are isosceles triangles and $\angle OAB = 2\angle ACB = 2\angle AFE = \angle O_1AE$. Now $\angle OBD = \angle O_1EP$ and $\angle BOD = \angle EO_1P$ imply that $\Delta OBD \sim \Delta O_1EP$. It follows that $\frac{EF}{BC} = \frac{AE}{AB} = \frac{AO_1}{AO} = \frac{EO_1}{BO} = \frac{O_1P}{OD}$. Hence, $\frac{EF}{BC} = \frac{AO_1 + O_1P}{AO + OD} = \frac{AP}{AD}$.

(2) One may see from the diagram that the lines *AP, BE*, *CF* are concurrent, i.e., *AP* passes through *H*, the orthocenter of Δ*ABC*. This is because ∠*CAP* = ∠*BAO* = ∠*CAH* (Example 3.4.1).

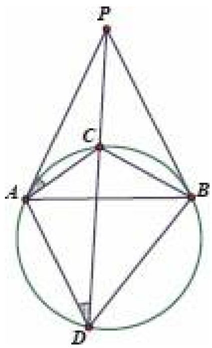
Example 4.1.3 Given a circle and a point *P* outside the circle, draw tangents *PA*, *PB* touching the circle at *A*, *B* respectively. *C* is a point on the minor arc $\overset{\frown}{AB}$ and *PC* extended intersects the circle at *D*.

Show that $\frac{BC}{AC} = \frac{BD}{AD}$.

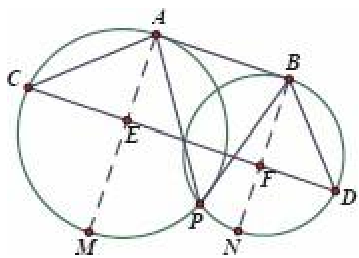
Proof. Refer to the following diagram. Since $\angle PAC = \angle PDA$, we have $\Delta PAC \sim \Delta PDA$.

Hence, $\frac{PA}{PD} = \frac{AC}{AD}$. Similarly, $\frac{PB}{PD} = \frac{BC}{BD}$. Since $PA = PB$, we must have $\frac{AC}{AD} = \frac{BC}{BD}$ and the conclusion follows.

□

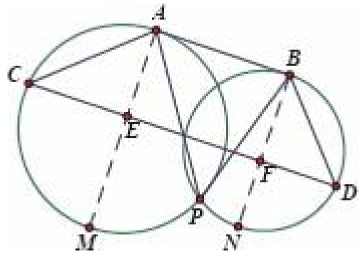


Example 4.1.4 Refer to the diagram below. *AB* is a common tangent of the two circles where *A, B* are the points of tangency. Given *CD* // *AB*, show that $\frac{AC}{BD} = \frac{AP}{BP}$.



Insight. Given the tangent line and parallel lines, it is natural to search for equal angles and similar triangles since we are to show $\frac{AC}{BD} = \frac{AP}{BP}$. It would be great if we can show $\Delta ACP \sim \Delta BDP$. However, this is not true ($\angle ACP = \angle BAP$ and $\angle BDP = \angle ABP$, but $\angle BAP$ and $\angle ABP$ are not necessarily the same). Can you see any pair of similar triangles which put *AC, BD, AP* and *BP* together?

It seems not easy. Apparently, the tangent line and the parallel lines do not give equal angles which leads to the similar triangle we need. Notice that we have not used the condition that *AB* is a **common** tangent. This implies *AB* is perpendicular to the diameters of both circles. Refer to the diagram below. Let AM, BN be the diameters of the two circles. Notice that the diameter AM gives a right angled triangle ΔACM where CE is the height on the hypotenuse.



Hence, $AC^2 = AE \cdot AM$ (Example 2.3.1).

Similarly, $BD^2 = BF \cdot BN$ by considering ΔBDN.

Recognize that $AEFB$ is a rectangle, which implies $AE = BF$ and hence, $\left(\frac{AC}{BD}\right)^2 = \frac{AM}{BN}$. Perhaps we can show $\left(\frac{AP}{BP}\right)^2 = \frac{AM}{BN}$ as well. This should not be difficult since AP, BP are also related to AM, AN by right angled triangles.

Proof. Let AM, BN, be the diameters of the two circles respectively, Clearly, $AM \perp AB$ and $BN \perp AB$. Let AM, BN intersect CD at E, F respectively. Since $CD // AB$, we have $AEFB$ a rectangle and $AE = BF$.

Since AM is a diameter, we have $\angle ACM = 90°$. Since, $CE \perp AM$, we have $AC^2 = AE \cdot AM$(Example 2.3.1). Similarly, $BD^2 = BF \cdot BN$. Now $AE = BF$ gives $\left(\frac{AC}{BD}\right)^2 = \frac{AM}{BN}$. (1)

On the other hand, $AP = AM \cos\angle MAP = AM \sin\angle BAP$. Similarly, $BP = BN \sin\angle ABP$. It follows that $\frac{AP}{BP} = \frac{AM \sin\angle BAP}{BN \sin\angle ABP}$. Notice that $\frac{\sin\angle BAP}{\sin\angle ABP} = \frac{BP}{AP}$ by Sine Rule. Hence, $\left(\frac{AP}{BP}\right)^2 = \frac{AM}{BN}$. (2)

The conclusion follows from (1) and (2).

□

Note:

(1) We intended to search for similar triangles but failed, and we complete the proof based on right angled triangles. This is because the tangent line and parallel lines did not give us equal angles directly, but a rectangle. Nevertheless, we managed to find the clues by carefully

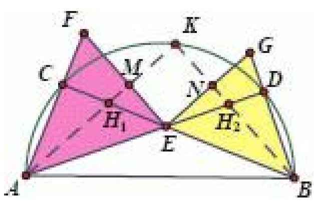
examining the conditions and setting up intermediate steps which lead to the conclusion. Without such repeated (and mostly failed) attempts, the insight will not appear spontaneously!

(2) One may also show $\left(\frac{AP}{BP}\right)^2 = \frac{AM}{BN}$ by drawing a line ℓ passing through and parallel to *AB*. Applying Example 2.3.1 to the right angled triangles ΔAPM and ΔBPN leads to the conclusion.

(3) If the two circles intersect at *P* and *Q*, one may show $\frac{AP}{BP} = \frac{AQ}{BQ}$. Indee a similar argument applies when showing $\left(\frac{AQ}{BQ}\right)^2 = \frac{AM}{BN}$.

Example 4.1.5 (CHN 10) Let *AB* be the diameter of a semicircle. *C, D* are points on the semicircle such that the chords *AD, BC* intersect at *E*. Let *F, G* be points on *AC* extended and *BD* extended respectively such that $AF \cdot BG = AE \cdot BE$. Let H_1, H_2 be the orthocenters of ΔAEF and ΔBEG respectively. If the lines AH_1, BH_2 intersect at *K*, show that *K* lies on the semicircle.

Insight. Refer to the diagram below. Since $AF \cdot BG = AE \cdot BE$, one immediately sees that $\Delta AEF \sim \Delta BGE$, as $\angle EAF = \angle GBE$(angles in the same arc).

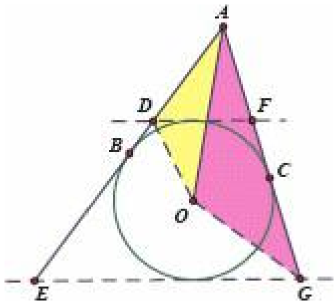


Notice that the orthocenters and the diameter give right angles. In particular, $AK \perp EF$ and $BK \perp EG$. We are to show *K* lies on the semicircle. Hence, we **should** have $\angle AKB = 90°$ and *MENK* **should** be a rectangle. Can you see it suffices to show $EF \perp EG$, i.e., $\angle BEG + \angle CEF = 90°$? This is easy because $\angle BEG = \angle AFE$ (since $\Delta AEF \sim \Delta BGE$) and $\angle AFE + \angle CEF = 90°$ (since $BC \perp AF$).
We leave it to the reader to write down the complete proof.

Note: *F* and *G* are constructed via $AF \cdot BG = AE \cdot BE$This is not a commonly seen condition. Indeed, once we focus on this condition and see the similar triangles, it is not far away from the conclusion. Seeking clues from such an uncommon and useful condition is an effective strategy. We will discuss this further in Chapter 6.

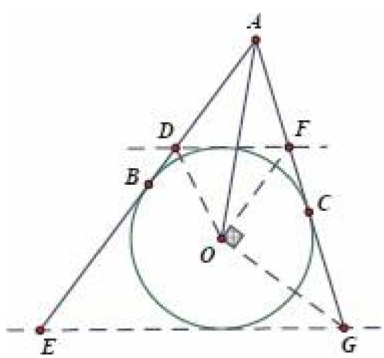
Example 4.1.6 Let A be a point outside $\odot O$. AB, AC are tangent to $\odot O$ at B, C respectively. Let ℓ_1, ℓ_2 be two lines tangent to $\odot O$ and $\ell_1 // \ell_2$. If the line AB intersects ℓ_1, ℓ_2 at D, E respectively, and the line AC intersects ℓ_1, ℓ_2 at F, G respectively. Show that $AD \cdot AG = AO^2$.

***Insight*.** We are to show $\frac{AD}{AO} = \frac{AO}{AG}$. Notice that $\odot O$ is tangent to the sides of ΔAEG, i.e., it is the incircle of ΔAEG and O is the incenter. Hence, AO bisects $\angle BAC$. Refer to the diagram below.



Notice that we **should** have $\Delta AOD \sim \Delta AGO$. Can we show either $\angle ADO = \angle AOG$ or $\angle AOD = \angle AGO$? Notice that $\angle ADO$ and $\angle AOD$ can be expressed in terms of $\angle BAC$ and $\angle ADF$. How about $\angle AGO$ and $\angle AOG$? Recall that $\angle FOG = 90°$ (Example 3.2.5).

***Proof*.** It is easy to see that AO bisects $\angle A$. In fact, O is the incenter of ΔAEG.



Now $\angle AOG = 90° + \frac{1}{2}\angle AEG$ (Theorem 1.3.3)

$= 90° + \frac{1}{2}\angle ADF$ because $DF // EG$.

Since $\angle ADO = \angle ADF + \frac{1}{2}(180° - \angle ADF) = 90° + \frac{1}{2}\angle ADF = \angle AOG$, we must have $\Delta AOD \sim \Delta AGO$. The conclusion follows.

□

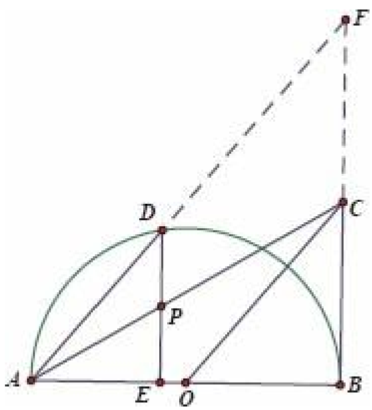
Note:

(1) If you cannot recall Theorem 1.3.3, simply calculate $\angle AOG$ by the fact that $\angle FOG = 90°$.

(2) One may also show $\angle AOD = \angle AGO$, where $\angle AGO = \frac{1}{2}\angle AGE = \frac{1}{2}\angle AFD$ and $\angle AOD = \angle ODE - \frac{1}{2}\angle A$.

Example 4.1.7 Let O be the center of the semicircle where AB is the diameter. Draw a line $\ell \perp AB$ at B. Let D be a point on the semicircle and draw $DE \perp AB$ at E. Draw $OC // AD$, intersecting ℓ at C. *If* AC and DE intersect at P, show that $PD = PE$.

Insight. Refer to the diagram below. It is easy to see that $DE // BC$ and hence, $\frac{PE}{BC} = \frac{AE}{AB}$.



However, one may not be able to relate this to PD in the diagram. What if we "fill up" the triangle by extending BC, intersecting AD extended at F?

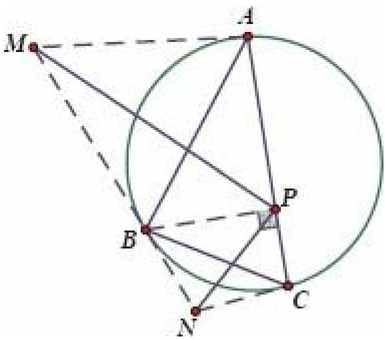
Notice that $\frac{PD}{CF} = \frac{AE}{AB} = \frac{PE}{BC}$. Now it suffices to show $BC = CF$.

Can you show it? (**Hint**: $OC // AD$ and $OA = OB$.) We leave the details to the reader.

Note: We did not construct any similar triangles, but simply applied the

Intercept Theorem where $AD \parallel CO$ and $DE \parallel BC$. Notice that $OA = OB$ is an elementary property, but it could be overlooked occasionally.

Example 4.1.8 Refer to the diagram below. Given ΔABC and its circumcircle Γ, MN is a line tangent to Γ at B such that MA, NC touch Γ at A, C respectively. Let P be a point on AC such that $BP \perp AC$. Show that BP bisects $\angle MPN$.



Insight. We are only given a few tangent lines of the circle. Notice that P is **not** the center of the circle: there are no other given right angles in the diagram and it may be difficult to find concyclicity related to P. Hence, showing $\angle BPM = \angle BPN$ by finding equal angles may not be an effective strategy.

If BP does bisect $\angle MPN$, we **should** have $\frac{PM}{PN} = \frac{BM}{BN}$ by the Angle Bisector Theorem. How could the tangent lines help us? Since we have $AM = BM$ and $BN = CN$, it suffices to show $\frac{PM}{PN} = \frac{AM}{CN}$. It *seems* that $\Delta AMP \sim \Delta CNP$ because $\angle PAM = \angle PCN$. (Can you see that the lines AM, CN are symmetric about the perpendicular bisector of AC?)

How can we show $\Delta AMP \sim \Delta CNP$? It seems we should find another pair of equal angles using the condition $BP \perp AC$, but this is equally difficult as showing the conclusion directly.

Notice that it is much easier to show the inverse: if we are given $\Delta AMP \sim \Delta CNP$, one could see that BP bisects $\angle MPN$ and $BP \perp AC$. Perhaps we should consider a proof by contradiction.

Proof. Choose P' on AC such that $\frac{AP'}{CP'} = \frac{AM}{CN}$. It is easy to see that $\angle P'AM = \angle P'CN$. Hence, $\Delta AMP' \sim \Delta CNP'$.

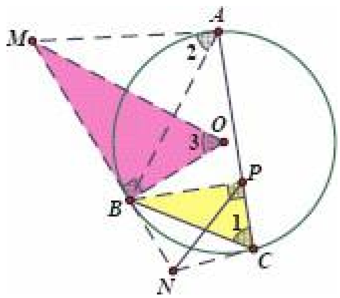
We have $\frac{P'M}{P'N} = \frac{AM}{CN}$ and $\angle AP'M = \angle CP'N$. (1)

It is easy to see that $BM = AM$ and $BN = CN$ (equal tangent segments). Hence, $\frac{P'M}{P'N} = \frac{BM}{CN}$, which implies BP bisects $\angle MPN$ by the Angle Bisector Theorem. Now $\angle BP'M = \angle BP'C$. (2)

(1) and (2) imply that $BP' \perp AC$, i.e., P and P' coincide. This completes the proof.

□

Note: One may still seek clues from $BP \perp AC$ and other right angles by introducing the center of Γ. Refer to the diagram below. It would be wise to erase unnecessary lines.



Can you see that $\Delta BOM \sim \Delta PCB$? (**Hint**: $\angle 3 = \frac{1}{2}\angle AOB = \angle 1$.)

Now $\Delta BOM \sim \Delta PCB$ implies $\frac{BM}{PB} = \frac{BO}{PC}$. (1)

A similar argument gives $\Delta BON \sim \Delta PAB$ and $\frac{BN}{PB} = \frac{BO}{PA}$. (2)

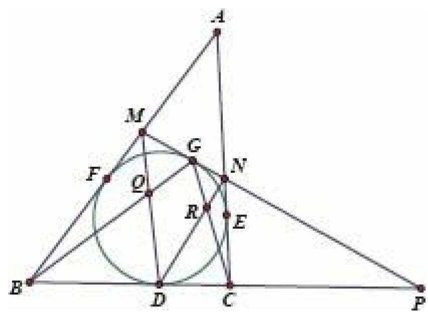
(1) and (2) imply that $\frac{AP}{CP} = \frac{BM}{BN} = \frac{AM}{CN}$ by equal tangent segments.

Now it is easy to see that $\Delta AMP \sim \Delta CNP$ and the conclusion follows. In fact, one familiar with angle properties in circle geometry may immediately see that $\angle 1 = \angle 2$ (Theorem 3.2.10) and $\angle 2 = \angle 3$ (because A, M, B, O are concyclic). Now it is easy to identify similar triangles and this alternative solution follows naturally.

In Chapter 2, we learnt Ceva's Theorem and Menelaus' Theorem, which are useful results solving problems on collinearity and concyclicity. When circles are introduced, one may find even more interesting results by applying Ceva's Theorem and Menelaus' Theorem, due to more equal angles and line segments. The following is a simple example.

Example 4.1.9 Given ΔABC where $AB > AC$, its incircle $\odot I$ touches BC, AC, AB at D, E, F respectively. P is a point on BC extended. Draw a line PG tangent to $\odot I$ at G (distinct from D), intersecting AB, AC at M, N respectively. Let BG, DM intersect at Q and CG, DN intersect at R. Show that if P, E, F are collinear, then P, Q, R are collinear.

Insight. Refer to the diagram below. We are to show collinearity and it seems we need to use either Ceva's Theorem or Menelaus' Theorem.



Now, which triangle should we start with?

Notice that we are given many tangent lines: those equal tangent segments could be helpful. If we choose the line PF intersecting ΔABC, Menelaus' Theorem gives $\frac{AF}{BF}\cdot\frac{BP}{CP}\cdot\frac{CE}{AE}=1$.

Since $AE = AF$, $BF = BD$ and $CE = CD$, we have $\frac{BP}{CP}\cdot\frac{CD}{BD}=1$. (1)

By applying Menelaus' Theorem to ΔBCG, it suffices to show that $\frac{GQ}{BQ}\cdot\frac{BP}{CP}\cdot\frac{CR}{GR}=1$. However, this is not easy even if we use (1), because we do not know much about $\frac{BQ}{GQ}$ and $\frac{CR}{GR}$. Can we avoid these terms?

Notice that one may apply Ceva's Theorem instead: not only will we have more equal tangent segments, but also get rid of those line segments which are not preferred (i.e., those not along the tangent lines).

Proof. Notice that $\frac{MF}{BF}\cdot\frac{BD}{PD}\cdot\frac{PG}{MG}=1$ because $MF = MG$, $BD = BF$ and $PD = PG$ (equal tangent segments). By Ceva's Theorem applied to ΔBPM, P, Q, F are collinear.

Similarly, we have $\frac{NE}{CE} \cdot \frac{CD}{PD} \cdot \frac{PG}{NG} = 1$. By Ceva's Theorem applied to ΔCPN, P, E, R are collinear. (Notice that the points of division D, G are on the extension of PC, PN respectively.)

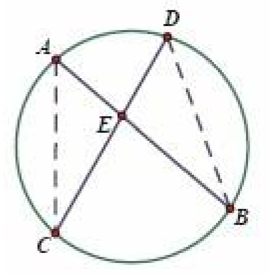
Since P, E, F are collinear, Q, R also lie on this line, i.e., P, Q, R are collinear. □

4.2 Intersecting Chords Theorem and Tangent Secant Theorem

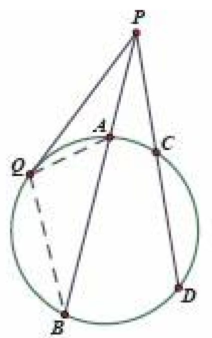
In most elementary geometry textbooks, Intersecting Chords Theorem and Tangent Secant Theorem are mentioned, but the application is not emphasized. Indeed, these are very useful results, with which we can show concyclicity **not** via equal angles.

Theorem 4.2.1 (Intersecting Chords Theorem) *Let AB and CD be two chords of a circle. If AB and CD intersect at E, we have* $AE \cdot BE = CE \cdot DE$.

Refer to the left diagram below. One sees the conclusion immediately from the fact that $\Delta ACE \sim \Delta DBE$.



Intersecting Chords Theorem



Tangent Secant Theorem

Theorem 4.2.2 (Tangent Secant Theorem) *Let P be a point outside the circle and a line passing through P intersects the circle at A and B. If PQ touches the circle at Q, we must have* $PQ^2 = PA \cdot PB$.

Refer to the right diagram above. One may see the conclusion from the fact that $\Delta PAQ \sim \Delta PQB$ (because $\angle PQA = \angle PBQ$).

Note:

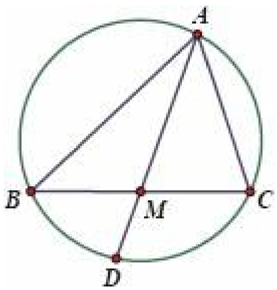
(1) An immediate corollary of the Tangent Secant Theorem is that if two lines passing through P intersect the circle at A, B and C, D respectively, we must have $PA \cdot PB = PC \cdot PD$, because both are equal to PQ^2.

(2) One easily sees that the inverse of the Intersecting Chords Theorem and

the Tangent Secant Theorem hold. (Can you show it, say by contradiction?) Hence, we may use these theorems, especially the inverse, to show concyclicity.

Example 4.2.3 In ΔABC, $AB = 9$, $BC = 8$ and $AC = 7$. Let M be the midpoint of BC. If AM extended intersects the circumcircle of ΔABC at D, find MD.

Ans. Refer to the diagram below.

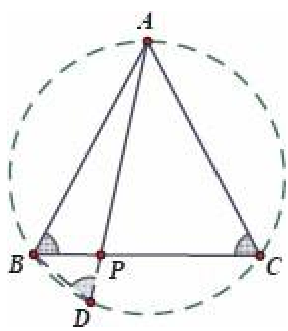


By Theorem 2.4.3, $AM^2 = \frac{1}{2}\cdot(9^2+7^2) - \frac{1}{4}\cdot 8^2 = 49$, i.e., $AM = 7$.

Now the Intersecting Chords Theorem gives $AM \cdot MD = BM \cdot CM$, where $BM = CM = 4$. Hence, $MD = \frac{4^2}{7} = \frac{16}{7}$.

Example 4.2.4 Let ΔABC be an isosceles triangle where $AB = AC$ and P is a point on BC. Show that $(AB + AP)(AB - AP) = BP \cdot CP$.

Insight. From the first glance, it is not clear how the line segments are related to each other. In particular, it seems not easy to obtain $AB + AP$ or $AB - AP$. However, $BP \cdot CP$ reminds us of the Intersecting Chords Theorem, if we draw the circumcircle of ΔABC. Refer to the diagram below.



Let AP extended intersects the circumcircle at D. We immediately have $BP \cdot CP = AP \cdot PD$. Notice that $(AB + AP)(AB - AP) = AB^2 - AP^2$. Hence, it suffices to show that $AB^2 = AP^2 + AP \cdot PD = AP \cdot (AP + PD) = AP \cdot AD$. Now we **should** have $\Delta ABP \sim \Delta ADB$, which is not difficult to show.

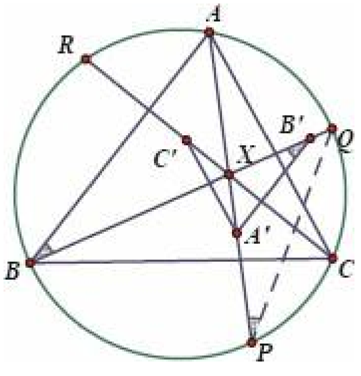
Proof. Let AP extended intersect the circumcircle of ΔABC at D. Since $\angle B = \angle C = \angle D$ (angles in the same arc), we have $\Delta ABP \sim \Delta ADB$. It follows that $\frac{AB}{AD} = \frac{AP}{AB}$, or $AB^2 = AP \cdot AD$.

Now $AB^2 = AP \cdot AD = AP \cdot (AP + PD) = AP^2 + AP \cdot PD$.

Hence, $AP \cdot PD = AB^2 - AP^2 = (AB + AP)(AB - AP)$. The conclusion follows as $AP \cdot PD = BP \cdot CP$ by the Intersecting Chords Theorem. □

Example 4.2.5 Let X be a point inside ΔABC and the lines AX, BX, CX intersect the circumcircle of ΔABC at P, Q, R respectively. Let A' be a point on PX. Draw $A'B' \,//\, AB$ and $A'C' \,//\, AC$, where B', C' are on the lines QX, RX respectively. Show that B', C', R, Q are concyclic.

Insight. Refer to the diagram below. Since X is an arbitrary point, the construction of the diagram seems symmetric, i.e., if we are to show B', C', R, Q are concyclic, we **might** have A', B', Q, P and A', C', R, P concyclic as well. **If** that is true, applying the Tangent Secant Theorem repeatedly gives $XB' \cdot XQ = XA' \cdot XP = XC' \cdot XR$!

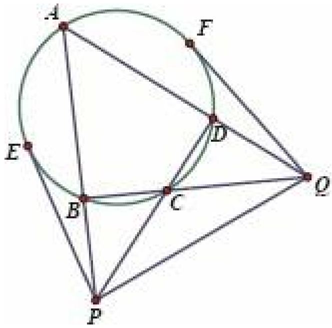


Proof. Since $A'B' \,//\, AB$, we have $\angle A'B'X = \angle ABQ = \angle APQ$ (angles in the same arc). Hence, A' B', Q, P are concyclic. By the Tangent Secant Theorem, $XA' \cdot XP = XB' \cdot XQ$. Similarly, A', C', R, P are concyclic and $XA' \cdot XP = XC' \cdot XR$. Now $XB' \cdot XQ = XC' \cdot XR$, which implies B', C', R, Q are concyclic. □

Note: One might also show $B'C' \,//\, BC$ by applying the Intercept Theorem repeatedly, which also leads to the condition. We leave the details to the reader.

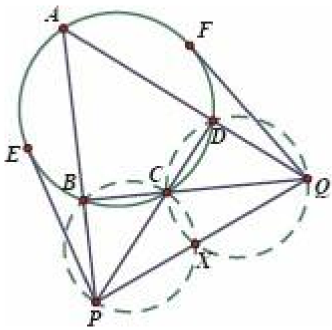
Example 4.2.6 *ABCD* is a quadrilateral inscribed in ⊙*O*. *AB* extended and *DC* extended intersect at *P*. *AD* extended and *BC* extended intersect at *Q*. Draw *PE* tangent to ⊙*O* at *E* and *QF* tangent to ⊙*O* at *F*. Show that *PE*, *QF* and *PQ* give the sides of a right angled triangle.

Insight. Clearly, we should show that *PE*, *QF*, *PQ* satisfy Pythagoras' Theorem. Refer to the diagram below. What do we know about PE^2, QF^2 or PQ^2?



By the Tangent Secant Theorem, $PE^2 = PA \cdot PB = PC \cdot PD$ and similarly, $QF^2 = QB \cdot QC$.

One sees that PQ^2 is related to those line segments above by Cosine Rule. However, it is difficult to use those line segments to express $\cos\angle A$ or $\cos\angle PCQ$.



Are there other methods to relate PE^2 and QF^2 to PQ^2? If the circumcircle of ΔCDQ intersects *PQ* at *X*, we must have $PE^2 = PC \cdot PD = PX \cdot PQ$.

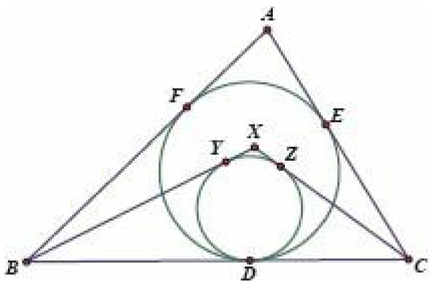
Hence, **if** $PQ^2 = PE^2 + QF^2$, we **should** have $QB \cdot QC = QF^2 = PQ^2 - PE^2 = PQ^2 - PX \cdot PQ = (PQ - PX) \cdot PQ = QX \cdot PQ$. Hence, *B*, *C*, *X*, *P* **should** be concyclic. Can we prove it?

Notice that we have $\angle CXP = \angle CDQ = \angle ABC$ by applying Corollary 3.1.5 repeatedly and hence, B, C, X, P are concyclic. (Can you see this is similar to the proof of the Simson's Line?) We leave it to the reader to write down the complete proof.

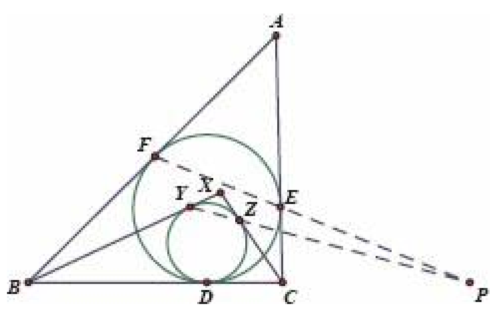
Note: One may *see* from the diagram that PQ is longer than PE and QF. (Drawing a reasonably accurate diagram would be helpful.) Even though this is not given, one should *aim* to show that $PQ^2 = PE^2 + QF^2$.

Example 4.2.7 (IMO 95) The incircle of ΔABC touches BC, AC, AB at D, E, F respectively. Let X be a point inside ΔABC such that the incircle of ΔXBC touches BC, XB, XC at D, Y, Z respectively. Show that E, F, Y, Z are concyclic.

Insight. Refer to the diagram below. Apparently, there are very few conditions given: we only know that E, F, Y, Z are all points of tangency. Although there are incircles (i.e., angle bisectors), but E, F, Y, Z are not related to the incenter or any angle bisectors.

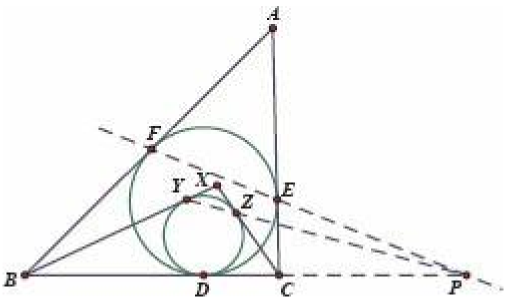


On the other hand, the diagram seems in an "upright" position because the two incircles share a common point of tangent. Do we have $YZ // EF$? If yes, then perhaps we can show that $EFYZ$ is an isosceles trapezium.
Regrettably, this is not true. Refer to the diagram below where FE extended and YZ extended intersect. Can you see a clue in this diagram? Perhaps we could show that $PE \cdot PF = PY \cdot PZ$.



Since the two circles have one common point of tangency D, if P lies on BC extended, we would have $PD^2 = PE \cdot PF = PY \cdot PZ$.

How can we show that P lies on BC extended? In other words, if we let P be the intersection of BC extended and YZ extended, can we show that E, F, P are collinear? This looks like Menelaus' Theorem. Refer to the diagram below. Do we have $\frac{AF}{BF} \cdot \frac{BP}{CP} \cdot \frac{CE}{AE} = 1$? (1)



Notice that $AE = AF$. In fact, there are many equal tangent segments in this diagram.

Perhaps we can also apply Menelaus' Theorem to ΔXBC, which might give us sufficient equalities leading to (1).

Proof. Suppose BC extended and YZ extended intersect at P. Apply Menelaus' Theorem to ΔXBC and the line YZ: $\frac{XY}{BY} \cdot \frac{BP}{CP} \cdot \frac{CZ}{XZ} = 1$.

Since $XY = XZ$ (equal tangent segments), we have $\frac{BP}{CP} = \frac{BY}{CZ}$. (*)

We claim that E, F, P are collinear, i.e., $\frac{AF}{BF} \cdot \frac{BP}{CP} \cdot \frac{CE}{AE} = 1$.

Notice that $AF = AE$. By (*), we have

$\frac{AF}{BF} \cdot \frac{BP}{CP} \cdot \frac{CE}{AE} = \frac{CE}{BF} \cdot \frac{BP}{CP} = \frac{CE}{BF} \cdot \frac{BY}{CZ} = 1$, because $BF = BD = BY$ and $CE = CD = CZ$ (equal tangent segments). Hence, E, F, P are collinear.

Now by the Tangent Secant Theorem, $PE \cdot PF = PD^2 = PX \cdot PY$, which implies E, F, Y, Z are concyclic.

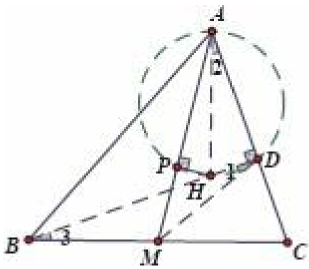
Notice that if BC and YZ do not intersect, i.e., BC // YZ, we must have $XB = XC$ (because $XY = XZ$) and hence, D is the midpoint of BC. Since $BF = BD = CD = CE$

and $AF = AE$, we have $AB = AF + BF = AE + CE = AE$. Now ΔABC and ΔXBC are both isosceles triangles. Hence, the line AX is the perpendicular bisector of EF and YZ. It is easy to see that $EFYZ$ is an isosceles trapezium, which implies E, F, Y, Z are concyclic. □

Example 4.2.8 (JPN 11) Given an acute angled triangle ΔABC and its orthocenter H, M is the midpoint of BC. Draw $HP \perp AM$ at P. Show that $AM \cdot PM = BM^2$.

Insight. It seems AM, AP are not closely related to BM. However, given the orthocenter and the midpoints, one immediately sees $BM = DM = CM$, where $BD \perp AC$ at D.

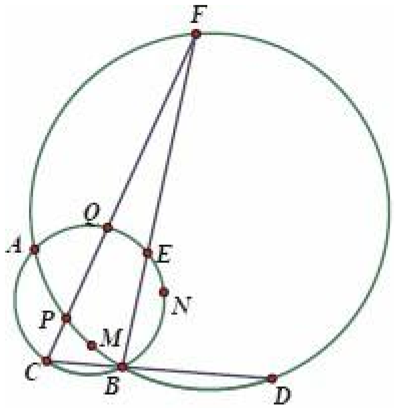
Since we are to show $AM \cdot PM = DM^2$, we **should** have MD tangent to the circumcircle of ΔADP by the Tangent Secant Theorem. Refer to the diagram below. It is easy to see H is on this circle as well. We have plenty of equal angles!

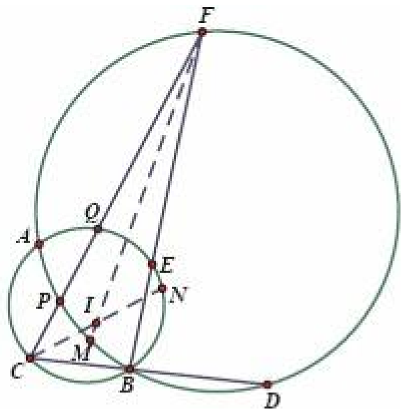


Proof. Let BD be the height on AC. In the right angled triangle ΔBCD, $DM = \frac{1}{2}BC = BM$. Now it suffices to show $AM \cdot PM = DM^2$.

Since $\angle APH = \angle ADH = 90°$, A, D, H, P are concyclic.
Notice that $\angle 2 = 90° - \angle C = \angle 3$ and $\angle 1 = \angle 3$ (because $BM = DM$).
Hence, $\angle 1 = \angle 2$, which implies MD is tangent to the circumcircle of ΔADP (Theorem 3.2.10).
By the Tangent Secant Theorem, $AM \cdot PM = DM^2$. □

Example 4.2.9 (CMO 10) Refer to the left diagram below. Two circles intersect at A and B. A line passing through B intersects the two circles at C, D respectively. Another line passing through B intersects the two circles at E, F, respectively. CF intersects the two circles at P, Q respectively. Let M, N be the midpoints of arcs $\overset{\frown}{PB}$, $\overset{\frown}{QB}$ respectively. Show that if $CD = EF$, then C, M, N, F are concyclic.



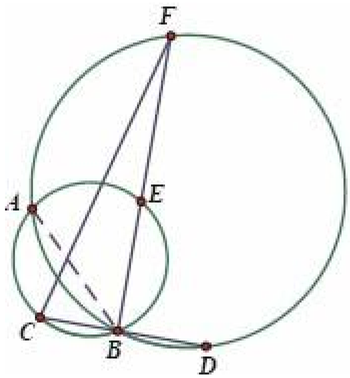


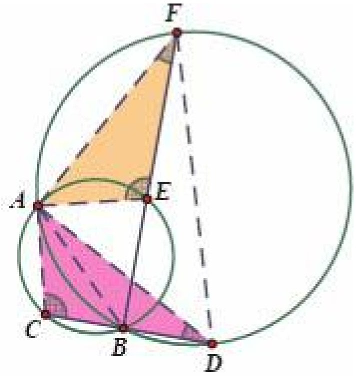
Insight. Clearly, we must use the condition $CD = EF$ in the proof. How about $EF \cdot BF = FQ \cdot CF$ and $BC \cdot CD = CP \cdot CF$?

Since $CD = EF$, we have $\frac{FQ}{BF} = \frac{CP}{BC}$. Notice that all these line segments ΔBCF. Perhaps we should focus on this triangle and see what we *may* discover.

Refer to the previous right diagram. How is ΔBCF related to the conclusion? Notice that CN and FM are the angle bisectors of ΔBCF (Corollary 3.3.3). Hence, they intersect at the incenter I of ΔBCF. Since we are to show C, M, N, F concyclic, we **should** have $CI \cdot IN = FI \cdot IM$. Although we cannot apply the Intersecting Chords Theorem directly because these are chords in two different circles, there is a common chord AB! Since we are to show $CI \cdot IN = FI \cdot IM$, we **should** have AB passing through I. (Suppose otherwise, say BI extended intersects the two circles at A and A' respectively. By the Intersecting Chords Theorem, $AI \cdot IB = CI \cdot IN = FI \cdot IM = A'I \cdot IB$, which implies that A and A' coincide.)

Now it suffices to show that AB is the angle bisector of $\angle CBF$. Refer to the left diagram below. This is much simpler!



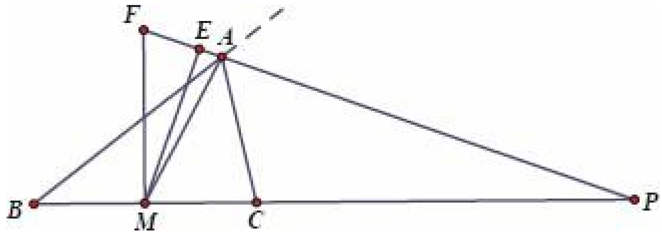


Note that we have not used the condition $CD = EF$ yet. Apparently, our previous exploration on $CD = EF$ was ineffective. Nevertheless, these two circles give many equal angles. Perhaps we can find congruent triangles.

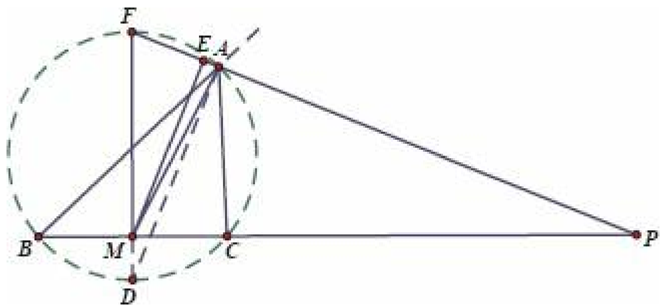
Proof. Refer to the right diagram above. We have $\angle ADC = \angle AFE$ (angles in the same arc) and $\angle ACD = \angle AEF$ (Corollary 3.1.5).
Given $CD = EF$, we conclude that $\Delta ACD \cong \Delta AEF$ (A.A.S.) and hence, $AD = AF$.
Now we have $\angle ABF = \angle ADF$ (angles in the same arc) $= \angle AFD$ (because $AD = AF$) $= \angle ABC$ (Corollary 3.1.5), i.e., BA is the angle bisector of $\angle CBF$.
Since M, N are the midpoints of arcs $\overset{\frown}{PB}, \overset{\frown}{QB}$ respectively, CN, FM are both angle bisectors of ΔCBF (Corollary 3.3.3). Let I be the incenter of ΔCBF. We have $CI \cdot IN = AI \cdot IB = FI \cdot IM$ by the Intersecting Chords Theorem. Hence, C, M, N, F are concyclic. □

Note: One sees many clues from the conditions given and hence, may explore in a wrong direction. For example, one may apply $\frac{FQ}{BF} = \frac{CP}{BC}$ and construct similar triangles, or seek angles in the same arc using the angle bisectors. Even though such (failed) attempts are not reflected in the final solution, these are inevitable during problem-solving and should not be considered a waste of effort. Indeed, beginners would learn much more from those attempts rather than merely reading the solution.

Example 4.2.10 (CGMO 10) Refer to the diagram below. In an acute angled triangle ΔABC, M is the midpoint of BC. Let AP bisect the exterior angle of $\angle A$, intersecting BC extended at P. Draw $ME \perp AP$ at E and draw $MF \perp BC$, intersecting the line AP at F. Show that $BC^2 = 4PF \cdot AE$.



Insight. It seems from the conclusion that the Intersecting Chords Theorem or the Tangent Secant Theorem should be applied, but where is the circle? Perhaps we can see concyclicity from the right angles given. Besides, we also have the angle bisector of the exterior angle. What does it remind you of? Recall that the angle bisectors of supplementary angles are perpendicular (Example 1.1.9)!



Refer to the diagram above. We draw the circumcircle of ΔABC and AD which bisects $\angle A$, intersecting the circumcircle of ΔABC at D. Notice that $AD \perp AP$. It *seems* that F lies on the circle as well. Can you prove it? (**Hint**: One may show that D, M, F are collinear and DF is indeed a diameter of the circle.)

Once we show that A, C, B, F are concyclic, by the Tangent Secant Theorem, $PB \cdot PC = PA \cdot PF$. How could we relate this to the conclusion $BC^2 = 4PF \cdot AE$? We have PF in both equations and $BC = PB - PC$. It is not clear at this stage how we should relate AE to the other line segments. Moreover, it seems the coefficient 4 does not appear naturally. Can we get rid of it?

Notice that M is the midpoint of BC, i.e., $BM = \frac{1}{2}BC$. Hence, it suffices to show $BM^2 = PF \cdot AE$.

We also note that $PB \cdot PC = (PM + BM)(PM - BM) = PM^2 - BM^2$, where $PM^2 = PE \cdot PF$ (Example 2.3.1). Apparently, we are very close to the conclusion.

Proof. Let D be the midpoint of the minor arc $\overset{\frown}{BC}$, i.e., $\overset{\frown}{BD} = \overset{\frown}{CD}$. It is easy to see that AD bisects $\angle BAC$ (Corollary 3.3.3). This implies that D lies on the perpendicular bisector of BC (because $BD = CD$). Since $MF \perp BC$, MF is also the perpendicular bisector of BC. It follows that D, M, F are collinear, the line of which passes through the center of the circumcircle of ΔABC.
Since $AD \perp AP$, we claim that F must lie on the circumcircle of ΔABC as well. Otherwise, say the line MD intersects the circumcircle of ΔABC at F', DF' must be a diameter of the circle and $\angle DAF' = 90°$, i.e., $AD \perp AF'$. This implies F' lies on the line AP, i.e., F and F' coincide.

Since A, C, B, F are concyclic, we have $PB \cdot PC = PA \cdot PF$, where $PB \cdot PC = (PM + BM)(PM - CM) = PM^2 - BM^2$ because $BM = CM$.

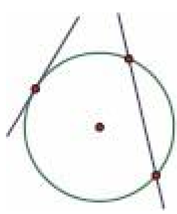
In the right angled triangle ΔPMF, $ME \perp PF$. Hence, $PM^2 = PE \cdot PF$.

It follows that $PA \cdot PF = PB \cdot PC = PM^2\ BM^2 = PE \cdot PF - BM^2$, i.e., $BM^2 = PE \cdot PF - PA \cdot PF = (PE - PA) \cdot PF = AE \cdot PF$. The conclusion follows as $BM = \frac{1}{2}BC$. □

Note: Once the circumcircle of ΔABC is drawn, it is easy to see that the line DM is a diameter of the circle, where AD bisects $\angle BAC$. Now the exterior angle bisector is used to construct right angles. Notice that applying the Angle Bisector Theorem may not be an effective strategy because AB, AC are not closely related to PF, AE.

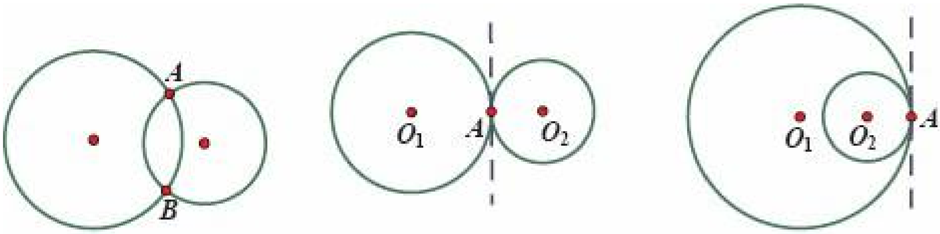
4.3 Radical Axis

Given a circle, a straight line could intersect the circle at two points, or touch the circle at one point, i.e., a tangent line. Refer to the diagram below.



Can you show that no straight line intersects a circle at more than two points? (**Hint**: Suppose otherwise, say a line intersect $\odot O$ at A, B, C, we have $OA = OB = OC$, i.e., both ΔOAB and ΔOBC are isosceles triangles. Show that this is impossible by considering the base angles.)

Given a circle, another circle may intersect it at two points, or touch it at one point, in which case we say the circles are tangent to each other. Refer to the following diagrams.

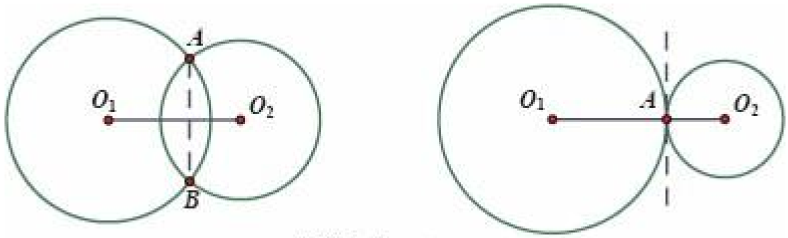


Circles Tangent to Each Other

Can you show that no two circles intersect at more than two points? (**Hint**: Suppose otherwise, say $\odot O_1$ and $\odot O_2$ intersect at A, B, C. It is easy to see that A, B, C cannot be collinear. Now consider the circumcircle of ΔABC.)

Given $\odot O_1$ and $\odot O_2$, if they intersect at A and B, then O_1O_2 must be the perpendicular bisector of AB (Theorem 3.1.20). In particular, if $\odot O_1$ and $\odot O_2$ touch each other at A, then O_1O_2 passes through A, i.e., O_1, O_2, A are collinear. Hence, one may consider two circles touching each other an extreme case of intersecting circles. Similarly, a tangent line of the circle is also an extreme case of a line intersecting the circle at two points, as reflected in the Tangent Secant Theorem. We may define radical axes when two or more circles intersect or touch each other.

Definition 4.3.1 If $\odot O_1$ and $\odot O_2$ intersect at A and B, we call the line AB the radical axis of $\odot O_1$ and $\odot O_2$. In particular, if $\odot O_1$ touches $\odot O_2$ at A, the radical axis of $\odot O_1$ and $\odot O_2$ is the common tangent of the two circles which passes through A.

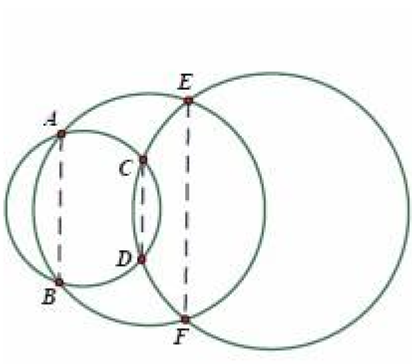


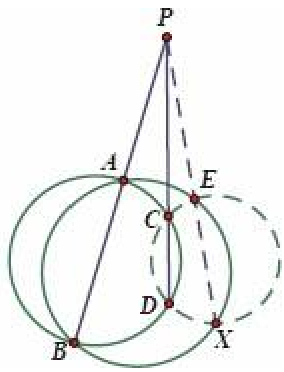
Radical axes

Note: One may also define a radical axis of two non-intersecting circles. However, we will only focus on radical axes of circles intersecting or tangent to each other, which are the most commonly seen applications in competitions.

Theorem 4.3.2 *If three circles are mutually intersecting each other, then the three radical axes are either parallel or concurrent*.

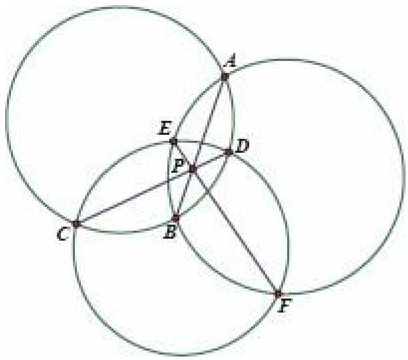
Proof. Let the three circles be Γ_1, Γ_2, Γ_3 such that Γ_1, Γ_2 intersect at A, B, Γ_2,Γ_3 intersect at C, D and Γ_1, Γ_3 intersect at E, F. If the radical axes AB, CD, EF are parallel, there is nothing to prove. Refer to the left diagram below. Otherwise, say without loss of generality that AB and CD intersect at P. Extend PE, intersecting Γ_2 at X. We claim that X and F coincide. Refer to the right diagram below.



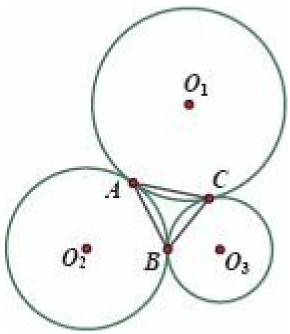


Since A, B, D, C are concyclic (on Γ_1), we have $PA \cdot PB = PC \cdot PD$ by the Tangent Secant Theorem. Similarly, A, B, X, E concyclic on Γ_2 implies $PA \cdot PB = PE \cdot PX$. It follows that $PC \cdot PD = PE \cdot PX$. Now C, D, X, E are concyclic and X must lie on the circumcircle of ΔCDE, which is Γ_3. Since X lies on both Γ_2 and Γ_3, X and F coincide. This implies P, E, F are collinear, i.e., the radical axes are concurrent. □

Note: This proof holds regardless of the relative positions of the three circles. Refer to the diagram below. Notice that $PA \cdot PB = PC \cdot PD = PE \cdot PF$ by the Intersecting Chords Theorem. Hence, we still have the radical axes AB, CD, EF concurrent.

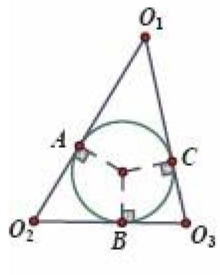


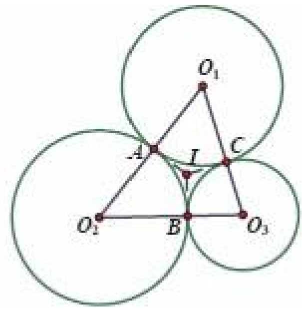
Example 4.3.3 Refer to the diagram below. $\odot O_1$, $\odot O_2$ and $\odot O_3$ are mutually tangent to each other at A, B, C respectively. Show that the circumcenter of ΔABC is the incenter of $\Delta O_1O_2O_3$.



Insight. We see that $O_1A = O_1C$, $O_2A = O_2B$ and $O_3B = O_3C$.

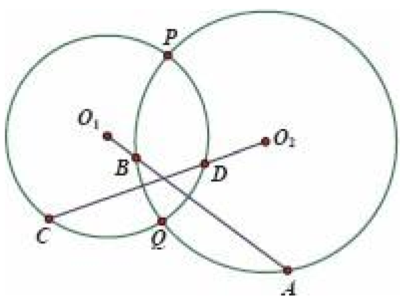
What do we know about the incenter and the incircle of $\Delta O_1O_2O_3$? Refer to the left diagram below. It seems A, B, C **should** be the feet of the perpendiculars from the incenter of $\Delta O_1O_2O_3$. What if we draw perpendicular lines from A to O_1O_2, B to O_2O_3 and C to O_3O_1? Can you see that the perpendicular from A to O_1O_2 is indeed a common tangent of $\odot O_1$ and $\odot O_2$, and similarly for B and C? These common tangents are concurrent! (Can you show this by the Tangent Secant Theorem? Refer to Exercise 4.11.)



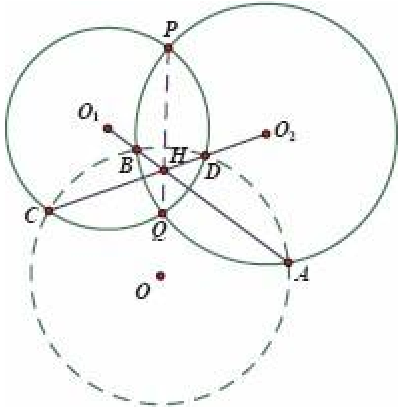


Proof. Refer to the right diagram above. Draw the perpendicular lines from A to O_1O_2 and from B to O_2O_3, intersecting at I. It is easy to show that $CI \perp O_1O_3$ (Exercise 4.11). Notice that $AI = BI = CI$ (equal tangent segments). Hence, I is the circumcenter of ΔABC. Observe that O_1I bisects $\angle O_1$ since $\Delta O_1AI \cong \Delta O_1CI$ (H.L.). Similarly, O_2I bisects $\angle O_2$. Hence, I is the incenter of $\Delta O_1O_2O_3$ This completes the proof. □

Example 4.3.4 Refer to the diagram below. $\odot O_1$ and $\odot O_2$ intersect at P, Q. O_1A intersects $\odot O_2$ at B. O_2C intersects $\odot O_1$ at D. Given that A, C, B, D are concyclic, show that the circumcenter of ΔABC lies on the line PQ.



Insight. Let us draw the circumcircle of ΔABC. Refer to the diagram below where A, B, C, D lie on the $\odot O$.



Can you see that the lines *AB*, *CD*, *PQ* are exactly the radical axes when ⊙*O*, ⊙O_1 and ⊙O_2 intersect each other?

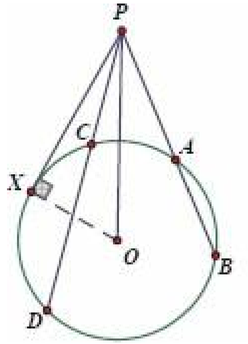
By Theorem 4.3.2, lines *AB*, *CD*, *PQ* must be concurrent, say at *H*.

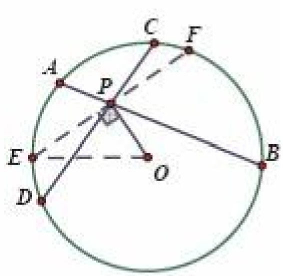
Notice that $AB \perp OO_2$ and $CD \perp OO_1$ (Theorem 3.1.20). Can you see that *H* is the orthocenter ΔOO_1O_2 ? Now can you see why *O* lies on the line *PQ*? (**Hint**: $OH \perp O_1O_2$ and $PH \perp O_1O_2$.) We leave it to the reader to complete the proof.

Note: Theorem 3.1.20 is an elementary but commonly used result. One may always apply it and seek clues when attempting questions with circles intersecting each other.

Definition 4.3.5 Let ⊙*O* be a circle centered at *O* with radius *r*. The power of a point *P* with respect to ⊙*O* is defined as $OP^2 - r^2$.

The concept of the power of a point with respect to a circle is closely related to the Intersecting Chords Theorem and the Tangent Secant Theorem. Refer to the following diagrams.



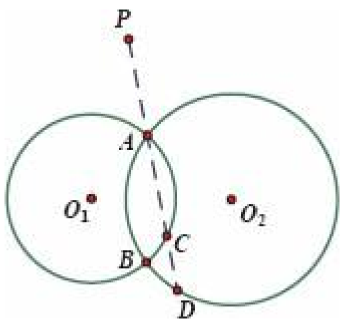


- If P is outside the circle where PX touches $\odot O$ at X, one sees that $PX^2 = OP^2 - OX^2$. By the Tangent Secant Theorem, we have $PA \cdot PB = PC \cdot PD = PX^2$, which equals the power of P with respect to $\odot O$.
- If P is inside the circle, draw $EF \perp OP$ at P, intersecting $\odot O$ at E and F. O sees that $OP^2 = OE^2 - PE^2$. By the Intersecting Chords Theorem, $PA \cdot PB = PC \cdot PD = PE \cdot PF = PE^2 = OE^2 - OP^2$, which is the negative of the power of P with respect to $\odot O$.

In conclusion, the power of a point P with respect to $\odot O$ is positive if P lies outside $\odot O$ and is negative if P lies inside $\odot O$. Clearly, the power of P is zero if it lies on $\odot O$.

Theorem 4.3.6 *Let $\odot O_1$ and $\odot O_2$ intersect at A, B. The power of a point P with respect to $\odot O_1$ and $\odot O_2$ is the same if and only if P lies on the line AB, which is also the radical axis of $\odot O_1$ and $\odot O_2$.*

Proof. Refer to the diagram below.



Let P be any point. Suppose the line PA intersects $\odot O_1$ and $\odot O_2$ at C and D respectively. Notice that the power of P with respect to $\odot O_1$ is $PA \cdot PC$, and the power with respect to $\odot O_2$ is $PA \cdot PD$.

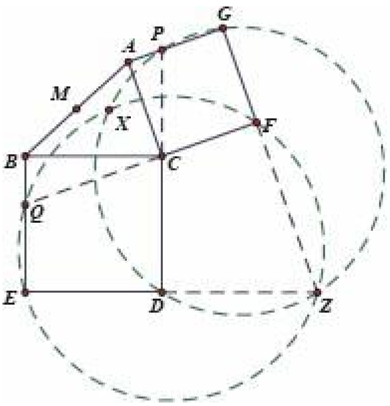
One sees that the power of P with respect to $\odot O_1$ and $\odot O_2$ is the same if and only if $PC = PD$, i.e., C, D coincide with B, the line PA passes through B and P lies on the radical axis AB.

Notice that this proof still holds if P lies inside $\odot O_1$ and $\odot O_2$. Now the power of P with respect to $\odot O_1$ and $\odot O_2$ are $-PA \cdot PC$ and $-PA \cdot PD$ respectively. Hence, the power of P with respect to $\odot O_1$ and $\odot O_2$ is the same if and only if $PC = PD$, i.e., if and only if P lies on the radical axis AB. □

Note: One may easily show Theorem 4.3.2 by applying Theorem 4.3.6: If a point P lies on the radical axis of $\odot O_1$ and $\odot O_2$ and the radical axis of $\odot O_2$ and $\odot O_3$, its power with respect to $\odot O_1$, $\odot O_2$ and $\odot O_3$ is the same. Hence, P must also lie on the radical axis of $\odot O_1$ and $\odot O_3$.

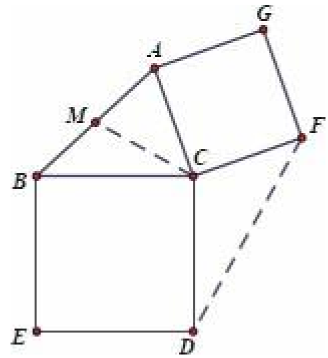
Example 4.3.7 (RUS 13) Given an acute angled triangle ΔABC, draw squares $BCDE$ and $ACFG$ outwards from BC, AC respectively. Let DC extended intersect AG at P and FC extended intersect BE at Q. X is a point inside ΔABC which lies on the circumcircles of both ΔPDG and ΔQEF. If M is the midpoint of AB, show that $\angle ACM = \angle BCX$.

Insight. Refer to the diagram below. It seems that the properties of $\angle ACM$ and $\angle BCX$ are not clear. Let the two circles intersect at X and Z. There are many right angles in the diagram and hence, a lot of concyclicity. In particular, one sees that the lines DE and FG intersect at Z. (Can you show it?)



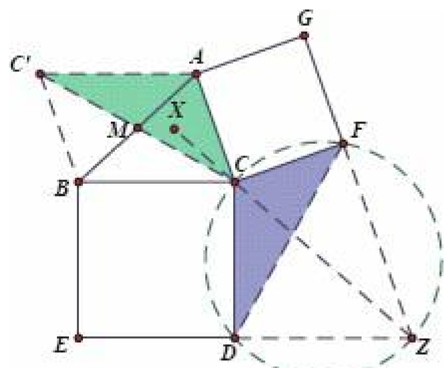
Observe the diagram. It *seems* that C lies on XZ, the common chord (and the radical axis) of the two circles. This is not difficult to show, by calculating the power of C with respect to the two circles, because we have many equal lengths in the diagram due to the squares.

It follows that $\angle BCX = \angle CZD$ (because C lies on XZ). One sees that C, D, Z, F are concyclic and hence, $\angle CZD = \angle CFD$. It suffices to show that $\angle ACM = \angle CFD$, where M is the midpoint of AB. Refer to the diagram below. It is much simpler! Does the diagram look familiar? (Refer to Example 1.2.11.)



Proof. Let the lines *FG* and *DE* intersect at *Z*. Since ∠*PDZ* = ∠*PGZ* = 90°, *P*, *D*, *Z*, *G* are concyclic. Similarly, *Q*, *E*, *Z*, *F* are concyclic because ∠*QEZ* = ∠*QFZ* = 90°. Let Γ_1, Γ_2 denote the circumcircles of Δ*PDG* and Δ*QEF* respectively. We see that *Z* lies on both Γ_1 and Γ_2, i.e., *XZ* is the common chord of Γ_1 and Γ_2. Notice that the power of *C* with respect to Γ_1 is $-PC \cdot CD$ and the power of *C* with respect to Γ_2 is $-QC \cdot CF$. Observe that $PC \cdot CD = PC \cdot BC$ and $QC \cdot CF = QC \cdot AC$.

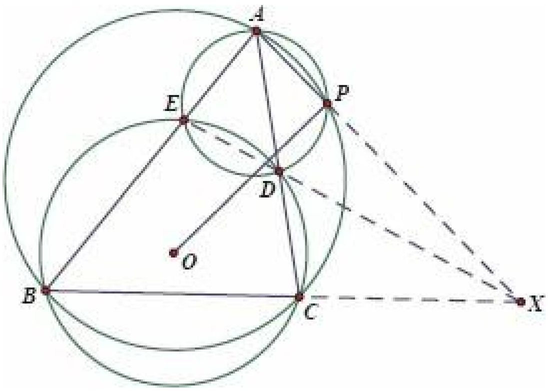
It is easy to see that Δ*BCQ* ~ Δ*ACP* because ∠*ACP* = 90° − ∠*ACB* = ∠*BCQ*. Hence, $\frac{PC}{AC} = \frac{QC}{BC}$, i.e., $PC \cdot BC = QC \cdot AC$. This implies the power of *C* with respect to Γ_1 and Γ_2 is the same. By Theorem 4.3.6, *C* lies on *XZ*, the radical axis of Γ_1 and Γ_2.
Now ∠*BCX* = ∠*CZD* (because *BC* // *DE*) = ∠*CFD* (angles in the same arc). Refer to the diagram below. Extend *CM* to *C*' such that *CM* = *C*'*M*. One sees that Δ*ACC*' ≅ Δ*CFD* (Example 1.2.11, or simply by S.A.S.). Now we have ∠*ACM* = ∠*CFD* = ∠*BCX*. □



Example 4.3.8 (IMO 85) In a non-isosceles acute angled triangle Δ*ABC*, *D*, *E* are points on *AC*, *AP* respectively such that *B*, *C*, *D*, *E* are concyclic on ⊙*O*. Let the circumcircles of Δ*ABC* and Δ*ADE* intersect at *A* and *P*. Show that *AP* ⊥

OP.

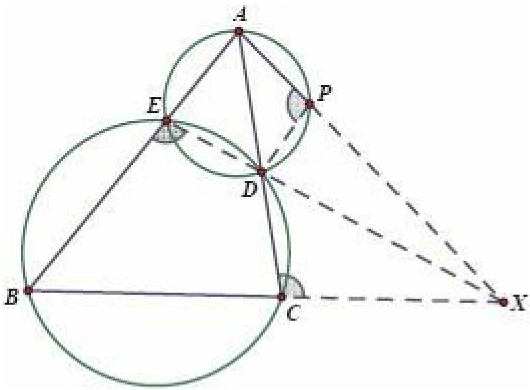
Insight. Refer to the diagram below. It is easy to see that BC and DE are not parallel. Since there are three circles, we immediately see that the radical axes are concurrent, say at X.



Notice that AO, OX (or more precisely, AO^2, OX^2) are closely related to the power of points A and X with respect to $\odot O$. One may also express the power of A with respect to $\odot O$ as $AD \cdot AC$ and the power of X with respect to $\odot O$ as $XB \cdot XC$. Since X lies on all radical axes (or by the Tangent Secant Theorem), we have $XB \cdot XC = XA \cdot XP$. How are these line segments helpful? Perhaps we can show $AP \perp OP$ by calculating $AO^2 - AP^2$ and $OX^2 - PX^2$. (Recall Theorem 2.1.9: $AP \perp OP$ if and only if $AO^2 - AP^2 = OX^2 - PX^2$.)

Proof. If $DE \,//\, BC$, $BCDE$ must be an isosceles trapezium (Exercise 3.1). Now $AB = AC$, which contradicts the fact that ΔABC is non-isosceles. Hence, DE and BC are not parallel, say intersecting at X.

We conclude that the radical axes BC, DE, AP are concurrent at X (Theorem 4.3.2). Let the radius of $\odot O$ be R. Now the power of X with respect to $\odot O$ is $OX^2 - R^2 = BX \cdot CX = AX \cdot PX$ and the power of A with respect to $\odot O$ is $AO^2 - R^2 = AD \cdot AC$. It follows that $AO^2 - OX^2 = AD \cdot AC - AX \cdot PX$. (1)



Refer to the diagram above. We have $\angle APD = \angle BED = \angle ACX$ (Corollary 3.1.5) and hence, C, D, P, X are concyclic.
By the Tangent Secant Theorem, $AC \cdot AD = AP \cdot AX$. (2)

(1) and (2) imply that $AO^2 - OX^2 = AP \cdot AX - AX \cdot PX$

$= AX \cdot (AP - PX) = (AP + PX)(AP - PX) = AP^2 - PX^2$

In conclusion, $AO^2 - AP^2 = OX^2 - PX^2$, which implies $AP \perp OP$ (Theorem 2.1.9). □

Note:

(1) One may see (2) from (1) and reverse engineering: Since we are to show $AO^2 - OX^2 = AP^2 - PX^2$, we **should** have

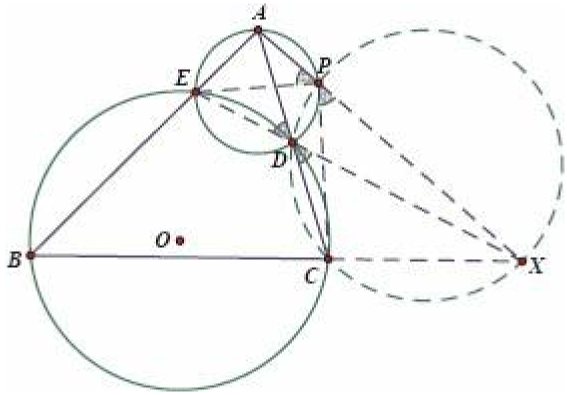
$AC \cdot AD - AX \cdot PX = AP^2 - PX^2$, or equivalently,

$AC \cdot AD = AX \cdot PX + (AP^2 - PX^2) = AP^2 + PX \cdot (AX - PX)$

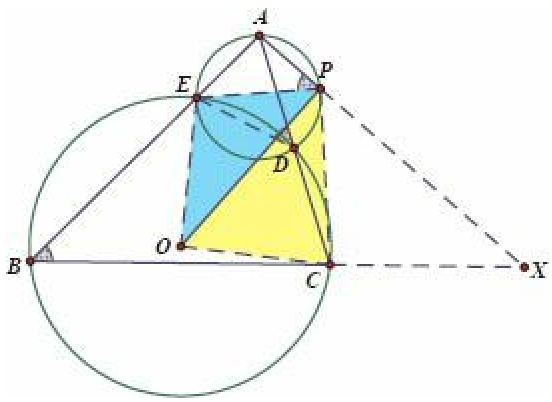
$= AP^2 + PX \cdot AP = AP \cdot (AP + PX) = AP \cdot PX$.

Hence, C, D, P, X **should** be concyclic. Once we see the necessity of this intermediate step, the proof is not difficult.

(2) One may also show the conclusion by angles. First, we show that E, D, X are collinear and C, D, P, X are concyclic as in the proof above. Now $\angle APE = \angle ADE = \angle CDX = \angle CPX$ (Corollary 3.1.3). Refer to the diagram below. We are to show $\angle OPA = \angle OPX = 90°$. Hence, it suffices to show that OP bisects $\angle CPE$.



Consider ΔOEP and ΔOCP. Refer to the diagram below. We have $OE = OC$ and we **should** have $\angle OPE = \angle OPC$. However, it seems that ΔOEP and ΔOCP are **not** congruent. Hence, we should have $\angle OEP + \angle OCP = 180°$! (Refer to the remarks before Example 3.3.6.) This implies C, O, E, P **should** be concyclic. Can we show it?



Notice that $\angle APE = \angle ADE = \angle B$ (Corollary 3.1.5). Since we have $\angle APE = \angle CPX$, it follows that $180° - \angle CPE = \angle APE + \angle CPX = 2\angle B = \angle COE$ (Theorem 3.1.1), i.e., $\angle CPE + \angle COE = 180°$. Hence, C, O, E, P are concyclic.

Now $\angle OEP + \angle OCP = 180°$ and hence, $\sin \angle OEP = \sin \angle OCP$.

By Sine Rule, $\dfrac{OE}{\sin \angle OPE} = \dfrac{OP}{\sin \angle OEP} = \dfrac{OP}{\sin \angle OCP} = \dfrac{OC}{\sin \angle OPC}$.

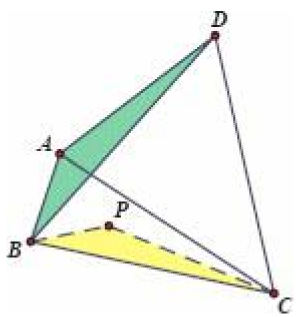
Since $OC = OE$, we must have $\sin \angle OPE = \sin \angle OPC$.
Clearly, $\angle OPE + \angle OPC = \angle CPE < 180°$.
It follows that $\angle OPE = \angle OPC$, which completes the proof.

4.4 Ptolemy's Theorem

Besides the Intersecting Chords Theorem and the Tangent Secant Theorem, Ptolemy's Theorem provides another way to determine concyclicity *without* finding equal angles. Moreover, it gives useful identity regarding the sides and diagonals of a cyclic quadrilateral.

Theorem 4.4.1 (Ptolemy's Theorem) *In a quadrilateral ABCD, $AB \cdot CD + BC \cdot AD \geq AC \cdot BD$ and the equality holds if and only if ABCD is cyclic.*

Proof. Refer to the diagram below. Choose P such that $\angle ABD = \angle CBP$ and $\angle ADB = \angle BCP$, i.e., we construct similar triangles $\Delta ABD \sim \Delta PBC$.

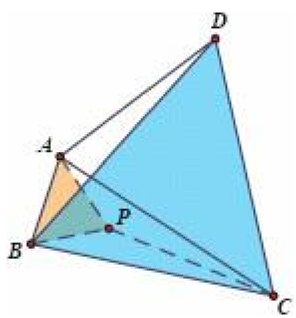


Hence, $\frac{AB}{BD} = \frac{BP}{BC}$ (1), and

$\frac{AD}{BD} = \frac{PC}{BC}$, i.e., $AD \cdot BC = BD \cdot PC$. (2)

(1) implies that there is another pair of similar triangles: $\Delta ABP \sim \Delta DBC$. This is because the angles between the corresponding sides are the same: $\angle ABP = \angle ABD + \angle PBD = \angle CBP + \angle PBD = \angle CBD$.

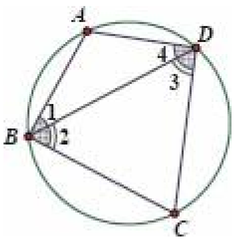
Refer to diagram below. We have $\frac{AB}{AP} = \frac{BD}{CD}$, i.e., $AB \cdot CD = AP \cdot BD$. (3)



(2) and (3) give that $AB \cdot CD + BC \cdot AD$
$= AP \cdot BD + BD \cdot PC$
$= (AP + PC) \cdot BD \geq AC \cdot BD$ because $AP + PC \geq AC$.
Notice that the equality holds if and only if P lies on AC, i.e., $\angle ADB = \angle BCA$ and $ABCD$ is cyclic. □

Ptolemy's Theorem is useful when solving problems regarding sides and diagonals about cyclic quadrilaterals. Refer to Example 3.1.10. One may see the conclusion immediately by applying Ptolemy's Theorem.

Example 4.4.2 Refer to the diagram below. $ABCD$ is a cyclic quadrilateral. Show that: $\sin(\angle 1 + \angle 2) \cdot \sin(\angle 2 + \angle 3) \cdot \sin(\angle 3 + \angle 4) \cdot \sin(\angle 4 + \angle 1) \geq 4\sin \angle 1 \cdot \sin \angle 2 \cdot \sin \angle 3 \cdot \sin \angle 4$.



Insight. One could see that $\sin(\angle 1 + \angle 2) = \sin \angle B = \sin \angle D$ because $\angle B + \angle D = 180°$ (Corollary 3.1.4). Hence, $\sin(\angle 1 + \angle 2) = \sin(\angle 3 + \angle 4)$.

Similarly, $\sin(\angle 2 + \angle 3) = \sin(180° - \angle C) = \sin \angle A = \sin(\angle 1 + \angle 4)$.
Now it suffices to show that $(\sin \angle A \cdot \sin \angle B)^2 \geq 4\sin \angle 1 \cdot \sin \angle 2 \cdot \sin \angle 3 \cdot \sin \angle 4$. (*)
However, it seems not easy to show (*) directly because we do not know how the *product* of $\sin\angle 1$, $\sin\angle 2$, $\sin\angle 3$, $\sin\angle 4$ is related to $\sin \angle A$ and $\sin \angle B$. Perhaps we should consider another strategy.
Notice that each of these angles (on the circumference) corresponds to a line segment in $ABCD$ by Sine Rule. For example, $AB = 2R\sin \angle 4$, $BD = 2R \sin \angle C = 2R \sin(\angle 2 + \angle 3)$, etc., where R is the radius of the circle.
Now (*) is equivalent to $(BD \cdot AC)^2 \geq 4AD \cdot CD \cdot BC \cdot AB$. We have all the four sides and the two diagonals of $ABCD$. Perhaps we can apply Ptolemy's Theorem.

Proof. By Sine Rule, $\sin(\angle 1 + \angle 2) = \frac{AC}{2R} = \sin(\angle 3 + \angle 4)$, where R is the radius of the circle. Similarly, we have $\sin(\angle 2 + \angle 3) = \sin \angle C = \sin(\angle 1 + \angle 4) = \frac{BD}{2R}$, $\sin \angle 1 = \frac{AD}{2R}$, $\sin \angle 2 = \frac{CD}{2R}$, $\sin \angle 3 = \frac{BC}{2R}$

and $\sin\angle 4 = \frac{AB}{2R}$. Now it suffices to show $(BD \cdot AC)^2 \geq 4AD \cdot CD \cdot BC \cdot AB$.

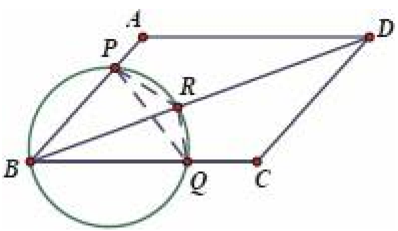
Ptolemy's Theorem gives $BD \cdot AC = AB \cdot CD + BC \cdot AD$Hence, it suffices to show that $AB \cdot CD + BC \cdot AD \geq 2\sqrt{AD \cdot CD \cdot BC \cdot AB}$. (1)

Notice that (1) follows from the inequality $x^2 + y^2 \geq 2xy$, where $x = \sqrt{AB \cdot CD}$ and $y = \sqrt{AD \cdot BC}$. This completes the proof. □

Note: $x^2 + y^2 \geq 2xy$ because $x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. Even though this is a commonly known fact and could be found in any elementary algebra textbook, one may not be able to recognize it immediately when it takes the form of (1).

Example 4.4.3 Given a parallelogram $ABCD$ where $\angle A > 90°$, a circle passing through B intersects AB, BC, BD at P, Q, R respectively. Show that $BP \cdot AB + BQ \cdot BC = BR \cdot BD$.

Insight. One notices that the conclusion looks like Ptolemy's Theorem. Refer to the diagram below.



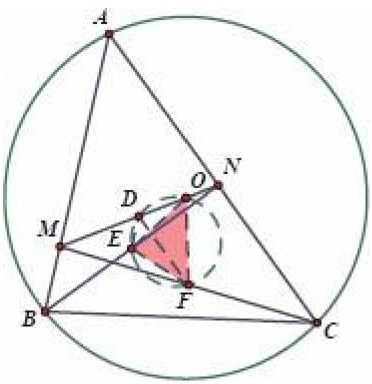
In fact, we are given a circle, even though applying Ptolemy's Theorem on that circle directly does not give the conclusion. Instead, we have $BP \cdot QR + BQ \cdot PR = BR \cdot PQ$. Notice that AB, BC, BD are replaced by QR, PR, PQ respectively.

Are these line segments in ratio? If they are, i.e., $\frac{AB}{QR} = \frac{BC}{PR} = \frac{BD}{PQ}$, then we immediately have the conclusion. Do we have any pair of similar triangles which leads to such equal ratio? Considering the line segments involved, it must be ΔPQR and another triangle.

Can you see $\Delta PQR \sim \Delta BDC$? It should not be difficult to show, by circle properties and parallel lines, that the corresponding angles of these triangles are all equal. For example, $\angle PRQ = 180° - \angle ABC = \angle BCD$. We leave it to the reader to complete the proof.

Example 4.4.4 Given an acute angled triangle ΔABC where O is the circumcenter, M, N are on AB, AC respectively such that O lies on MN. Let D, E, F be the midpoints of MN, BN, CM respectively. Show that O, D, E, F are concyclic.

Insight. Refer to the diagram below. Since we are to show that O, D, E, F are concyclic, it is natural to consider angles. Can we show that $\angle EDF = \angle EOF$? Since D, E, F are midpoints, we must have $DE \,//\, AB$ and $DF \,//\, AC$. Hence, $\angle EDF = \angle A$. Can we show $\angle EOF = \angle A$? (1) Similarly, $AC \,//\, DF$ gives $\angle ANM = \angle ODF$. Can we show $\angle ODF = \angle ANM = \angle OEF$? (2) Since O, E, F, D **should** be concyclic, (1) and (2) **should** be true, i.e., we **should** have $\Delta OEF \sim \Delta ANM$. Can we show this, say by the ratio of corresponding sides? Although DE, DF are related to BM, CN, we cannot apply Ptolemy's Theorem because we have **not** shown O, E, F, D are concyclic.



Apparently, there are many clues, but none of them is useful unless O, E, F, D are concyclic. Perhaps we can draw the circumcircle of ΔDEF, which intersects MN at O' and show that O' coincide with O. By applying Ptolemy's Theorem to $O'DEF$ and replacing the lengths by those in ΔABC (by similar triangles or the Midpoint Theorem), we might be close to the conclusion.

Proof. Let the circumcircle of ΔDEF intersect MN at O'. It is easy to see that $DE//AB$ and $DF//AC$. Hence, $\angle A = \angle EDF = \angle EO'F$ and $\angle O'EF = \angle O'DF = \angle ANM$, which imply $\Delta O'EF \sim \Delta ANM$.

Since O', D, E, F are concyclic, Ptolemy's Theorem gives

$$O'D \cdot EF + O'F \cdot DE = O'E \cdot DF \text{, or } O'D = \frac{O'E}{EF} \cdot DF - \frac{O'F}{EF} \cdot DE. \quad (1)$$

Since $\Delta O'EF \sim \Delta ANM$, $\dfrac{O'E}{EF} = \dfrac{AN}{MN}$ and $\dfrac{O'F}{EF} = \dfrac{AM}{MN}$.

Now

$$O'D = \frac{1}{MN}(AN \cdot DF - AM \cdot DE) = \frac{1}{2MN}(AN \cdot CN - AM \cdot BM),$$

where $DE = \frac{1}{2}BM$ and $DF = \frac{1}{2}CN$. (3)

Notice that $AN \cdot CN$ and $AM \cdot BM$ are the negative of the power of N, M with respect to the circumcircle of ΔABC respectively. Hence, we have $AN \cdot CN = R^2 - NO^2$ and $AM \cdot CM = R^2 - MO^2$, where R denotes the radius of the circumcircle of ΔABC.

Now $O'D = \frac{1}{2MN}(MO^2 - NO^2) = \frac{1}{2MN}(MO + NO)(MO - NO)$ where $MO + NO = MN$. It follows that $O'D = \frac{MO - NO}{2} = OD$. (*)

This implies O and O' coincide. □

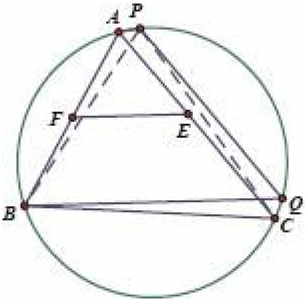
Note:

(1) One sees that (*) holds regardless of the positions of O and O' on MN, i.e., if $MO < NO$, both $O'D$ and OD are negative, which means O and O' lie between M and D. If $MO > NO$, O and O' lie between N and D.

(2) Considering the power of M, N upon (3) is expected: We have not used the condition that O is the circumcenter of ΔABC and we are to remove A, B, C in the expression of $O'D$!

Example 4.4.5 Given ΔABC, E, F are on AC, AB respectively such that BE, CF bisect $\angle B, \angle C$ respectively. P, Q are on the minor arc $\overset{\frown}{AC}$ of the circumcircle of ΔABC such that $AC \parallel PQ$ and $BQ \parallel EF$. Show that $PA + PB = PC$.

Insight. Refer to the diagram below. We are to show the relationship among PA, PB and PC, which lie in the quadrilateral $PABC$. On the other hand, we might obtain equal angles from the circle and parallel lines.



For example, we have $\angle BAC = \angle BPQ$ and $\angle AEF = \angle PQB = \angle PCB$. It follows that $\Delta AEF \sim \Delta PCB$.

In the quadrilateral $PABC$, PA, PB, PC are related to AB, BC, AC by Ptolemy's Theorem. PB, PC are also related to AE, AF by similar triangles. Since AE, AF are angle bisectors, they can be expressed in terms of AB, BC, AC (Example 2.3.8). It seems that we are close to the conclusion.

Proof. We have $\angle AEF = \angle PQB$ (since $EF \mathbin{//} BQ$ and $AC \mathbin{//} PQ$) $= \angle PCB$ (angles in the same arc). Since $\angle BAC = \angle BPC$ (angles in the same arc), we have $\Delta AFE \sim \Delta PBC$. Hence, $\frac{PB}{PC} = \frac{AF}{AE}$. (1)

Let $BC = a$, $AC = b$ and $AB = c$. We see $AE = \frac{bc}{a+c}$ and $AF = \frac{bc}{a+b}$ (Example 2.3.8).

It follows that $= \frac{AF}{AE} = \frac{a+c}{a+b}$. (2)

We are to show $PA + PB = PC$. By (1) and (2), it suffices to show that $PA + PB = PB \cdot \frac{a+b}{a+c}$, or $PA = PB \cdot \left(\frac{a+b}{a+c} - 1\right) = PB \cdot \frac{b-c}{a+c}$.

Ptolemy's Theorem implies $PA \cdot a + PC \cdot c = PB \cdot b$.

Hence, $PA \cdot a + \left(PB \cdot \frac{a+b}{a+c}\right) \cdot c = PB \cdot b$, which gives $PA \cdot a = PB \cdot \left(b - \frac{a+b}{a+c} \cdot c\right) = PB \cdot \left(\frac{b-c}{a+c} \cdot a\right)$. It follows that $PA = PB \cdot \frac{b-c}{a+c}$, which completes the proof. □

4.5 Exercises

1. Given ΔABC and its circumcircle $\odot O$, P is a point outside $\odot O$ such that $\odot P$ touches $\odot O$ at C. AC extended intersects $\odot P$ at D and BC extended intersects $\odot P$ at E. Show that if A, B, D, E are concyclic, then $AC \cdot CP = OC \cdot CE$.

2. Let AB be the diameter of a semicircle centered at O. $BP \perp AB$ at B and AP intersects the semicircle at C. Let D be the midpoint of BP. If $ACDO$ is a parallelogram, find $\sin \angle PAD$.

3. Given a cyclic quadrilateral *ABCD*, *E* is a point on *AB* such that $DE \perp AC$. Draw *BF*//*DE*, intersecting *AD* extended at *F*. Show that if $\angle B = 90°$, then $\frac{AB^3}{AD^3} = \frac{AF}{AE}$.

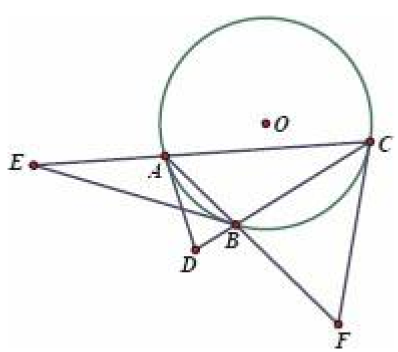
4. (CHN 96) In a quadrilateral *ABCD*, its diagonals *AC* and *BD* intersect at *M*. Draw a line *EF* // *AD* passing through *M*, intersecting *AB*, *CD* at *E*, *F* respectively. Let *EF* extended intersect *BC* extended at *O*. Draw a circle centered at *O* with radius *OM* and *P* is a point on this circle. Show that $\angle OPF = \angle OEP$.

5. Given a circle and a point *P* outside the circle, draw *PA*, *PB* touching the circle at *A*, *B* respectively. *C* is a point on the minor arc $\overset{\frown}{AB}$ and *PC* extended intersects the circle at *D*. Let *E* be a point on *AC* extended and *F* a point on *AD* such that *EF*// *PA*. If *EF* intersect *AB* at *Q*, show that *QE* = *QF*.

6. Let *P* be a point outside $\odot O$ and *PA*, *PB* touch $\odot O$ at *A*, *B* respectively. *C* is a point on the minor arc $\overset{\frown}{AB}$ and *PC* extended intersects $\odot O$ at *D*. If *M* is the midpoint of *AB*, show that *O*, *M*, *C*, *D* are concyclic.

7. Given a semicircle with the diameter *AB*, *C* is a point on the semicircle and *D* is the midpoint of the minor arc $\overset{\frown}{BC}$. Let *AD*, *BC* intersect at *E*. If *CE* = 3 and $BD = 2\sqrt{5}$, find *AB*.

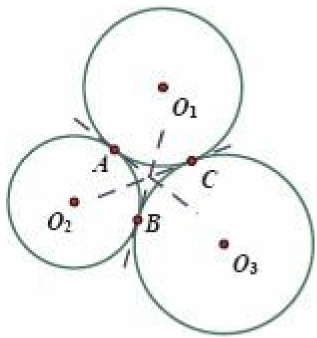
8. Refer to the diagram below. Let $\odot O$ be the circumcircle of ΔABC. *D* is a point on the line *BC* such that the line *AD* touches $\odot O$ at *A*. *E* is a point on *E* the line *AC* such that the line *BE* touches $\odot O$ at *B*. *F* is a point on the line *AB* such that the line *CF* touches $\odot O$ at *C*. Show that *D*, *E*, *F* are collinear.



9. (CGMO 05) Given an acute angled triangle ΔABC and its circumcircle, P is a point on the minor arc $\overset{\frown}{BC}$. AB extended intersects CP extended at E. AC extended intersects BP extended at F. If the perpendicular bisector of AC intersects AB at J and the perpendicular bisector of AB intersects AC at K, show that $\frac{CE^2}{BF^2} = \frac{AJ \cdot JE}{AK \cdot KF}$.

10. An acute angled triangle ΔABC is inscribed inside $\odot O$. BO extended intersects AC at D. CO extended intersects AB at E. If the line DE intersects $\odot O$ at P, Q respectively and it is given that $AP = AQ$, show that $DE // BC$.

11. Refer to the diagram below. We have $\odot O_1$, $\odot O_2$ and $\odot O_3$ mutually tangent to each other at A, B, C respectively, while ℓ_1, ℓ_2, ℓ_3 are the common tangents passing through A, B, C respectively. Show that ℓ_1, ℓ_2, ℓ_3 are concurrent.



12. Let O_1, O_2 be two points inside $\odot O$. Draw $\odot O_1$ and $\odot O_2$, which touch $\odot O$ at A, B respectively. If $\odot O_1$ and $\odot O_2$ intersect at C, D and A, B, C are collinear, show that $OD \perp CD$.

13. In ΔABC, $\angle B = 2\angle C$, show that $AC^2 = AB \cdot (AB + BC)$.

14. $\odot O_1$ is tangent to two parallel lines ℓ_1, ℓ_2. Let O_2 be a point outside $\odot O_1$. $\odot O_2$ is tangent to $\odot O_1$ and ℓ_1 at A, B respectively. Let O_3 be a point outside $\odot O_1$ and $\odot O_2$. $\odot O_3$ is tangent to $\odot O_1$, ℓ_2 and $\odot O_2$ at C, D, E respectively. Show that the intersection of AD and BC is the circumcenter of ΔACE.

15. Let O be the circumcenter of ΔABC. P, Q are points on AC, AB respectively. Let M, N, L be the midpoints of BP, CQ, PQ respectively. Show that if PQ is tangent to the circumcircle of ΔMNL, then we have $OP = OQ$.

Chapter 5

Basic Facts and Techniques in Geometry

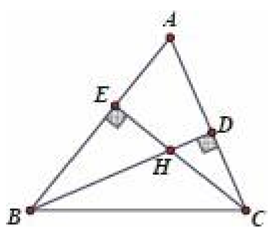
5.1 Basic Facts

We have learnt a number of theorems and corollaries through the previous chapters. Besides those well-known results, we have also seen many examples, some of which are indeed commonly used facts in geometry. One familiar with these basic facts could find it significantly more effective when seeking clues and insights during problem-solving. Hence, we shall have a summary of these basic facts in this section.

❖ Most Commonly Used Facts

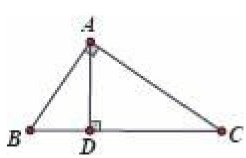
The following are standard results which could be used directly in problem solving, i.e., one may simply state these results without proof.

- In an acute angled triangle ΔABC, BD, CE are heights. We have $\angle ABD = \angle ACE$.



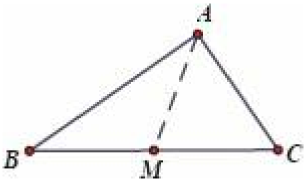
  Moreover, B, C, D, E are concyclic and A, D, H, E are concyclic, where H is the orthocenter of ΔABC.

- In ΔABC, $\angle A = 90°$ and $AD \perp BC$ at D. We have $\angle BAD = \angle C$ and $\angle CAD = \angle B$.

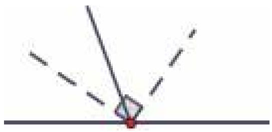


Moreover, we have $AB^2 = BD \cdot BC$ and $AD^2 = BD \cdot CD$.

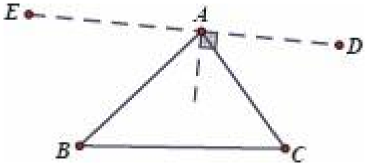
- Given ΔABC where M is the midpoint of BC, we have $AM = \frac{1}{2}BC$ if and only if $\angle A = 90°$.



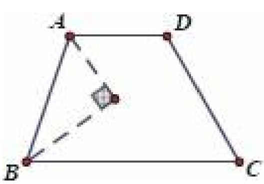
- Angle bisectors of neighboring supplementary angles are perpendicular to each other.



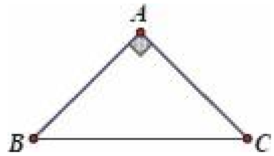
Hence, if D, E are the ex-centers opposite B, C respectively in ΔABC, then DE passes through A and is perpendicular to the angle bisector of $\angle A$.



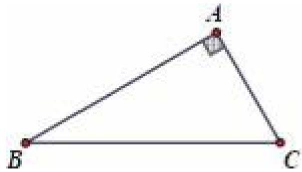
- Let $ABCD$ be a trapezium where $AD // BC$. The angle bisectors of $\angle A$ and $\angle B$ are perpendicular to each other.



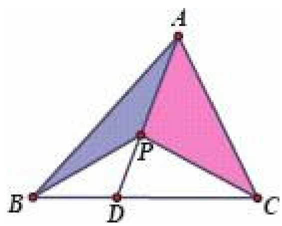
- In a right angled isosceles triangle ΔABC where $\angle A = 90°$ and $AB = AC$, we have $\frac{AB}{BC} = \frac{1}{\sqrt{2}}$.



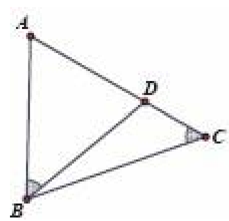
In a right angled triangle ΔABC where $\angle A = 90°$ and $\angle B = 30°$, we have $AC : AB : BC = 1 : \sqrt{3} : 2$.

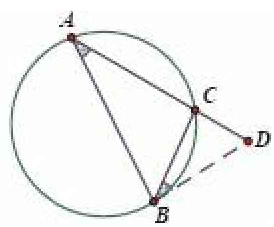


- In ΔABC, D is a point on BC and P is a point on AD. We have $\frac{[\Delta ABP]}{[\Delta ACP]} = \frac{BD}{CD}$.



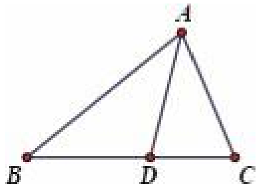
- Given ΔABC and D is on AC such that $\angle ABD = \angle C$, we have $\Delta ABD \sim \Delta ACB$. Refer to the left diagram below.



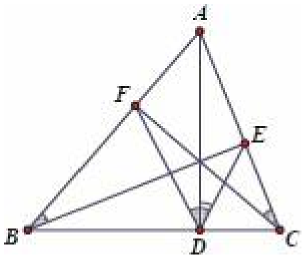


In particular, given ΔABC, if D is a point on AC extended and BD touches the circumcircle of ΔABC at B, then $\Delta ABD \sim \Delta BCD$. Refer to the right diagram above.

- Let AD be the angle bisector of $\angle A$ in ΔABC. We have $BD = BC \cdot \frac{AB}{AB + AC}$.

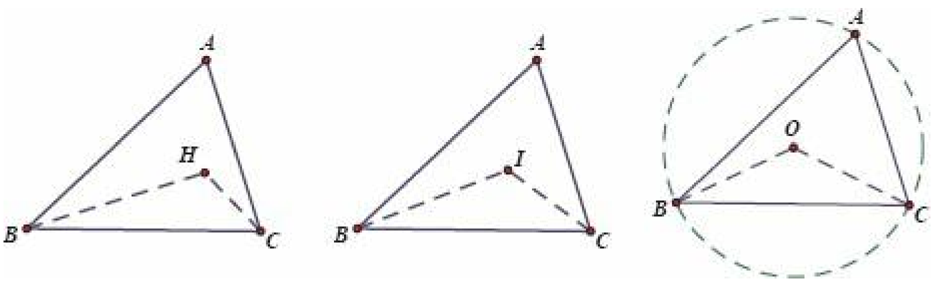


- Let *AD*, *BE*, *CF* be the heights of an acute angled triangle ΔABC. We have $\angle ABE = \angle ADF = \angle ADE = \angle ACF$.

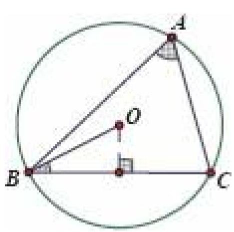


In particular, the orthocenter of ΔABC is the incenter of ΔDEF.

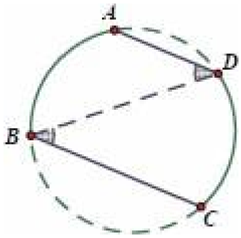
- Let *H*, *I*, *O* be the orthocenter, incenter and circumcenter of an acute angled triangle ΔABC respectively. We have $\angle BHC = 180° - \angle A$, $\angle BIC = 90° + \frac{1}{2}\angle A$ and $\angle BOC = 2\angle A$.



- Given an acute angled triangle ΔABC and its circumcenter *O*, we must ha $\angle A + \angle OBC = 90°$.



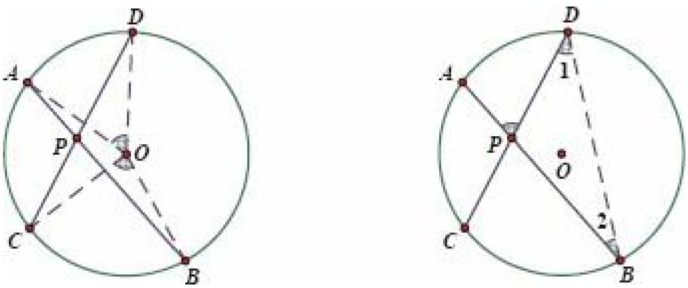
- Refer to the diagram on the below. Given a circle with $\overset{\frown}{AB} = \overset{\frown}{CD}$, we must have *AD* // *BC*.



Proof. Notice that $\overset{\frown}{AB} = \overset{\frown}{CD}$ implies ∠*ADB* = ∠*CBD*. Hence, *AD* // *BC*.

□

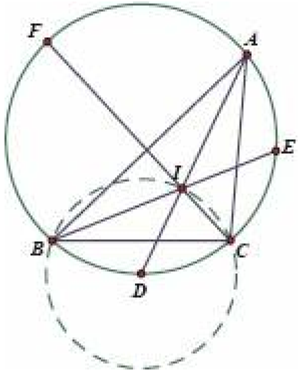
- Refer to the left diagram below. *AB*, *CD* are two chords in ⊙*O* and *AB*, *C* intersect at *P*. We have ∠*AOD* + ∠*BOC* = 2∠*APD*.



Proof. Refer to the right diagram above. We have ∠*BOC* = 2∠1 and ∠*AOD* = 2∠2 (Theorem 3.1.1).
Now ∠*APD* = ∠1 + ∠2 and the conclusion follows.

□

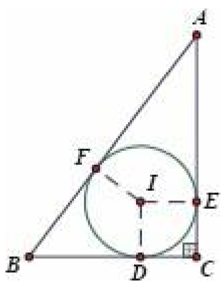
- Given Δ*ABC* and its circumcircle, *D*, *E*, *F* are the midpoints of arcs $\overset{\frown}{BC}$, $\overset{\frown}{AC}$, $\overset{\frown}{AB}$ respectively. Refer to the diagram on the below.



We have *AD*, *BE*, *CF* the angle bisectors of Δ*ABC* and hence, concurrent at *I*, the incenter of Δ*ABC*.

Notice that *D*, *E*, *F* are the circumcenters of Δ*BCI*, Δ*ACI*, Δ*ABI* respectively (Example 3.4.2).

- Given a right angled triangle Δ*ABC* with ∠*C* = 90°, we have $r = \frac{1}{2}(AC + BC - AB)$, where *r* is the radius of the incircle of Δ*ABC*.



Proof. Let *I* be the incenter of Δ*ABC*.
Suppose the incircle of Δ*ABC* touches *BC*, *AC*, *AB* at *D*, *E*, *F* respectively.
Refer to the diagram above.
It is easy to see that *AE* = *AF*, *BD* = *BF* and *CDIE* is a square.

It follows that $r = CD = \frac{1}{2}(CD + CE) = \frac{1}{2}(AC + BC - AB)$.

□

Note: Let *BC* = *a*, *AC* = *b*, *AB* = *c*. We have $r = \frac{1}{2}(a + b - c)$.

By Theorem 3.2.9, we have $r = \frac{2S}{a + b + c}$ where *S* = [Δ*ABC*].

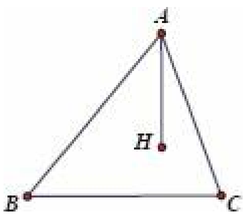
Indeed, $2r \cdot (a + b + c) = (a + b - c)(a + b + c) = (a + b)^2 - c^2 = (a + b)^2 - (a^2 + b^2) = 2ab = 4S$.

❖ Useful Facts

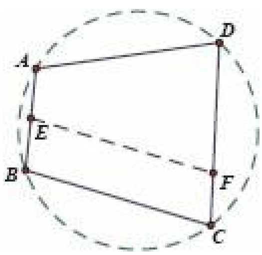
One familiar with the following facts may see clues and intermediate steps in problem-solving quickly, which might tremendously simplify the conclusion to be shown. While experienced contestants simply state these well-known facts during competitions, beginners are recommended not to omit any necessary proof to these results (which were illustrated in the previous chapters).

Occasionally, one may derive an intermediate step, but find it irrelevant to the problem given. If it seems not a useful clue, one should put it aside and refrain from wasting time exploring that piece further.

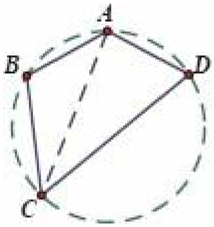
- Let H be the orthocenter of an acute angled triangle ΔABC. We have $AH = \dfrac{BC}{\tan A}$.



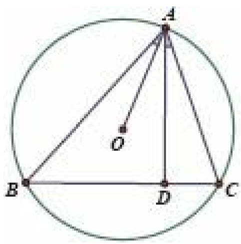
- Let $ABCD$ be a cyclic quadrilateral. E, F are on AB, CD respectively such th $BC \parallel EF$. We must have A, E, F, D concyclic.



- In a quadrilateral $ABCD$, $AB = AD$ and $BC \neq CD$, if AC bisects $\angle C$, then ABC is cyclic.

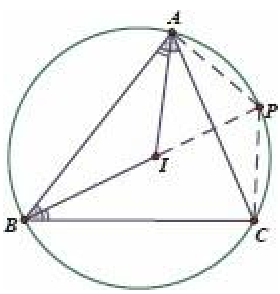


- Let O be the circumcenter of an acute angled triangle ΔABC and AD is a height. We have $\angle CAD = \angle BAO$.

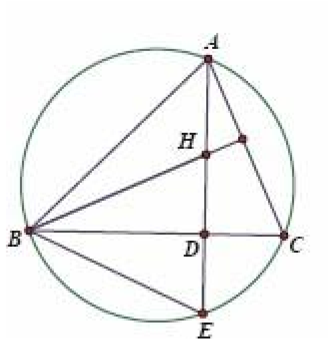


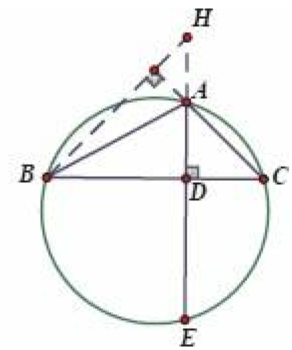
- Let I be the incenter of ΔABC. If BI extended intersects the circumcircle (ΔABC at P, we have $AP = PI = CP$.

  Hence, P is the circumcenter of ΔAIC.



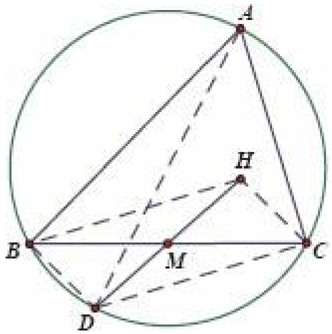
- Let H be the orthocenter of an acute angled triangle ΔABC. $AD \perp BC$ at L and AD extended intersects the circumcircle of ΔABC at E. We have $DE = DH$. Refer to the left diagram below.

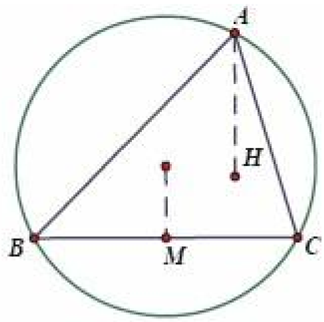




Notice that the conclusion still holds if ΔABC is a right angled triangle (i.e., A is the orthocenter and BC is the diameter of the circumcircle) or an obtuse angled triangle. Refer to the right diagram above. The proof is similar and we leave it to the reader.

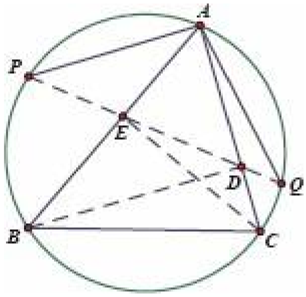
- Let H be the orthocenter of ΔABC and M be the midpoint of AB. Let HM extended intersect the circumcircle of ΔABC at D. We have that $BDCH$ is a parallelogram and hence, AD is a diameter of the circumcircle of ΔABC. Refer to the left diagram below.



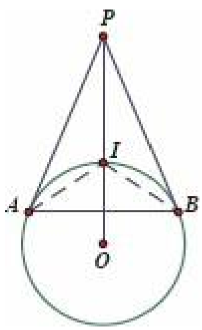


Moreover, we have $AH = 2OM$, where O is the circumcenter of ΔABC. Refer to the right diagram above.

- Let BD, CE be the heights of an acute angled triangle ΔABC. If the line DE intersects the circumcircle of ΔABC at P, Q respectively, we have $AP = AQ$.



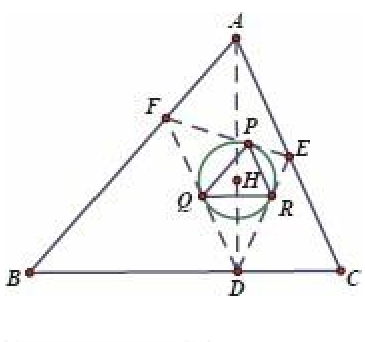
- Let P be a point outside ⊙O and PA, PB touch ⊙O at A, B respectively. V have that the incenter of ΔPAB is the midpoint of the minor arc $\overparen{AB}$.



We shall see how these facts (together with theorems and standard results) could be helpful in problem-solving.

Example 5.1.1 In an acute angled triangle ΔABC, AD, BE, CF are heights. Let the incircle of ΔDEF touch EF, DF, DE at P, Q, R respectively. Show that $\Delta PQR \sim \Delta ABC$.

Proof. Refer to the diagram on the below Let H be the orthocenter of ΔABC. It is well-known that H is also the incenter of ΔDEF. In particular, DH bisects $\angle EDF$ and we have



$$\angle BDF = \angle CDE = \frac{1}{2}\left(180^\circ - \angle EDF\right).$$

Since $DR = DQ$, we have $\angle DQR = \frac{1}{2}(180° - \angle EDF) = \angle BDF$, which implies $BC // QR$. Similarly, $PQ // AB$ and $PR // AC$.

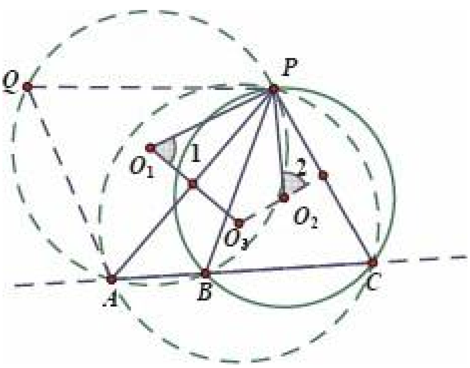
We must have $\angle A = \angle P$, $\angle B = \angle Q$ and hence the conclusion.

□

Note: One may attempt to show $\angle A = \angle P$ by observing that $\angle P = \angle DQR$ and $\angle A = \angle CDE$. Can we show that $\angle DQR = \angle DRQ = \angle CDE$? Notice that **if** we have $\angle DRQ = \angle CDE$, it follows immediately that $BC // QR$ and similarly, $PQ // AB$ and $PR // AC$. Indeed, we **should** have these parallel lines.

Example 5.1.2 (CHN 10) Let ℓ be a straight line and P is a point which does not lie on ℓ. A, B, C are distinct points on ℓ. Let the circumcenters of ΔPAB, ΔPBC, ΔPCA be O_1, O_2, O_3 respectively. Show that P, O_1, O_2, O_3 are concyclic.

Insight. Refer to the diagram on the below. This is indeed a complicated diagram and if we draw all the perpendicular bisectors explicitly, it will be unreadable!



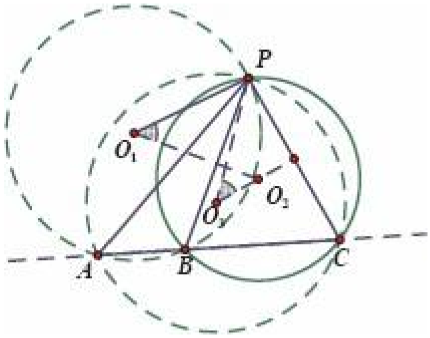
Since we are asked to show P, O_1, O_2, O_3 are concyclic, it is natural to search for equal angles. For example, can we show $\angle 1 = \angle 2$?

Notice that both $\angle 1$ and $\angle 2$ are at the center of a circle. Moreover, it is easy to see that $O_1O_3 \perp PA$ and $O_2O_3 \perp PC$, i.e., $\angle 1 = \frac{1}{2}\angle AO_1P$ and $\angle 2 = \frac{1}{2}\angle CO_2P$. Can we show $\angle AO_1P = \angle CO_2P$?

Notice that $\angle CO_2P = 2\angle CBP$, where $\angle CBP = \angle AQP$ (Corollary 3.1.5) $= \frac{1}{2}\angle AO_1P$ (Theorem 3.1.1). This completes the proof.

Alternatively, one may also show that P, O_1, O_2, O_3 are concyclic via

$\angle PO_1O_2 = \angle PO_3O_2$ Since $O_1O_2 \perp PB$, we have



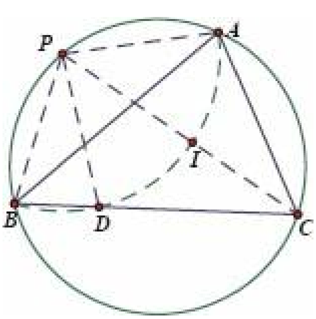
$$\angle PO_1O_2 = \frac{1}{2}\angle PO_1B = \angle PAB.$$

Meanwhile, $\angle PO_3O_2 = \frac{1}{2}\angle PO_3C = \angle PAC$. This completes the proof.

Note: This could be considered a very easy problem if one is familiar with the basic properties in circle geometry, including recognizing the angles needed while disregarding the unnecessary line segments. Indeed, if one decides to show the concyclicity via angle properties, it is natural to consider either of the approaches above.

Example 5.1.3 Given ΔABC and its incenter I, the circumcircles of ΔAIB and ΔAIC intersect BC at D, E respectively. Show that $DE = AB + AC - BC$.

Insight. One recalls that the circumcenter of ΔAIB is indeed the intersection of CI extended with the circumcircle of ΔABC. Refer to the diagram on the below. Let CI extended intersect the circumcircle of ΔABC at P. We have $PA = PB = PI = PD$. Notice that CI bisects $\angle C$.



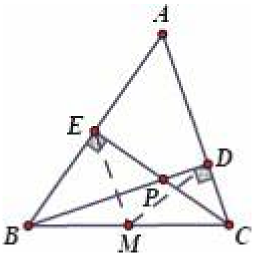
It is not difficult to see $\Delta ACP \cong \Delta DCP$. (Can you show it?) Hence, $AC = CD$. Similarly, $AB = BE$. Now it is easy to see the conclusion because $BE + CD - DE$

= BC. We leave the details to the reader.

Warning: One should **not** conclude $\Delta ACP \cong \Delta DCP$ via $PA = PD$, $\angle DCP = \angle ACP$ and $PC = PC$. This is NOT S.A.S.! Instead, one may show that $\angle PAC = 180° - \angle PBC = 180° - \angle PDB = \angle PDC$ and apply A.A.S.

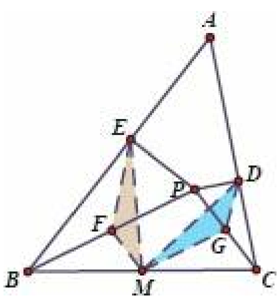
Example 5.1.4 (CMO 11) Let P be a point inside ΔABC such that $\angle PBA = \angle PCA$. Draw $PD \perp AB$ at D and $PE \perp AC$ at E. Show that the perpendicular bisector of DE passes through the midpoint of BC.

Insight. Refer to the diagram on the below. It seems the conclusion is easy to show **if** P is the orthocenter of ΔABC (i.e., when BD and CE intersect at P), in which case we have $ME = MD = \frac{1}{2}BC$ where M is the midpoint of BC. The conclusion follows immediately.



Of course, P may not be the orthocenter of ΔABC, but we **should** still have $MD = ME$. How can we show it? We cannot apply the previous argument since M is not the midpoint of the hypotenuse in a right angled triangle. What if we construct one, say the midpoint of BP?

Proof. Refer to the diagram on the below. Let M be the midpoint of BC. Let F, G be the midpoints of BP, CP respectively.



In the right angled triangle ΔBEP, $EF = \frac{1}{2}BP$.

In ΔBCP, MG is a midline and hence, $MG = \frac{1}{2}BP$ and $MG // BP$. It follows that $EF = MG$.

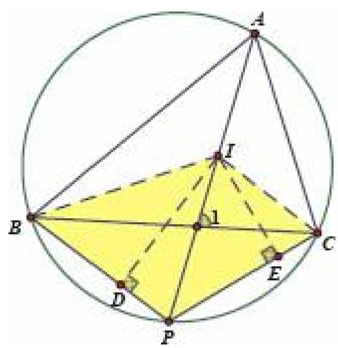
Similarly, $FM // CP$ and $FM = DG$. Now $FPGM$ is a parallelogram.
Notice that $\angle EFM = \angle EFP + \angle PFM = 2\angle PBA + \angle PFM$.
Similarly, $\angle MGD = 2\angle PCA + \angle PGM$. Since $\angle PFM = \angle PGM$ (in the parallelogram $FPGM$) and given that $\angle PBA = \angle PCA$, we must have $\angle EFM = \angle MGD$.
Now $\Delta EFM \cong \Delta MGD$ (S.A.S.), which implies $MD = ME$. It follows that M lies on the perpendicular bisector of DE.

□

Note: The condition $\angle PBA = \angle PCA$ seems not easy to apply at first. We leave it aside. Once we see that ΔEFM and ΔMGD **should** be congruent, it becomes natural to show equal angles using this condition.

Example 5.1.5 Let I be the incenter of ΔABC. AI extended intersects the circumcircle of ΔABC at P. Draw $ID \perp BP$ at D and $IE \perp CP$ at E. Show that $ID + IE = AP \sin \angle BAC$.

Insight. Refer to the diagram on the below. We immediately recall that $PB = PC = PI$. However, one may find it difficult to construct a line segment equal to $ID + IE$. Since ID, IE are heights, perhaps we could use the area method. Notice that



$$[BPCI] = [\Delta BPI] + [\Delta CPI] = \frac{1}{2}BP \cdot ID + \frac{1}{2}CP \cdot IE = \frac{1}{2}BP \cdot (ID + IE).$$

On the other hand, $[BPCI] = \frac{1}{2}BC \cdot PI \sin\angle 1 = \frac{1}{2}BC \cdot BP \sin\angle 1$. (*)

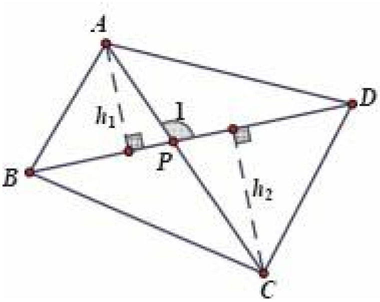
It follows that $ID + IE = BC \sin \angle 1$. Now it suffices to show that $BC \sin \angle 1 = AP \sin \angle BAC$.

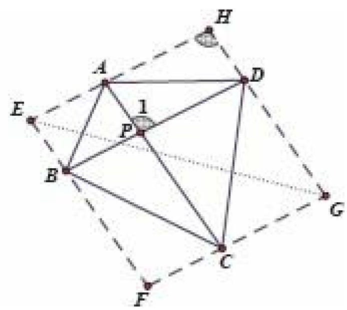
Is it reminiscent of Sine Rule? Shall we show that $\frac{BC}{\sin\angle BAC}=\frac{AP}{\sin\angle 1}$?

Indeed, applying Sine Rule repeatedly gives $\frac{BC}{\sin\angle BAC}=\frac{AB}{\sin\angle ACB}$ $=\frac{AB}{\sin\angle APB}=\frac{AP}{\sin\angle ABP}$.

One sees the conclusion by showing $\angle ABP = \angle 1$. We leave it to the reader to complete the proof. (**Hint**: $\angle PBC = \angle PAC = \angle PAB$.)

Note: We use the fact that $[BPCI]=\frac{1}{2}BC\cdot PI\sin\angle 1$ in (*). Indeed, this holds for a general quadrilateral. Refer to the left diagram below where *AC*, *BD* intersect at *P*. We must have $[ABCD]=\frac{1}{2}AC\cdot BD\sin\angle 1$.



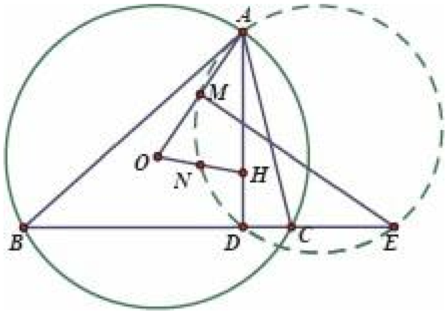


Notice that $[ABCD]=[\Delta ABD]+[\Delta BCD]=\frac{1}{2}BD\cdot h_1+\frac{1}{2}BD\cdot h_2$, where h_1, h_2 are heights from *A*, *C* to *BD* respectively. Notice that $h_1 = AP \sin \angle 1$ and $h_2 = CP \sin \angle 1$ because $\sin \angle 1 = \sin \angle APB$. Hence, $[ABCD]=\frac{1}{2}BD\cdot(h_1+h_2)=\frac{1}{2}BD\cdot(AP+CP)\cdot\sin\angle 1=\frac{1}{2}AC\cdot BD\sin\angle 1$.

Alternatively, one may also draw lines passing through *A*, *C* and parallel to *BD*, and lines passing through *B*, *D* and parallel to *AC*. Refer to the right diagram above. Notice that *EFGH* is a parallelogram. One sees that $[ABCD]=\frac{1}{2}EFGH=[\Delta EGH]=\frac{1}{2}EH\cdot GH\sin\angle H$. Hence, we still have $[ABCD]=\frac{1}{2}AC\cdot BD\sin\angle 1$ since $GH=AC$, $EH=BD$ and $\angle 1=\angle H$.

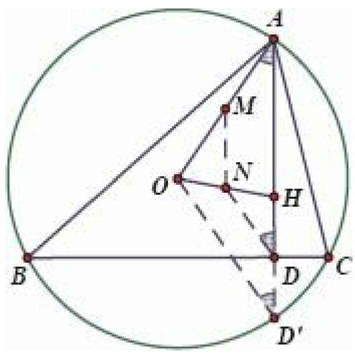
Example 5.1.6 (CWMO 12) In an acute angled triangle ΔABC, *D* is on *BC* such that $AD \perp BC$. Let *O* and *H* be the circumcenter and orthocenter of ΔABC respectively. The perpendicular bisector of *AO* intersects *BC* extended at *E*. Show that the midpoint of *OH* is on the circumcircle of ΔADE.

Insight. Refer to the diagram on the below. Let *N* be the midpoint of *OH*. One sees that *M* lies on the circumcircle of ΔADE. i.e., *A*, *M*, *D*, *E* are concyclic since $\angle AME = \angle ADE = 90°$. We are to show *N* also lies on this circle.



It seems easier to show the concyclicity involving *M* instead of *E*, as we know more about *M* than *E*.

Can we show that *A*, *M*, *N*, *D* are concyclic? Notice that *M*, *N* are both midpoints and we have $MN // AH$. Hence, we **should** have *MNDA* an isosceles trapezium. How can we show it?



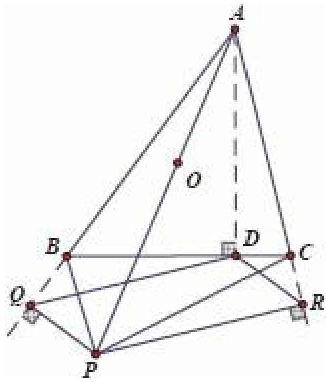
Notice that we have used the condition about *M* and the perpendicular bisector of *AO*, the midpoint *N* and the orthocenter *H*, but we have not used the condition about *O* and the circumcircle. How could we relate *O* and the circumcircle to *A*, *M*, *N* and *D*?

Recall that if *AD* extended intersects the circumcircle at *D*'. we have *DH* = *DD*'. Refer to the diagram above.

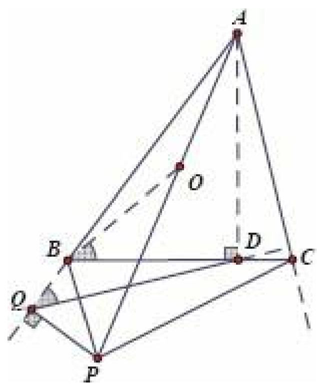
It follows that $DN // OD'$. This implies $\angle ADN = \angle AD'O = \angle OAD$. Hence, *ADNM* is an isosceles trapezium and the conclusion follows.

Example 5.1.7 Given a non-isosceles acute angled triangle ΔABC, *O* is its circumcenter. *P* is a point on *AO* extended such that $\angle BPA = \angle CPA$. Refer to the diagram on the below. Draw $PQ \perp AB$ at *Q*, $PR \perp AC$ at *R* and $AD \perp$

BC at *D*. Show that *PQDR* is a parallelogram.



Insight. It is natural to consider showing PQ // DR and PR // DQ. Given that $PQ \perp AB$ and $PR \perp AC$, it suffices to show $DQ \perp AC$ and $DR \perp AB$. Let us focus on one of them, say $DQ \perp AC$: most probably a similar argument applies for the other. How can we show that $\angle BAC + \angle AQD = 90°$?

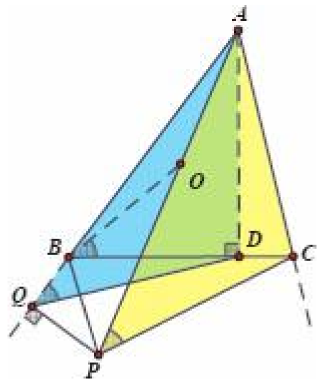


Recall $\angle BAC = 90° - \angle OBC$, i.e., it suffices to show $\angle AQD = \angle OBC$. Refer to the diagram above. How are these two angles related? It seems not very clear.

On the other hand, we are given $\angle BPA = \angle CPA$. How can we use this condition? Can you see that *O*, *B*, *P*, *C* are concyclic (Example 3.1.11)? Now we have $\angle OBC = \angle APC$. Can we show $\angle APC = \angle AQD$?

One may also notice that $\angle CAO = \angle BAD$. It seems that we **should** have $\Delta PAC \sim \Delta QAD$. Refer to the diagram below.

Proof. It is easy to see $\angle CAD = \angle BAO$ (Example 3.4.1). Hence, $\Delta AQP \sim \Delta ADC$ and $\frac{AQ}{AP} = \frac{AD}{AC}$.



Notice that $\angle CAO = \angle BAD$. We must have $\Delta QAD \sim \Delta PAC$. It follows that $\angle AQD = \angle APC$.

Since ΔABC is non-isosceles, we must have $PB \neq PC$. Otherwise OP is the perpendicular bisector of BC, which implies $AB = AC$. It follows that O, B, P, C are concyclic (Example 3.1.11), which implies that $\angle OBC = \angle APC = \angle AQD$. Notice that $\angle OBC + \angle BAC = 90°$ because O is the circumcenter of ΔABC. It follows that $\angle AQD + \angle BAC = 90°$, which implies that $DQ \perp AC$, i.e., $DQ \,//\, PR$.
Similarly, $DR \,//\, PQ$ and the conclusion follows.

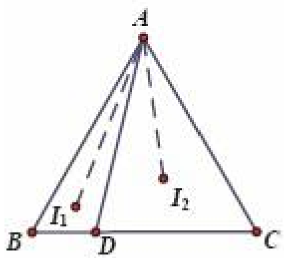
□

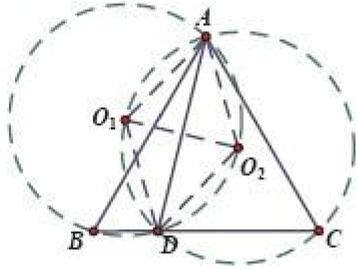
Note: One sees that familiarity with basic facts in geometry is important in solving this problem.

Example 5.1.8 In an equilateral triangle ΔABC, D is a point on BC. Let O_1, I_1 be the circumcenter and incenter of ΔABD respectively, and O_2, I_2 be the circumcenter and incenter of ΔACD respectively. If the lines O_1I_1 and O_2I_2 intersect at P, show that D is the circumcenter of ΔO_1PO_2.

Insight. Apparently, the construction of the diagram is not simple. Perhaps we shall consider the circumcenters and incenters separately.

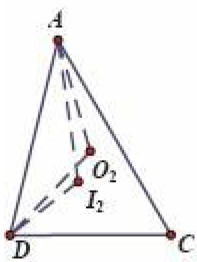
Refer to the following diagrams. Can you see $\angle I_1AI_2 = 30°$? Can you see $\odot O_1$ and $\odot O_2$ have the same radius (by Sine Rule) and hence, AO_1DO_2 is a rhombus?

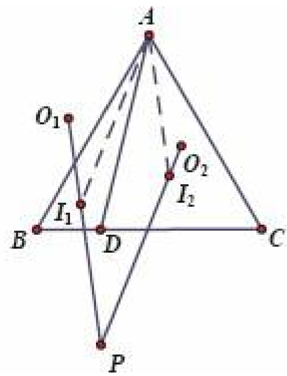




If we focus on one triangle, say ΔACD with its incenter and circumcenter, we have $\angle AO_2D = 2\angle C$ and $\angle AI_2D = 90° + \frac{1}{2}\angle C$.

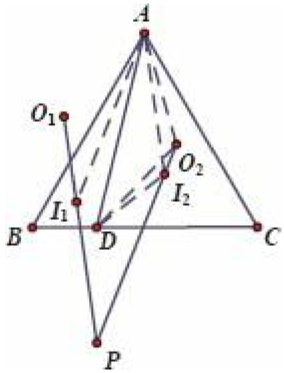
But these two angles are the same since $\angle C = 60°$! This implies A, O_2, I_2, D are concyclic. Refer to the left diagram below.





One sees that ΔO_1O_2D is an equilateral triangle. Hence it suffices to show that $\angle P = 30°$. Refer to the previous right diagram. We **should** have $\angle P = \angle I_1 AI_2$. It *seems* that AI_1PI_2 is a parallelogram. Can we show it? (We have not used the concyclicity of A, O_2, I_2, D.)

Proof. Since O_2, I_2 are the circumcenter and A incenter of ΔACD respectively, we have $\angle AO_2D = 2\angle C$ and $\angle AI_2D = 90° + \frac{1}{2}\angle C$.



It follows that $\angle AO_2D = \angle AI_2D = 120°$ because $\angle C = 60°$. Hence, A, O_2, I_2, D are concyclic and we have $\angle AI_2O_2 = \angle ADO_2 = 30°$ (because $AO_2 = DO_2$ and $\angle AO_2D = 120°$).

Notice that $\angle I_1AI_2 = \frac{1}{2}\angle BAC = 30°$, which implies $\angle I_1AI_2 = \angle AI_2O_2$, i.e., $AI_1 \parallel O_2P$. Similarly, $AI_2 \parallel O_1P$ and AI_1PI_2 is a parallelogram.

In particular, $\angle P = \angle I_1AI_2 = 30°$.

On the other hand, let the circumradius of ΔABD and ΔACD be r_1, r_2 respectively. By Sine Rule, $\frac{AB}{\sin\angle ADB} = 2r_1$ and $\frac{AC}{\sin\angle ADC} = 2r_2$.

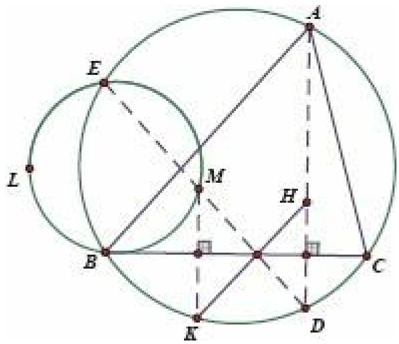
Notice that $\sin\angle ADB = \sin\angle ADC$ (because $\angle ADB = 180° - \angle ADC$) and $AB = AC$. It follows that $r_1 = r_2$ and hence, AO_1DO_2 is a rhombus. In particular, ΔO_1O_2D is an equilateral triangle.

Now $\angle P = 30° = \frac{1}{2}\angle O_1DO_2$ implies that P lies on the circle centered at D with radius O_1D. This completes the proof.

□

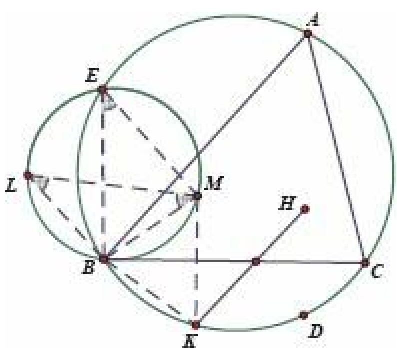
Example 5.1.9 (EGMO 12) Given an acute angled triangle ΔABC, its circumcircle Γ and orthocenter H, K is a point on the minor arc $\overset{\frown}{BC}$. Let L be the reflection of K about the line AB and M be the reflection of K about the line BC. The circumcircle of ΔBLM intersects Γ at B and E. Show that the lines KH, EM and BC are concurrent.

Insight. Refer to the diagram on the below. It seems not easy to show the concurrency using Ceva's Theorem. However, we notice that H and D are L symmetric about BC, where D is the intersection of AH extended and Γ. On the other hand, it is given that M and K are symmetric about BC.



Now it is easy to see that *MD*, *KH* and *BC* are concurrent, because *BC* is the perpendicular bisector of *HD* and *MK*, where *HD* // *MK*. Since we are to show the lines *KH*, *EM* and *BC* are concurrent, it suffices to show that *E*, *M*, *D* are collinear. Notice that there are many equal angles in the diagram due to the two circles and the symmetry of *K*, *L* and *K*, *M*. Is there any angle related to say the point *E*?

How about $\angle BEM$? One sees immediately that $\angle BEM = \angle BLM$. Refer to the diagram on the below. Since *L*, *M* are reflections of *K* about *AB*, *BC* respectively, we have $BK = BM = BL$. It follows that $\angle BLM = \angle BML$. Can we show that $\angle BAD = \angle BED = \angle BEM$?



Unfortunately, neither $\angle BLM$ nor $\angle BML$ seems directly related to $\angle BAD$. Perhaps we can write $\angle BLM = 90° - \frac{1}{2}\angle MBL$. Notice that $\angle MBL = \angle ABL + \angle ABM$ and these angles, after applying the reflections, might be related to $\angle BAD$.

Proof. Let *AH* extended intersect Γ at *D*. We know that *D* is the reflection of *H* about *BC*. Since *M* is the reflection of *K* about *BC*, *BC* is the perpendicular bisector of both *MK* and *HD*. Hence, *MK* // *HD*. Now *DHMK* is an isosceles trapezium and it is easy to see that *KH*, *DM*, *BC* are concurrent. We claim that *E*, *M*, *D* are collinear.

Since L, M are reflections of K about AB, BC respectively, one sees that $BK = BM = BL$, which implies $\angle BLM = \angle BML$.

Now we have $\angle BEM = \angle BLM = 90° - \frac{1}{2}\angle MBL$

$$= 90° - \frac{1}{2}(\angle ABL + \angle ABM) = 90° - \frac{1}{2}[\angle ABK + (\angle ABC - \angle CBM)]$$

$$= 90° - \frac{1}{2}[(\angle ABC + \angle CBK) + (\angle ABC - \angle CBK)]$$

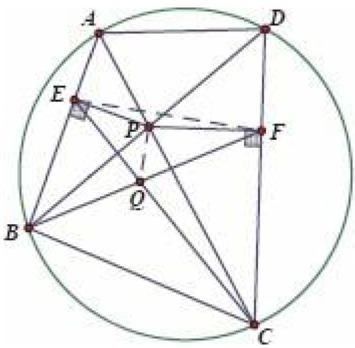
$= 90° - \angle ABC = \angle BAD = \angle BED$. Hence, E, M, D are collinear. We conclude that KH, EM and BC are concurrent.

□

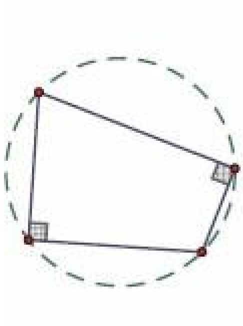
Note: One may find it difficult to show that E, M, D are collinear by $\angle BME + \angle BMD = 180°$. Indeed, we do not know much about $\angle BMD$ or $\angle BKH$ because K is an arbitrary point.

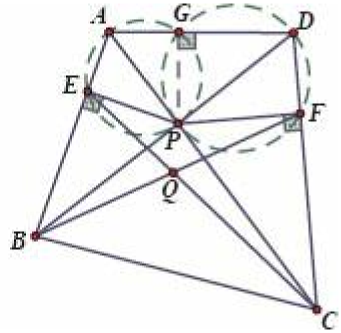
Example 5.1.10 (USA 12) Let $ABCD$ be a cyclic quadrilateral whose diagonals AC, BD intersect at P. Draw $PE \perp AB$ at E and $PF \perp CD$ at F. BF and CE intersect at Q. Show that $PQ \perp EF$.

Insight. Refer to the diagram on the below. Apparently, the construction of the diagram is straightforward, but it is not clear how we could show $PQ \perp EF$. Even if we extend QP, intersecting EF, it seems difficult to find the angles at the B intersection. Perhaps we shall leave the conclusion aside and study the diagram further.

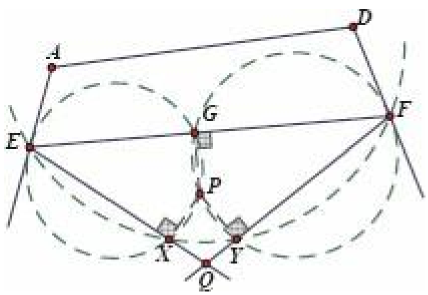


Notice that E and F are introduced by perpendicular lines. $ABCD$ is cyclic. If we introduce more perpendicular lines, we should obtain more concyclicity by the right angles. Refer to the left diagram below.





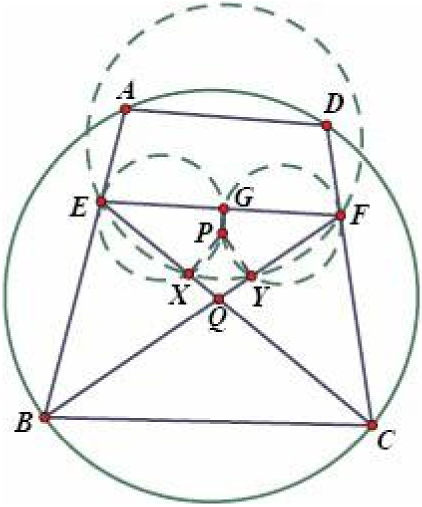
Let us draw say $PG \perp AD$ at G. Refer to the right diagram above. We immediately obtain two circles, i.e., A, E, P, G and D, F, P, G are concyclic. Even though this seems not directly related to our conclusion, it gives us an inspiration: what if we draw $PG \perp EF$ at G instead? Perhaps we could still obtain concyclicity and it would suffice to show that P, G, Q are collinear, or PG passes through Q. Refer to the diagram below.

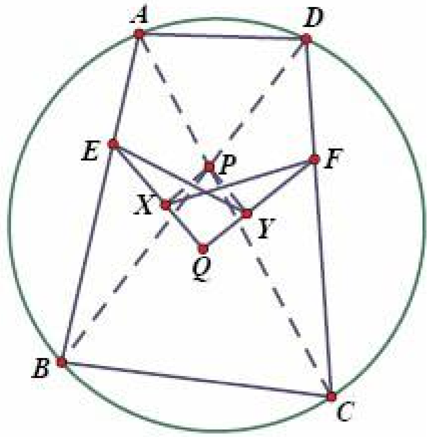


Since BF, CE intersect at Q, it suffices to show PG, BF, CE are concurrent. Is it reminiscent of radical axes? Let Γ_1, Γ_2 denote the circumcircles of ΔEGP and ΔFGP respectively. We see that PG is the radical axis of Γ_1, Γ_2.

If we can find another circle Γ_3 such that EQ, FQ are the radical axes of Γ_1, Γ_3 and Γ_2, Γ_3 respectively, the conclusion follows. Let Γ_1 intersect the line EQ at X and Γ_2 intersect the line FQ at Y. It is easy to see that X, Y are the feet of the perpendiculars from P to EQ, FQ respectively. Can we show that E, X, Y, F are concyclic? This should not be difficult to as we have an abundance of concyclicity in the diagram (for example, B, E, P, Y are concyclic because $\angle BEP = \angle BYP = 90°$) and hence, many pairs of equal angles.

***Proof*.** Draw $PG \perp EF$ at G, $PX \perp CE$ at X and $PY \perp BF$ at Y. Clearly, E, G, P, X are concyclic and F, G, P, Y are concyclic. We claim that E, X, Y, F are concyclic, which implies the radical axes EX, FY and PG are concurrent at Q (Theorem 4.3.2) and hence, leads to the conclusion. Refer to the left diagram below.





It suffices to show $\angle EXF = \angle EYF$. Since $\angle PXE = \angle PYF = 90°$, it suffices to show $\angle PXF = \angle PYE$. Refer to the right diagram above. Notice that $\angle BEP = \angle BYP = 90°$. Hence, B, E, P, Y are concyclic and $\angle PYE = \angle ABD$. Similarly, $\angle PXF = \angle ACD$. This completes the proof as $\angle ABD = \angle ACD$ (angles in the same arc).

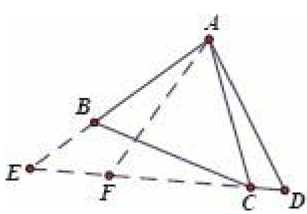
□

5.2 Basic Techniques

Knowing the basic facts and important theorems well is important for solving geometry problems, but is still insufficient. In fact, it is common to see beginners who diligently learn many theorems, but do not know how to apply those results and solve geometry problems. Indeed, many beginners are not aware of the commonly used *techniques* (instead of theorems), which are not found in most textbooks.

The following is an elementary example: **NO** advanced knowledge is required to solve this problem. Can you see the clues without referring to the solution?

Example 5.2.1 Given a quadrilateral $ABCD$ where $AD = BC$ and $\angle BAC + \angle ACD = 180°$, show that $\angle B = \angle D$.



Insight. It seems not easy to apply the condition $\angle BAC + \angle ACD = 180°$ since the angles are far apart. Can we put them together? If we extend the line *CD*, say the lines *AB* and *CD* intersect at *E*, can you see that we obtain an isosceles triangle?
If $\angle BAC = \angle ACD = 90°$, it is easy to see that *ABCD* is a parallelogram and we have $\angle B = \angle D$ immediately. Otherwise, say without loss of generality that $\angle BAC < 90°$, *AB* extended and *DC* extended intersect at *E*. Refer to the diagram above. We have $AE = CE$. It seems not clear how $AD = BC$ leads to the conclusion because they are far apart. Can we put them together? If we draw $AF = BC$, where *F* is on *DC* extended, we obtain an isosceles trapezium!

Proof. If $\angle BAC = \angle ACD = 90°$, we have $\Delta BAC \cong \Delta DCA$ (H.L.) and hence, *ABCD* is a parallelogram and $\angle B = \angle D$.
Suppose $\angle BAC < 90°$. Let *DC* extended and *AB* extended intersect at *E*. Since $\angle BAC + \angle ACD = 180°$, we have $\angle BAC = \angle ECA$ and $AE = CE$. Choose *F* on the line *CD* such that $AF = AD$. We have $\angle D = \angle AFD$. Now $BC = AD = AF$ gives $\Delta ABC \cong \Delta CFA$ (S.A.S.). It follows that $\angle B = \angle AFD = \angle D$.
If $\angle BAC > 90°$, the lines *AB* and *CD* intersect at the other side of *AC* and a similar argument applies.

□

Note: We used "cut and paste" to find clues in this problem: since $\angle BAC$ and $\angle ACD$ are supplementary, if we put them together, a straight line is obtained. We also put the line segments *AD*, *BC* together, which gives an isosceles trapezium. Notice that simply applying any theorem directly to this problem will not give the conclusion.

Basic and commonly used techniques in solving geometry problems include the following:

- Cut and paste

  When given equal line segments, equal or supplementary angles, and sum of angles or line segments which are far apart, one may cut and paste, moving those angles or line segments together. This technique may give straight lines, isosceles triangles or congruent triangles.

- Construct congruent and similar triangles.

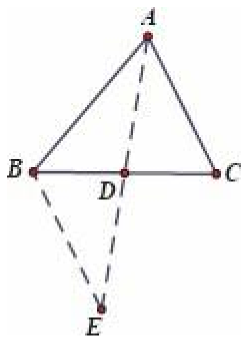
  One strategy to show equal angles or line segments is to place them in congruent or similar triangles. If no such triangles exist in the diagram, consider drawing auxiliary lines and construct one! Notice that any other angles or line segments known to be equal may give inspiration on which triangles **could** be congruent or similar.

- Reflection about an angle bisector

  When given an angle bisector, it is naturally a line of symmetry. Reflecting about the angle bisector may bring angles and line segments together and hence, it may be an effective technique besides "cut and paste".

- Double the median.

  Refer to the diagram on the below. Given ΔABC and its median AD, extending AD to E with $AD = DE$ gives ΔABE where $BE = AC$ and $\angle ABE = 180° - \angle A$.

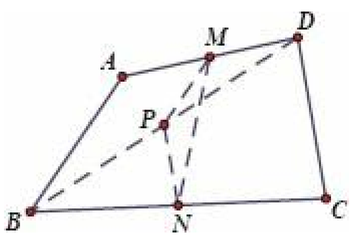


  Hence, $\sin \angle A = \sin \angle ABE$ and $[\Delta ABC] = [\Delta ABE]$.

  Moreover, (twice) the median of ΔABC becomes a side of ΔABE. This may be a useful technique when constructing congruent and similar triangles.

- Midpoints and Midpoint Theorem

  When midpoints are given, it is natural to apply the Midpoint Theorem, which not only gives parallel lines, but also moves the (halved) line segments around. In particular, if connecting the midpoints does not give a midline of the triangle, one may choose more midpoints and draw the midlines. Refer to the diagram below.



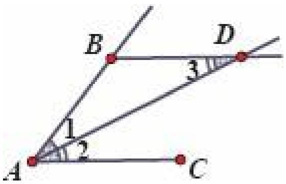
Given a quadrilateral $ABCD$ where M, N are the midpoints of AD, BC respectively, simply connecting MN does not give any conclusion. If we choose P, the midpoint of BD, then $PM = \frac{1}{2}AB$ and $PN = \frac{1}{2}CD$.

If we know more about AB and CD, say $AB = CD$, then we conclude that ΔPMN is an isosceles triangle.

On the other hand, if midpoints are given together with right angled triangles, one may consider the median on the hypotenuse. Example 5.1.4 illustrates this technique.

- Angle bisector plus parallel lines

One may easily see an isosceles triangle from an angle bisector plus parallel lines. Refer to the diagram on the below. If AD bisects $\angle A$, we have $\angle 1 = \angle 2$.
If $AC // BD$, $\angle 2 = \angle 3$. It follows that $AB = BD$.



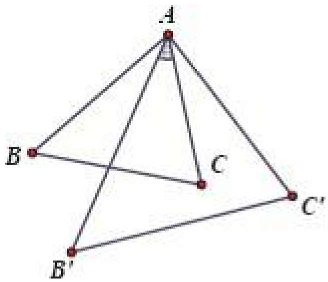
Notice that this technique could also be applied reversely. In the diagram above, if we know $AB = BD$, then by showing $AC // BD$, we conclude that AD bisects $\angle A$.

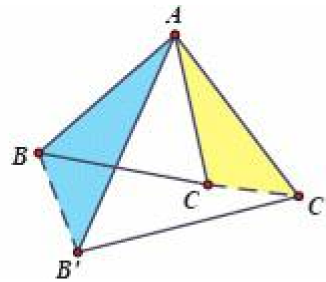
- Similar triangles sharing a common vertex

A pair of similar triangles sharing a common vertex may immediately give another pair of similar triangles. Refer to the following diagrams where $\Delta ABC \sim \Delta AB'C'$.

Since $\frac{AB}{AB'} = \frac{AC}{AC'}$ and $\angle BAC = \angle B'AC'$, by subtracting $\angle B'AC$, we see that $\angle BAB' = \angle CAC'$. It follows that $\Delta ABB' \sim \Delta ACC'$.

Notice that this technique applies for the inverse as well. If we have $\Delta ABB' \sim \Delta ACC'$, we may also conclude that $\Delta ABC \sim \Delta AB'C'$.





One may recall that we applied this technique in the proof of Ptolemy's Theorem, as well as in Example 5.1.7.

- Angle-chasing

This is an elementary but effective technique when we explore angles related to a circle, especially when an incircle or circumcircle of a triangle is given (because the incenter and circumcenter give us even more equal angles). If more than one circle is given, it is a basic technique to apply the angle properties repeatedly and identify equal angles far apart or apparently unrelated. Indeed, experienced contestants are very familiar with the angle properties and are sharp in observing and catching equal angles. (For example, can you write down the proof of Simson's Line quickly?)

However, one should avoid long-winded angle-chasing which leads nowhere. If that happens, one may seek clues from the line segments instead, say identifying similar triangles, or applying the Intersecting Chords Theorem and the Tangent Secant Theorem.

- Watch out for right angles.

When right angles are given, it is worthwhile to spend time and effort digging out more information about them, because right angles may lead to a number of approaches:

(1) If a right angled triangle with a height on the hypotenuse is given, we will have similar triangles.
(2) If there are other heights or the orthocenter of a triangle, we may find parallel lines.
(3) One may see concyclicity when a few right angles are given.
(4) If a right angle is extended on the circumference of a circle, it corresponds to a diameter of the circle.

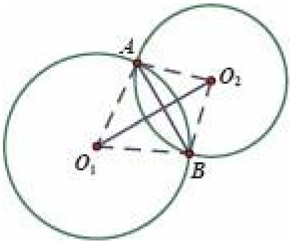
One should always refer to the context of the problem and determine which approach might be effective.

- Perpendicular bisector of a chord

  Introducing a perpendicular from the center of a circle to a chord is a simple technique but occasionally, it may be decisively useful. Notice that the perpendicular bisector gives both right angles and the midpoint of the chord.

- Draw a line connecting the centers of two intersecting circles.

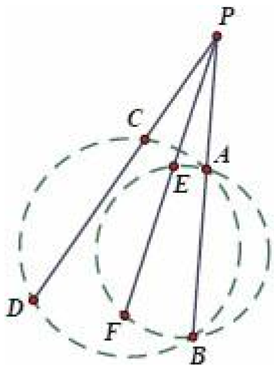
  This is a very basic technique where the line connecting the centers of the two circles is a line of symmetry.
  Refer to the diagram on the below. Notice that $O_1O_2 \perp AB$ and O_1O_2 is the angle bisector of both $\angle AO_1B$ and $\angle AO_2B$. Even though this is an elementary result, one may apply it to solve difficult problems.

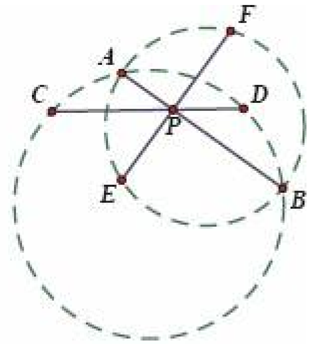


  Example 3.2.13 illustrates this technique. It is noteworthy that beginners tend to overlook this elementary property during problem-solving, especially when the diagram is complicated.

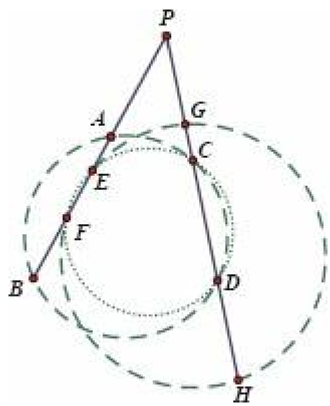
- Relay: Tangent Secant Theorem and Intersecting Chords Theorem

  When more than one circle is given and there is a common chord or concurrency, one may apply the Tangent Secant Theorem or the Intersecting Chords Theorem repeatedly to acquire more concyclicity. Refer to the diagrams below. Can you see C, D, E, F are concyclic in both diagrams? Can you see that $PC \cdot PD = PA \cdot PB = PE \cdot PF$?



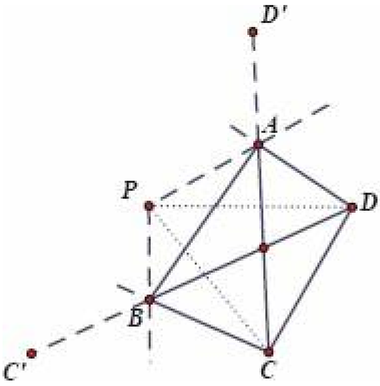


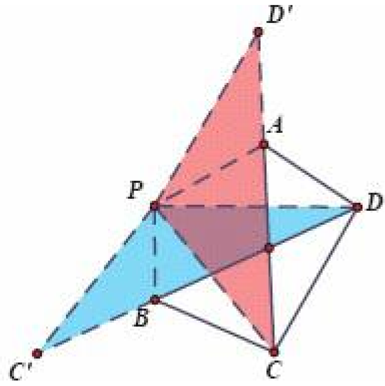
Refer to the diagram on the below. If *A*, *B*, *C*, *D* are concyclic, *C*, *D*, *E*, *F* are concyclic and *E*, *F*, *G*, *H* are concyclic, can you see that *A*, *B*, *G*, *H* are concyclic? (**Hint**: $PA \cdot PB = PC \cdot PD = PE \cdot PF = PG \cdot PH$.)



One may recall that we applied these basic techniques extensively when solving problems in the previous chapters. We shall illustrate these techniques with more examples in this section.

Example 5.2.2 (ITA 11) Given a quadrilateral *ABCD*, the external angle bisectors of $\angle CAD$, $\angle CBD$ intersect at *P*. Show that if $AD + AC = BC + BD$, then $\angle APD = \angle BPC$.





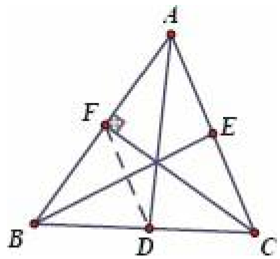
Insight. Refer to the left diagram above, where *AP*, *BP* are the external angle bisectors of $\angle CAD$, $\angle CBD$ respectively. How can we apply the condition $AD + AC = BC + BD$? Cut and paste!

Extend *DB* to *C*' such that $BC' = BC$ and extend *CA* to *D*' such that $AD = AD'$. Can you see that *C*, *C*' are symmetric about the line *PB*, and *D*, *D*' are symmetric about the line *PA*? (**Hint**: $\Delta BCC'$ is an isosceles triangle and *PB* is the perpendicular bisector of *CC*'.) Now $BC = BC'$ and $AD = AD'$. Refer to the right diagram above. Can you see that $AD + AC = BC + BD$ implies $CD' = C'D$? Can you see that $PC = PC'$, $PD = PD'$ and hence, $\Delta PC'D \cong \Delta PCD'$?

Now $\angle C'PD = \angle CPD'$ and the conclusion follows. We leave the details to the reader.

Example 5.2.3 (GER 08) Given an acute angled triangle ΔABC, *AD* is the angle bisector of $\angle A$, *BE* is a median and *CF* is a height. Show that *AD*, *BE*, *CF* are concurrent if and only if *F* lies on the perpendicular bisector of *AD*.

Insight. We are to show $AF = DF$ if and only if *AD*, *BE*, *CF* are concyclic. Since *AD* bisects $\angle A$, the isosceles triangle ΔADF gives $AC \,//\, DF$. Refer to the diagram on the below. How can we show the concurrency?



What if we apply Ceva's Theorem to the height *CF*, the median *BE* and the angle bisector *AD*? By the Angle Bisector Theorem and $AE = CE$, we may obtain the ratio of line segments leading to $AC \,//\, DF$.

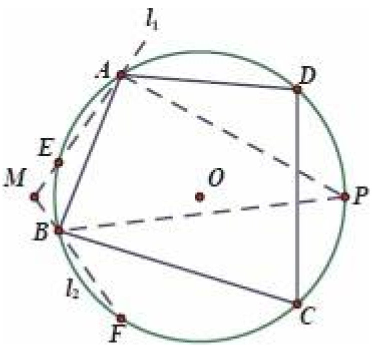
***Proof*.** By Ceva's Theorem, *AD*, *BE*, *CF* are concurrent if and only if $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$. Since *BE* is a median, it is equivalent to $\frac{AF}{BF}=\frac{CD}{BD}$, or $DF \parallel AC$.

We claim that $DF \parallel AC$ if and only if $AF = DF$. In fact, since *AD* bisects $\angle A$, $DF \parallel AC$ if and only if $\angle ADF = \angle CAD = \angle BAD$, which is equivalent to $AF = DF$. In conclusion, *AD*, *BE*, *CF* are concurrent if and only if $AF = DF$, i.e., *F* lies on the perpendicular bisector of *AD*.

□

Example 5.2.4 (BRA 08) Given a quadrilateral *ABCD* inscribed inside ⊙*O*, draw lines ℓ_1, ℓ_2 such that ℓ_1 and the line *AB* is symmetric about the angle bisector of $\angle CAD$, and ℓ_2 and the line *AB* is symmetric about the angle bisector of $\angle CBD$. If ℓ_1 and ℓ_2 intersect at *M*, show that $OM \perp CD$.

***Insight*.** It is easy to see that the angle bisectors of $\angle CAD$ and $\angle CBD$ meet at the midpoint of the arc $\overset{\frown}{CD}$, say *P*. Refer to the diagram on the below. Notice that the reflections ℓ_1, ℓ_2 and the circle give a lot of equal angles.



How can we show $OM \perp CD$?
It may not be wise to find the angle directly because we do not know where *OM* and *CD* intersect. Shall we explore the angles around the circles and seek more clues? If for example *OM* is the perpendicular bisector of *AF* (i.e., $AM = FM$), then it suffices to show $AF \parallel CD$.

***Proof*.** Let *P* be the midpoint of $\overset{\frown}{CD}$. Clearly, *AP*, *BP* are the angle bisectors of $\angle CAD$, $\angle CBD$ respectively.

Let ℓ_1 and ℓ_2 intersect ⊙*O* at *A*, *E* and *B*, *F* respectively.

Since ℓ_1 and *AB* are symmetric about *AP*, we must have $\angle BAP = 180° -$

$\angle EAP = \angle ECP$ (because A, E, C, P are concyclic). (1)

Since P is the midpoint of $\overset{\frown}{CD}$, we have $\angle PCD = \angle PAC$. (2)

(1) and (2) imply that $\angle BAP - \angle PAC = \angle ECP - \angle PCD$, which gives $\angle BAC = \angle DCE$, i.e., $\overset{\frown}{BC}$ and $\overset{\frown}{DE}$ extend the same angle on the circumference. This implies $BC = DE$ and hence, $BCDE$ is an isosceles trapezium with $BE // CD$.

Since ℓ_2 and AB are symmetric about BP, a similar argument applies which gives $AF // CD$ and $ADCF$ is an isosceles trapezium. Now it is easy to see that $AEBF$ is also an isosceles trapezium. Notice that $AM = MF$ and hence, OM is the perpendicular bisector of AF. Since $AF // CD$, we must have $OM \perp CD$.

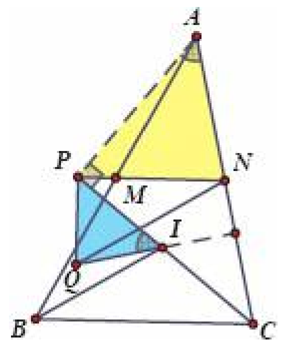
□

Note: If the diagram becomes complicated and the angles on the circumference do not give clear insight, it might be easier to consider the corresponding arcs. Notice that we showed $\overset{\frown}{BC} = \overset{\frown}{DE}$ in the proof above, which simplifies the argument. In fact, one would easily see the isosceles trapeziums via equal arc lengths.

Example 5.2.5 Let I be the incenter of ΔABC. M, N are the midpoints of AB, AC respectively. NM extended and CI extended intersect at P. Draw $QP \perp MN$ at P such that $QN // BI$. Show that $QI \perp AC$.

Insight. Refer to the following diagram. Notice that the angle bisector CI and the parallel lines $BC // MN$ give $PN = CN$.

Since $CN = \frac{1}{2}AC$, we must have $\angle APC = 90°$.



Given $PQ \perp PN$, i.e., $\angle QPN = 90° = \angle APC$, one immediately sees that $\angle QPI = \angle APN$. Since we are to show $QI \perp AC$, we **should** have $\angle PIQ = 90° - \angle ACI = \angle PAC$.

Hence, we **should** have $\Delta APN \sim \Delta IPQ$.

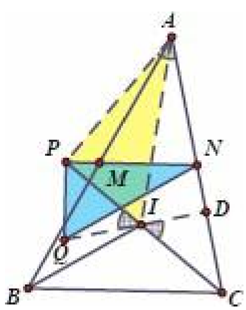
Can we show it, say via $\frac{AP}{PN}=\frac{IP}{PQ}$? Notice that AP, PN, IP, PQ are the sides of the right angled triangles ΔAPI and ΔPQN. Indeed, if we can show that $\Delta API \sim \Delta NPQ$, it follows that $\Delta APN \sim \Delta IPQ$.

We have not used the condition $QN \parallel BI$ yet. Perhaps this could help us to find an equal pair of angles in ΔAPI and ΔNPQ.

Proof. Since CI bisects $\angle C$ and $BC \parallel MN$, we have $\angle NCP = \angle BCP = \angle NPC$, i.e., $PN = CN$. Since N is the midpoint of AC, we have $PN = AN = CN$ and hence, $\angle APC = 90°$ (Example 1.1.8).

Since I is the incenter of ΔABC, we have $\angle AIC = 90° + \frac{1}{2}\angle ABC$ and hence, $\angle AIP = 180° - \angle AIC = 90° - \frac{1}{2}\angle ABC = 90° - \angle CBI$.

Notice that $\angle CBI = \angle PNQ$ (because $MN \parallel BC$ and $BI \parallel QN$). Hence, $\angle AIP = 90° - \angle PNQ = \angle PQN$. Since $\angle APC = \angle QPN = 90°$, we must have $\Delta API \sim \Delta NPQ$. Refer to the diagram below.



Now we have $\frac{AP}{PN}=\frac{IP}{PQ}$ and $\angle QPI = \angle APN$.

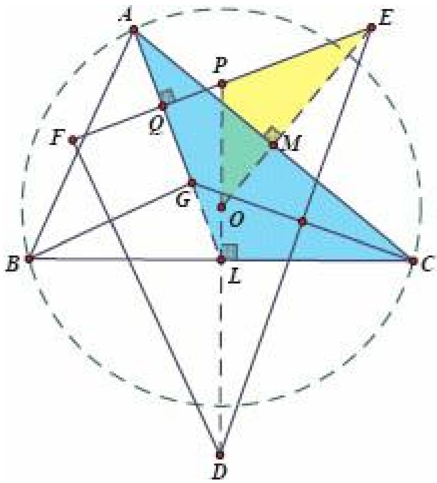
It follows that $\Delta APN \sim \Delta IPQ$.

Let QI extended intersect AC at D. We have $\angle CID = \angle PIQ = \angle PAC = 90° - \angle ACI$, i.e., $\angle CDI = 90°$. This completes the proof.

□

Example 5.2.6 (HEL 09) Let O, G denote the circumcenter and the centroid of ΔABC respectively. Let the perpendicular bisectors of AG, BG, CG intersect mutually at D, E, F respectively. Show that O is the centroid of ΔDEF.

Insight. Refer to the diagram below. What can we say about O and ΔDEF ? O is the circumcenter of ΔABC where ΔDEF is constructed by the perpendicular bisectors of AG, BG, CG. Can you see the *link* between perpendicular bisectors and circumcenters? Indeed, one immediately concludes that D, E, F are the circumcenters of ΔBCG, ΔACG, ΔABG respectively.



How can we show that O is the centroid of ΔDEF ? Let DO extended intersect EF at P. If we can show that $EP = FP$ (which **should** be true), perhaps it is similar to show that EO, FO pass through the midpoints of DF, DE respectively. Notice that we have many right angles in the diagram, which give a lot of concyclicity.

Let L be the midpoint of BC. Can you see that $\angle CAL = \angle OEP$ and $\angle ACB = \angle EOP$? What can you say about ΔACL and ΔEOP ? How about ΔABC and ΔDEF ?

Proof. It is easy to see that D, E, F are the circumcenters of ΔBCG, ΔACG, ΔABG respectively. Let L, M, N be the midpoints of BC, AC, AB respectively. Notice that the lines DL, EM, FN are the perpendicular bisectors of BC, AC, AB respectively and hence, intersect at O. Let DL extended intersect EF at P. We claim that P is the midpoint of EF.

Let AG intersect EF at Q. Since $AG \perp EF$ and $EM \perp AC$, A, E, M, Q are concyclic and hence, $\angle CAL = \angle OEP$. (1)

Since $\angle CLO + \angle CMO = 180°$, we also have C, L, O, M concyclic and hence, $\angle ACL = \angle EOP$. (2)

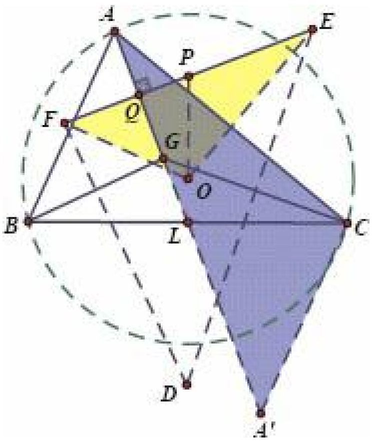
(1) and (2) give $\Delta ACL \sim \Delta EOP$ and hence, $\frac{EP}{AL}=\frac{OP}{CL}$. (3)

Similarly, one sees that $\Delta ABL \sim \Delta FOP$ and hence, $\frac{FP}{AL}=\frac{OP}{BL}$. (4)

(3) and (4) imply $EP = FP$, i.e., DO extended passes through the midpoint of EF. Similarly, EO extended and FO extended pass through the midpoints of DF and DE respectively. We conclude that O is the centroid of ΔDEF.

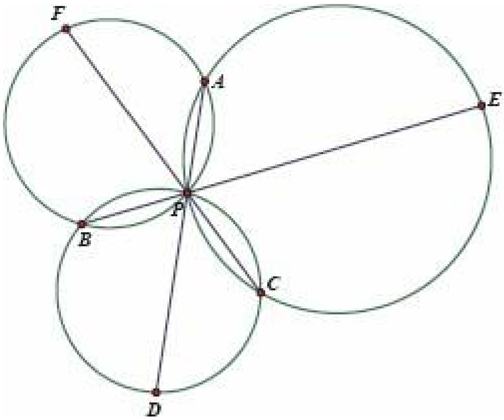
□

Note: Even though we did not explicitly double the median in the proof above, it is essentially the technique we applied. Refer to Example 1.2.11, where ΔABC and ΔAEF are related in a similar manner as ΔABC and ΔOEF in this example. Refer to the diagram on the below. If we extend AL to A' such that $AL = AL'$, can you see $\Delta ACA' \sim \Delta EOF$? Notice that P and L are corresponding points because $\angle ACL = \angle EOP$.



One may also show that $\Delta DEP \sim \Delta CGL$ and $\Delta DFP \sim \Delta BGL$, which also leads to the conclusion.

Example 5.2.7 Let Γ_1, Γ_2, Γ_3 be three circles such that Γ_1, Γ_2 intersect at A and P, Γ_2, Γ_3 intersect at C and P, and Γ_1, Γ_3 intersect at B and P. Refer to the following diagram. If AP extended intersects Γ_3 at D, BP extended intersects Γ_2 at E and CP extended intersects Γ_1 at F, show that that $\frac{AP}{AD}+\frac{BP}{BE}+\frac{CP}{CF}=1$.

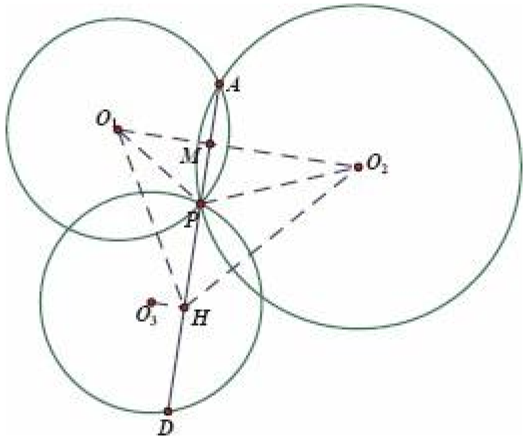


Insight. We focus on $\frac{AP}{AD}$ first. Since we do not have much information about the line segments, we may consider re-writing the ratio by areas of triangles.

For example, $\frac{AP}{AD}=\frac{[\Delta APF]}{[\Delta ADF]}$.

However, applying this to $\frac{BP}{BE}$ and $\frac{CP}{CF}$ gives ratios of no common denominator and hence, it is not easy to calculate the sum.

Perhaps we should use the triangles independent of *AD*, *BE*, *CF*. Notice that *AP*, *BP*, *CP* are common chords of circles. How about connecting the centers of the circles? It gives us the perpendicular bisector of the common chords. Refer to the diagram below, where we denote the centers of Γ_1, Γ_2, Γ_3 by O_1, O_2, O_3 respectively. If we draw $O_3H \perp AD$, it is the perpendicular bisector of *DP*. Hence, $\frac{AP}{AD}=\frac{MP}{HP}$. This seems closely related to $\Delta O_1O_2O_3$.



Proof. Let O_1, O_2, O_3 denote the centers of Γ_1, Γ_2, Γ_3 respectively. Let O_1O_2 intersect AP at M. Clearly, $AM = PM$. Draw $O_3 \perp DP$ at H.

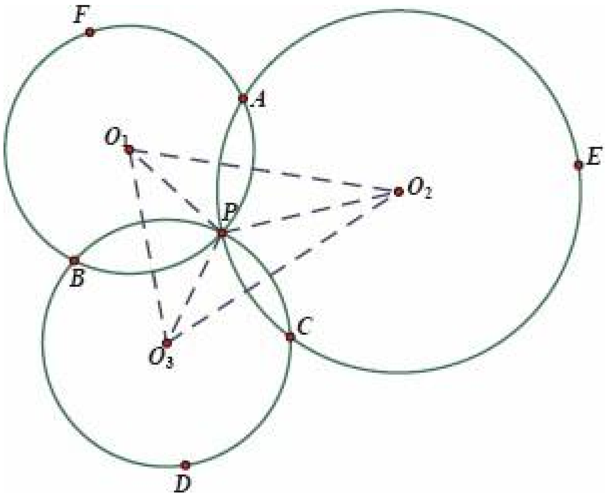
It is easy to see that $DH = PH$. Hence, $MH = \frac{1}{2}(AP + DP) = \frac{1}{2}AD$.

Now $\frac{AP}{AD} = \frac{\frac{1}{2}AP}{\frac{1}{2}AD} = \frac{PM}{HM} = \frac{[\Delta O_1O_2P]}{[\Delta O_1O_2H]}$.

Notice that $[\Delta O_1O_2H] = [\Delta O_1O_2O_3] = \frac{1}{2}O_1O_2 \cdot MH$, because $O_1O_2 \perp AD$ and $O_3H \perp AD$, i.e., $O_1O_2 // O_3H$. It follows that $\frac{AP}{AD} = \frac{[\Delta O_1O_2P]}{[\Delta O_1O_2O_3]}$.

Similarly, $\frac{BP}{BE} = \frac{[\Delta O_1O_3P]}{[\Delta O_1O_2O_3]}$ and $\frac{CP}{CF} = \frac{[\Delta O_2O_3P]}{[\Delta O_1O_2O_3]}$.

Refer to the diagram below.

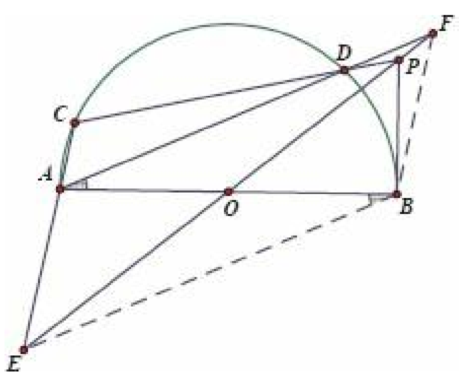


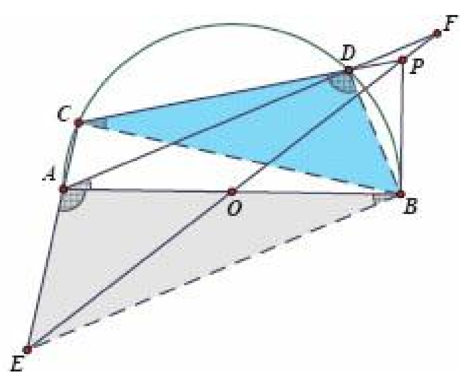
The conclusion follows as $\frac{[\Delta O_1O_2P]+[\Delta O_1O_3P]+[\Delta O_2O_3P]}{[\Delta O_1O_2O_3]}=1$.

□

Example 5.2.8 (CHN 07) Let *AB* be the diameter of a semicircle centered at *O*. Given two points *C*, *D* on the semicircle, *BP* is tangent to the circle, intersecting *CD* extended at *P*. If the line *PO* intersects *CA* extended and *AD* extended at *E*, *F* respectively, show that *OE* = *OF*.

Insight. Clearly, *AO* = *BO*. One sees that *AEBF* **should** be a parallelogram. How can we show it? Refer to the left diagram below. Perhaps the most straightforward way is to show $\angle ABE = \angle BAF$.



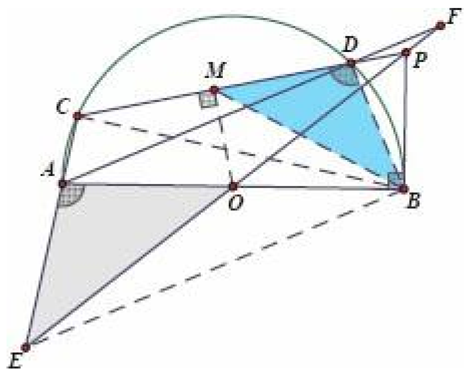


By applying circle properties, we obtain many equal angles, for example $\angle BAD = \angle BCD$ and $\angle BDC = \angle BAE$. It seems that we **should** have $\Delta BDC \sim \Delta EAB$. Refer to the right diagram above. Can we show it by considering the sides, say $\frac{BD}{CD} = \frac{AE}{AB}$? Unfortunately, this is not easy because we do not know much about *CD* or *AE*.

On the other hand, we have not used the condition $BP \perp AB$. This is when drawing a perpendicular to the chord becomes handy: we bisect *CD* and obtain a right angle as well. Notice that the midpoint of *CD* and *O* **should** be corresponding points in ΔBDC and ΔEAB.

Proof. Draw $OM \perp CD$ at *M*. We have $CM = DM$. Since $BP \perp AB$, we have *B*, *O*, *M*, *P* concyclic and hence, $\angle BMP = \angle BOP = \angle AOE$. (1) Since *A*, *B*, *D*, *C* are concyclic, we have $\angle BDC = \angle BAE$. (2)

(1) and (2) imply that $\Delta BDM \sim \Delta EAO$ and hence, $\frac{AE}{AO} = \frac{BD}{DM}$. Refer to the following diagram.



Since *O* and *M* are midpoints of *AB*, *CD* respectively, we have

$$\frac{AE}{AB} = \frac{AE}{2AO} = \frac{BD}{2DM} = \frac{BD}{CD}. \quad (3)$$

(2) and (3) imply that $\Delta BDC \sim \Delta EAB$. Hence, $\angle BCD = \angle ABE$.

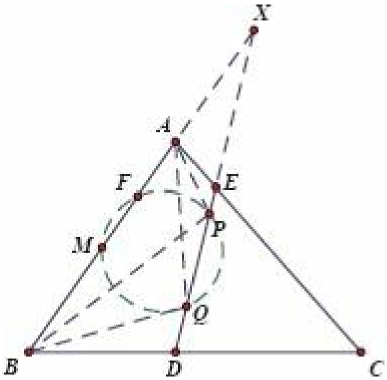
Since $\angle BCD = \angle BAD$, we must have $\angle BAD = \angle ABE$.

One sees that $\Delta AOF \cong \Delta BOE$ (A.A.S.) and hence, $OE = OF$.

□

Example 5.2.9 (IRN 09) Given an acute angled triangle ΔABC where *AD*, *BE*, *CF* are heights, draw $FP \perp DE$ at *P*. Let *Q* be the point on *DE* such that $QA = QB$. Show that $\angle PAQ = \angle PBQ = \angle PFC$.

Insight. Refer to the diagram on the below. Clearly, $\angle PAQ = \angle PBQ$ if and only if *A*, *B*, *Q*, *P* are concyclic. We are given many perpendicular lines, but we should **not** draw all the lines explicitly: otherwise, the diagram will be in a mess. Notice that there are a few concyclicity due to the right angles. For example, A, B, D, E are concyclic.

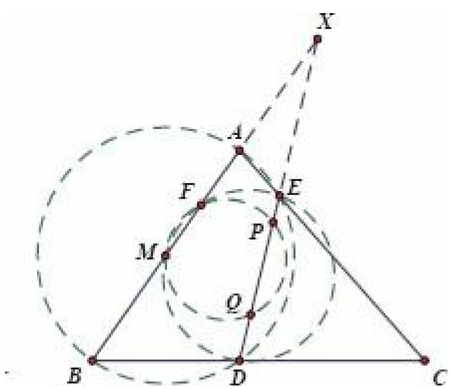


Can you see that the circumcircle of ΔPQF passes through M, the midpoint of AB? (**Hint**: QM is the perpendicular bisector of AB.) Can you see that that the circumcircle of ΔDEF pass through M as well? (**Hint**: Consider the nine-point circle of ΔABC.) Suppose BA extended and DE extended intersect at X. Perhaps we can apply the Tangent Secant Theorem repeatedly and show that A, B, Q, P are concyclic.

How about $\angle PFC$? Can you see that $\angle PFC = \angle X$, because $FP \perp DE$ and $CF \perp AB$?

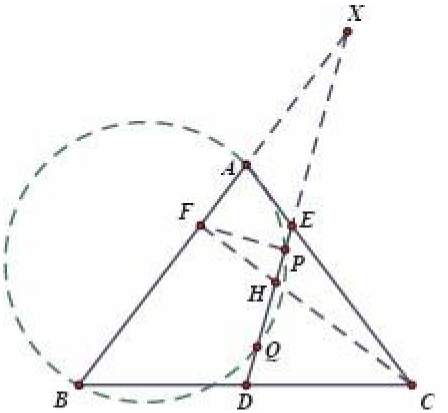
Proof. Clearly, Q lies on the perpendicular bisector of AB. Let M be the midpoint of AB. We must have $QM \perp AB$. Since $FP \perp DE$, F, M, Q, P are concyclic. Let the lines AB and DE intersect at X. By the Tangent Secant Theorem, $XP \cdot XQ = XF \cdot XM$. (1)

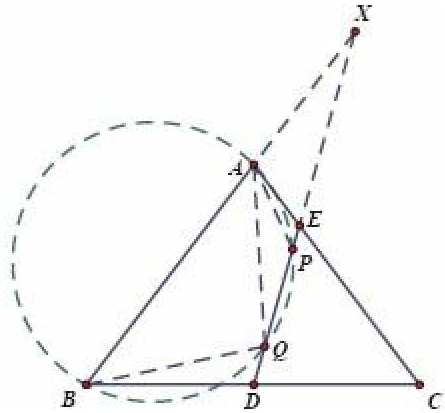
It is well known that A, B, D, E are concyclic and hence, we have $XA \cdot XB = XD \cdot XE$. (2) Notice that D, E, F, M are concyclic because they lie on the nine-point circle of ΔABC. Hence, $XD \cdot XE = XF \cdot XM$. (3) Refer to the diagram on the below. (1), (2) and (3) give $XA \cdot XB = XP \cdot XQ$.



Hence, A, B, Q, P are concyclic and $\angle PAQ = \angle PBQ$.

Let H denote the orthocenter of ΔABC. Consider the right angled triangle ΔFHX. Since $FP \perp HX$, we have $\angle PFC = \angle X$. Refer to the left diagram below. It suffices to show $\angle X = \angle PAQ$.





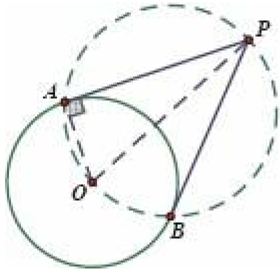
Notice that $\angle X = \angle PAB - \angle APX$, where $\angle APX = \angle ABQ = \angle BAQ$. It follows that $\angle X = \angle PAB - \angle BAQ = \angle PAQ$. Refer to the right diagram above. This completes the proof.

□

5.3 Constructing a Diagram

Most geometry problems in competitions held recently were presented in descriptive sentences without any diagram. Contestants are expected to construct the diagram on their own, usually with a straightedge and a compass allowed. Indeed, a well-constructed diagram is very important, if not indispensable, for solving a geometry problem: it not only helps in seeking geometric insight (for example, catching equal angles around a circle), but also gives inspiration on what **could** or **should** be true.

Constructing a diagram with only a straightedge and a compass involves a lot of skills. For example, given $\odot O$ and a point P outside the circle, do you know how to introduce tangent lines from P to $\odot O$ accurately? (**Hint**: Draw a circle Γ where OP is a diameter. Let $\odot O$ and Γ intersect at A, B. Can you see that PA, PB are the tangent lines from P to $\odot O$, because the diameter OP extends right angles on the circumference of Γ? Refer to the diagram on the below.)



In this section, we shall introduce a few techniques (related to the diagram) which one may find useful.

- Turn the paper around.

  If one thinks there might be symmetry in the diagram constructed but cannot see it clearly, a wise strategy is rotating the diagram (by turning the paper around) to the *upright* position, for example, with respect to the angle bisector, the perpendicular bisector or the line connecting the centers of two intersecting circles. Usually, the symmetry would become clearer in this view.

  This technique is also helpful for beginners to catch the geometric insight. It is common that beginners cannot identify similar triangles or equal tangent segments if a (complicated) diagram is drawn in an *oblique* manner. Hence, by turning the paper around, one may observe the diagram more thoroughly and find clues more easily.

- Coincidence and equivalent conclusions

  Occasionally, finding a direct proof could be difficult (or infeasible due to technical difficulties). Hence, one may consider showing an equivalent conclusion instead by coincidence. For example, if showing that a line ℓ passes through a specific point X on a circle Γ is difficult, one may let ℓ intersect Γ at X' and show that X and X' coincide. In fact, this technique is often applied when showing collinearity and concurrency, and is also illustrated in Example 1.4.3.

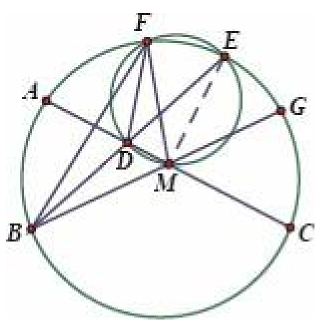
- Uniquely determined points

  It is an advanced technique to examine the diagram and check **how** it could be constructed and which points (and angles, line segments, etc.) are uniquely determined by the given conditions. For example, given a circle Γ and a point O outside Γ, if we are to construct $\odot O$ which touches Γ, then it is easy to see that the point of tangency, called P, is uniquely determined. In fact, P lies on the line connecting O and the center of Γ. Notice that OP, the radius of ⊙O, is also uniquely determined.

Although this technique may not help the problem-solving directly, it gives clues on how the diagram **could** vary and which points and line segments are more closely related. Acquiring such insight may greatly help us understand the diagram, identify the links and design an effective strategy leading to the solution.

Example 5.3.1 (RUS 09) Let ⊙O be the circumcircle of ΔABC. D is on AC such that BD bisects $\angle B$. Let BD extended intersect ⊙O at E. Draw a circle Γ with a diameter DE, intersecting ⊙O at E and F. Draw a line ℓ such that the line BF and ℓ are symmetric about the line BD. Show that ℓ passes through the midpoint of AC.

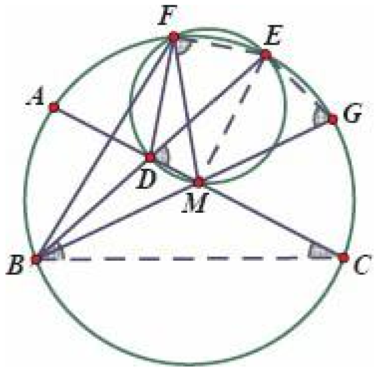
Insight. Refer to the diagram on the below. Notice that there are a few symmetries in the diagram due to the angle bisector.

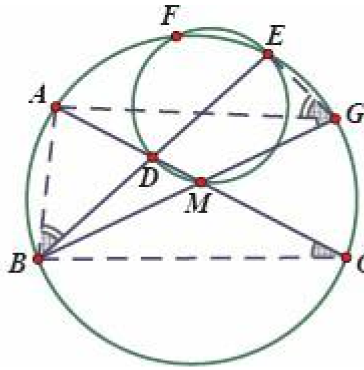


Suppose ℓ intersects AC at M (which **should** be the midpoint of AC). It *seems* from the diagram that M lies on Γ as well! Can we show it?
On the other hand, it may not be easy to show $AM = CM$ directly because we do not know much about the point M. How about choosing M as the midpoint of AC ? Would it be easier to show BD bisects $\angle MBF$? (We can probably apply the angle properties about ⊙O and Γ.)

Notice that E is the midpoint of the arc $\overset{\frown}{AC}$ and hence, EM is the perpendicular bisector of AC.

Proof. Let M be the midpoint of AC. Let BM extended intersect ⊙O at G. Since BE bisects $\angle ABC$, E must be the midpoint of $\overset{\frown}{AC}$. Hence, EM is the perpendicular bisector of AC. We claim that BM coincides with ℓ, i.e., BE bisects $\angle FBG$. Notice that it suffices to show that F and G are symmetric about EM, or equivalently, $\angle EFM = \angle G$. Refer to the left diagram below.





Since $EM \perp AC$, M must lie on Γ where DE is a diameter. It follows that $\angle EFM = \angle EDM = \angle CBD + \angle C = \angle ABD + \angle C$.

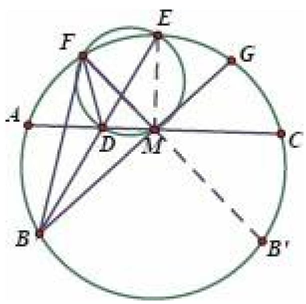
Refer to the right diagram above. Notice that $\angle ABD = \angle AGE$ and $\angle C = \angle AGB$. It follows that $\angle ABD + \angle C = \angle AGE + \angle AGB = \angle BGE$. This completes the proof.

□

Note:

(1) Given $\overset{\frown}{AC}$ and ⊙O, E and M are determined regardless of the choice of B. By choosing D, other points including B, F and G are determined. Hence, it is a wise strategy to explore the properties of angles around D.

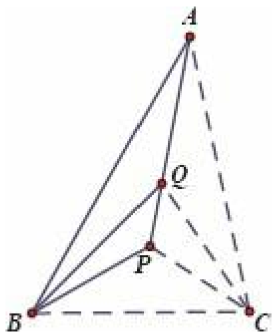
(2) By rotating the diagram, one may see the symmetry about the line EM. Refer to the diagram on the below. Let FM extended intersects ⊙O at B'. Notice that BG and $B'F$ are symmetric about the line EM.



Example 5.3.2 Let P be a point inside ΔABC such that $\angle APB - \angle ACB = \angle APC - \angle ABC$. Let I_1, I_2 be the incenters of ΔAPB, ΔAPC respectively. Show that AP, BI_1, CI_2 are concurrent.

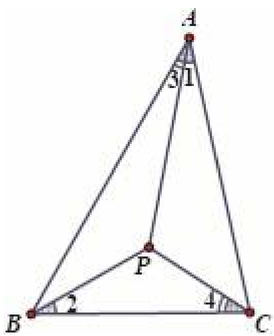
Insight. Apparently, the conditions given are unusual, not easy to apply and unrelated to the conclusion. In fact, we do not even know **how** to construct such a diagram. Let us focus on the conclusion: we are to show AP,

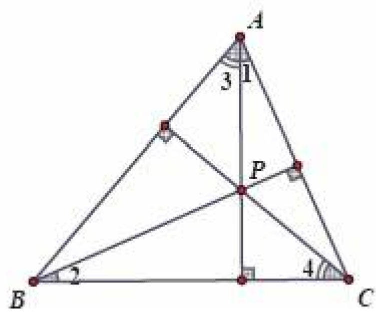
BI_1, CI_2 are concurrent. Since BI_1, CI_2 are angle bisectors of $\angle ABP$, $\angle ACP$ respectively, it suffices to show that these angle bisectors intersect AP at the same position.



Refer to the left diagram above. Let us draw ΔABP first where BQ is the angle bisector of $\angle ABP$. We shall find a point C such that CQ bisects $\angle ACP$. What conditions must C satisfy? For example, we must have $\frac{AC}{CP}=\frac{AQ}{PQ}$. In this case, we see that it suffices to show $\frac{AB}{BP}=\frac{AC}{CP}$, which leads to the conclusion.

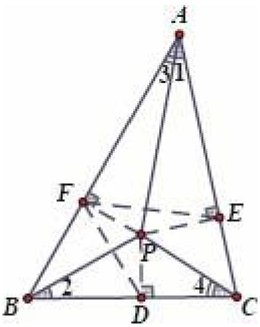
Now we are to apply the condition $\angle APB - \angle ACB = \angle APC - \angle ABC$. Notice that these angles are far apart. Can we bring them together? Refer to the left diagram below. Notice that $\angle APB - \angle ACB = \angle 1 + \angle 2$ and $\angle APC - \angle ABC = \angle 3 + \angle 4$. Hence, $\angle 1 + \angle 2 = \angle 3 + \angle 4$.





It seems these angles are still far apart. Recall that **if** P is the orthocenter, then we have $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$. Refer to the right diagram above. This is because the perpendicular lines imply concyclicity and give equal angles. For a general P, there are no perpendicular lines given, but perhaps we can introduce some!

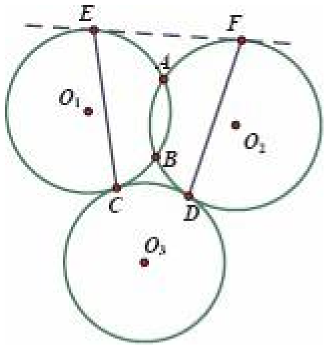
Proof. Refer to the diagram on the below. Since $\angle APB - \angle ACB = \angle 1 + \angle 2$ and $\angle APC - \angle ABC = \angle 3 + \angle 4$, we have $\angle 1 + \angle 2 = \angle 3 + \angle 4$.
Let D, E, F be the feet of the perpendiculars from P to BC, AC, AB respectively. Since $\angle AFP = \angle AEP = 90°$, A, F, P, E are concyclic and $\angle 1 = \angle EFP$. Similarly, $\angle 2 = \angle DFP$. It follows that $\angle 1 + \angle 2 = \angle EFP + \angle DFP = \angle DFE$. A similar argument gives $\angle 3 + \angle 4 = \angle DEF$. Now $\angle DEF = \angle DFE$ and we must have $DE = DF$.



By Sine Rule, $\frac{DF}{\sin \angle ABC} = BP$ (since BP is a diameter of the circumcircle of ΔBDF). Similarly, $\frac{DE}{\sin \angle ACB} = CP$. Since $DE = DF$, we have $\frac{BP}{CP} = \frac{\sin \angle ACB}{\sin \angle ABC} = \frac{AB}{AC}$ by applying Sine Rule to ΔABC. Hence, $\frac{AB}{BP} = \frac{AC}{CP}$. By the Angle Bisector Theorem, the angle bisectors of $\angle ABP$ and $\angle ACP$ must intersect AP at the same point. (Otherwise, say they intersect AP at X, Y respectively, we have $\frac{AX}{PX} = \frac{AP}{BP} = \frac{AC}{CP} = \frac{AY}{PY}$, which implies X, Y coincide.) This completes the proof.

□

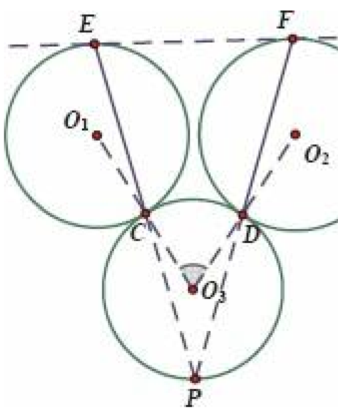
Example 5.3.3 Refer to the diagram on the below. $\odot O_1$ and $\odot O_2$ intersect at A and B. $\odot O_3$ touches $\odot O_1$ and $\odot O_2$ at C, D respectively. A common tangent of $\odot O_1$ and $\odot O_2$ touches the two circles at E, F respectively.



If the lines *CE* and *DF* intersect at *P*, show that *P* lies on the line *AB*.

Insight. One sees that *AB* is the radical axis of $\odot O_1$ and $\odot O_2$. Hence, it suffices to show that the powers of *P* with respect to $\odot O_1$ and $\odot O_2$ are the same, i.e., $PC \cdot PE = PD \cdot PF$ (or by the Tangent Secant Theorem if one is not familiar with the power of a point with respect to circles). However, the difficulty is that we do not know the position of *P* and hence, we cannot calculate *PC*, *PD*, *PE*, *PF* directly.

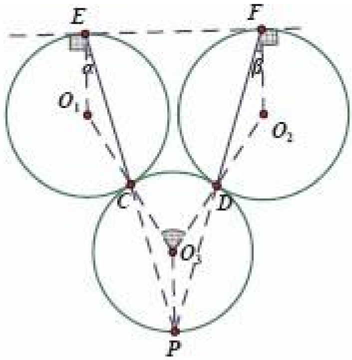
Refer to the diagram on the below. (We omit *A*, *B* to have a clearer view of the angles.) It *seems* from the diagram that *P*, the intersection of the lines *CE* and *DF*, lies on $\odot O_3$. Can we prove it?



Since $\odot O_3$ is tangent to $\odot O_1$ and $\odot O_2$, the line connecting the centers of the circles must pass through the point of tangency, i.e., O_1O_3 passes through *C* and O_2O_3 passes through *D*. Notice that $\angle CO_3D$ is an angle at the center of $\odot O_3$. Now *P* lies on $\odot O_3$ if and only if $\angle CO_3D = 2\angle CPD$. Can we show this?

Notice that $\angle CO_3D$ could be calculated via the pentagon $O_1O_3O_2FE$ (which has two right angles) and $\angle CPD$ could be calculated via ΔEPF.

If we denote $\angle O_1EC = \alpha$ and $\angle O_2FD = \beta$, all the interior angles in the pentagon $O_1O_3O_2FE$ and ΔEPF could be expressed in α,β, (using the fact that ΔO_1CE and ΔO_2DF are isosceles triangles). Refer to the diagram on the below.



On the other hand, if P indeed lies on $\odot O_3$, we have similar isosceles triangles $\Delta O_1CE \sim \Delta O_3CP$ and $\Delta O_2DF \sim \Delta O_3DP$. Now and $\frac{PC}{CE}$ and $\frac{PD}{DF}$ could be expressed using the radii of $\odot O_1$, $\odot O_2$ and $\odot O_3$. We should not be far away from the conclusion.

Proof. First, we claim that P lies on $\odot O_3$. Let $\angle O_1EC = \alpha$ and $\angle O_2FD = \beta$. Consider ΔEPF. We have

$\angle CPD = 180° - \angle CEF - \angle DFE = 180° - (90° - \alpha) - (90° - \beta) = \alpha + \beta$.

On the other hand, by considering the pentagon $O_1O_3O_2FE$, we have $\angle CO_3D = 540° - \angle O_1EF - \angle O_2FE - \angle CO_1E - \angle DO_2F$

$= 540° - 90° - 90° - (180° - 2\alpha) - (180° - 2\beta) = 2(\alpha + \beta) = 2\angle CPD$.

Hence, P lies on $\odot O_3$ (Theorem 3.1.1).

Now it is easy to see that $\Delta O_1CE \sim \Delta O_3CP$ since both are isosceles triangles and $\angle O_1CE = \angle O_3CP = \alpha$. Similarly, $\Delta O_2DF \sim \Delta O_3DP$.

Let the radii of $\odot O_1$, $\odot O_2$ and $\odot O_3$ be a, b, c respectively. Let $CE = x$ and $DF = y$. We have $\frac{PC}{CE} = \frac{c}{a}$, i.e., $PC = \frac{xc}{a}$. Similarly, $PD = \frac{yc}{b}$.

Consider ΔO_1CE. We have $\frac{x}{a} = 2 \cdot \frac{x/2}{a} = 2\cos\alpha$. Similarly, $\frac{y}{b} = 2\cos\beta$.

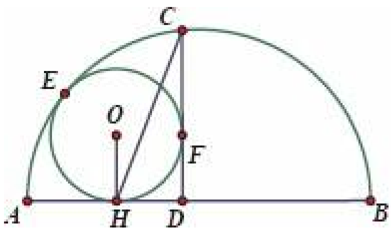
Now $\frac{PC}{PD} = \frac{2\cos\alpha \cdot c}{2\cos\beta \cdot c} = \frac{\cos\alpha}{\cos\beta}$. On the other hand, applying Sine Rule in ΔPEF gives $\frac{PE}{PF} = \frac{\sin\angle PFE}{\sin\angle PEF} = \frac{\cos\beta}{\cos\alpha}$.

It follows that $\frac{PC}{PD} = \frac{PF}{PE}$, or $PC \cdot PE = PD \cdot PF$.

Now the power of P with respect to $\odot O_1$ and $\odot O_2$ are the same, which implies that P lies on the line AB, the radical axis of $\odot O_1$ and $\odot O_2$.

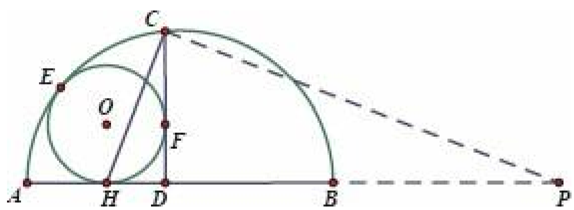
□

Example 5.3.4 (CHN 10) Refer to the diagram on the below. $\odot O$ is tangent to AB at H. Draw a semicircle with the diameter AB, touching $\odot O$ at E. C is a point on the semicircle such that $CD \perp AB$ at D and CD touches $\odot O$ at F. Show that $CH^2 = 2DH \cdot BH$.



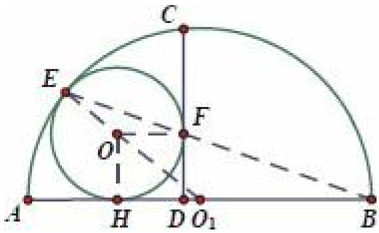
Insight. Clearly, $\angle BCH$ is not 90°, but if it were, we could have concluded $CH^2 = DH \cdot BH$. Now we are to show $CH^2 = 2DH \cdot BH$.

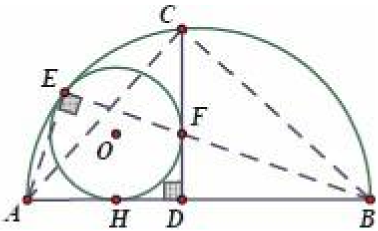
Hence, if one draws $PC \perp CH$ at C, intersecting AB extended at P, we **should** have $PH = 2BH$, i.e., $BH = BP$. Can we show it? Refer to the diagram on the below. It seems we do not have many clues about BH and BP, although there are many points of tangency given in the diagram.



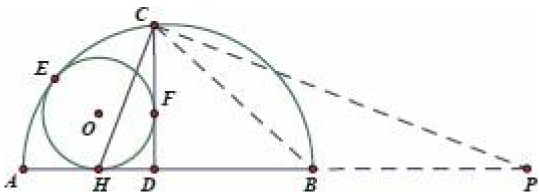
One may find equal angles or apply the Tangent Secant Theorem, but those are not directly related to BH or BP. Perhaps we should study the diagram more carefully and see how it **could** be constructed.

Suppose we are given $\odot O$. Notice that if we choose AB casually, the semicircle may not touch $\odot O$. In fact, once the center of the semicircle, called O_1, is chosen, the positions of A, B, E (and C, F) are uniquely determined. Refer to the left diagram below. Can you see that $OFDH$ is a square?



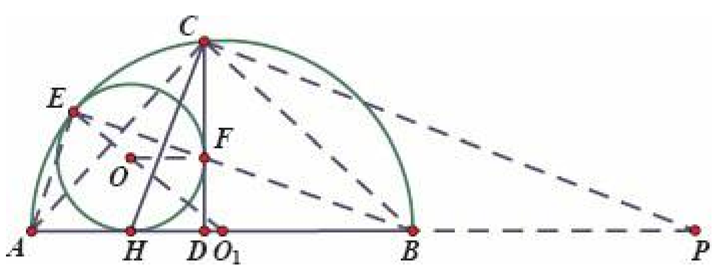


Since E is the point of tangency, O, O_1, E are collinear. Since $OF \,//\, AB$, the isosceles triangles ΔOEF and ΔO_1EB are similar, which implies B, E, F are collinear! Now we have plenty of clues to apply the Tangent Secant Theorem. Refer to the right diagram above. One sees that $BE \cdot BF = BH^2$. Can you see that $BE \cdot BF = BA \cdot BD$ because A, D, F, E are concyclic? Can you see that $BA \cdot BD = BC^2$?



It follows that $BC = BH$. This is almost what we want. Refer to the diagram on the below. Can you see why $BH = BP$?

Proof. Let O_1 be the midpoint of AB. Extend AB to P such that $BH = BP$. Connect OF and O_1E. It is easy to see that O_1E passes through O. Refer to the diagram below.



Since $OF \perp CD$ and $AB \perp CD$, we must have $OF \parallel AB$ and hence, $\angle EOF = \angle EO_1B$. Since ΔOEF and ΔO_1EB are both isosceles triangles, we have $\angle OFE = \angle O_1BE$. It follows that B, E, F are collinear.

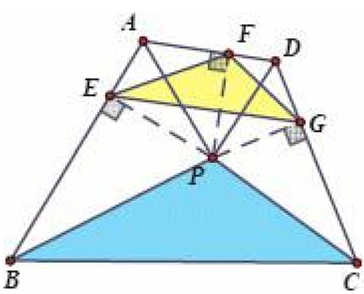
Connect AC, BC and AE. Notice that $\angle ACB = \angle AEB = \angle ADC = 90°$. Hence, A, D, F, E are concyclic. Now $BC^2 = BD \cdot BA$ (Example 2.3.1) $= BE \cdot BF = BH^2$ (Tangent Secant Theorem). Hence, $BC = BH$.

Notice that $BC = BP = BH$ implies $CH \perp CP$ (Example 1.1.8). Now we have $CH^2 = DH \cdot PH = DH \cdot 2BH$. This completes the proof.

□

Example 5.3.5 Let P be a point inside the cyclic quadrilateral $ABCD$ such that $\angle BPC = \angle BAP + \angle CDP$. Draw $PE \perp AB$ at E, $PF \perp AD$ at F and $PG \perp CD$ at G. Show that $\Delta FEG \sim \Delta PBC$.

Insight. Refer to the diagram on the below. It seems $\angle BPC = \angle BAP + \angle CDP$ is not a straightforward condition. How can we show $\Delta FEG \sim \Delta PBC$? It should be via equal angles or sides of equal ratio. One easily sees that A, E, P, F are concyclic and D, F, P, G are concyclic. Can you see that $\angle BPC = \angle EFG$? What else can we derive from $\angle BPC = \angle BAP + \angle CDP$? Even though this condition is not straightforward, it seems the only source for us to understand the diagram. (Notice that E, F, G could be obtained simply by drawing circles using AP, DP as diameters.) Hence, we shall explore further about this condition.



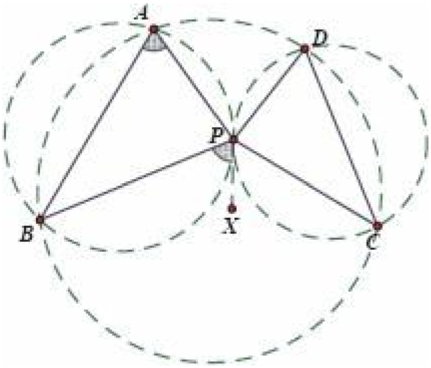
One sees that $\angle BPC$, $\angle BAP$, $\angle CDP$ are either an angle around P, or an angle inside $ABCD$, both of whom might give 360° :

$$\angle APB + \angle CPD = 360° - \angle BPC - \angle APD \quad (1)$$

$$\angle APB + \angle CPD = (180° - \angle ABP - \angle BAP) + (180° - \angle CDP - \angle DCP) \quad (2)$$

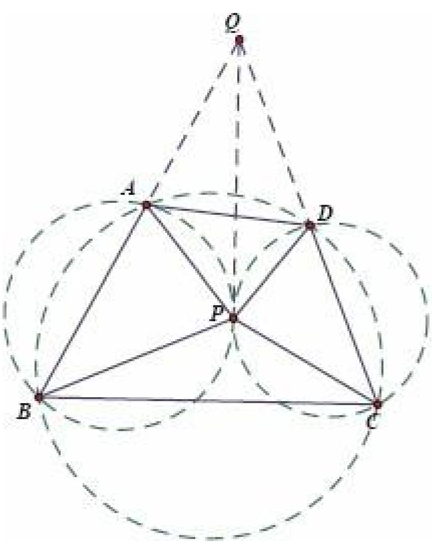
(1) and (2) give $\angle APD = \angle ABP + \angle DCP$. This is a symmetric version of what is given. Is it useful? Perhaps we shall examine the construction of our diagram, i.e., **how** can we locate a point P such that $\angle BPC = \angle BAP + \angle CDP$? By taking $\angle BPX = \angle BAP$, we must have PX tangent to the circumcircle of

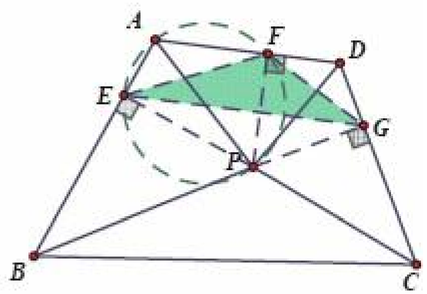
Δ*ABP* at *P* (Theorem 3.2.10). Refer to the diagram on the below.



Now we construct another circle tangent to the circumcircle of Δ*ABP* at *P* (which is simple because the line connecting the centers of the two circles must be perpendicular to *PX*). This circle intersects the circumcircle of Δ*ABC* at *D* because $\angle CDP = \angle CPX$.
In conclusion, given Δ*ABC* and *P*, *D* is uniquely determined and *PX* **should** be a common tangent of the circumcircles of Δ*ABP* and Δ*CDP*.

Since *ABCD* is cyclic, we now have three circles, whose radical axes should be concurrent (Theorem 4.3.2). Refer to the left diagram below. Can you see similar triangles in this diagram involving *BP* and *CP*, for example, Δ*QAP* ~ Δ*QPB* and Δ*QDP* ~ Δ*QPC* ? Recall that we are to show $\frac{EF}{BP}=\frac{FG}{CP}$. What do we know about *EF* and *FG*?

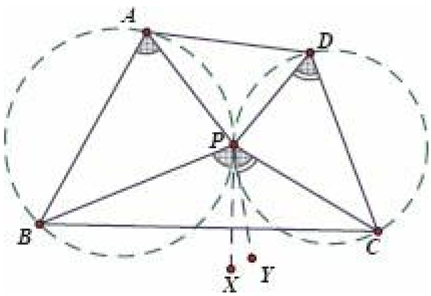




Refer to the right diagram above. Since *A*, *E*, *P*, *F* are concyclic where *AP* is a diameter, we have $EF = AP \sin \angle BAD$ (Sine Rule). Similarly, $FG = DP \sin \angle ADC$. Now AP, BP, CP, DP are related by similar triangles. It seems we have gathered all the links!

Please note that in the formal proof, one should also consider the case if $AB // CD$ or if AB, CD intersect at the other side of line AD, i.e., our argument should not depend on the diagram.

***Proof*.** First, we claim that the circumcircles of ΔABP and ΔCDP touch at P. Let PX be tangent to the circumcircle of ΔABP at P. We have $\angle BPX = \angle BAP$. Refer to the diagram on the below.

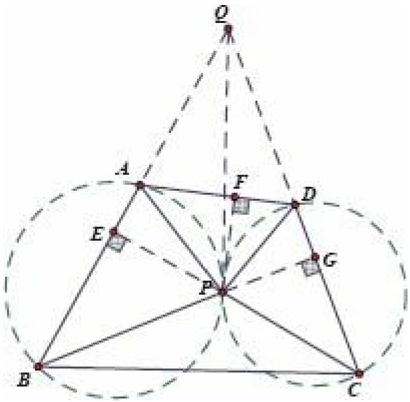


We also draw PY tangent to the circumcircle of ΔCDP at P, which implies $\angle CDP = \angle CPY$. It follows that $\angle BPC = \angle BAP + \angle CDP = \angle BPX + \angle CPY$, i.e., P, X, Y are collinear. This is only possible if the circumcircles of ΔABP and ΔCDP are tangent at P.

Consider the lines AB and CD.

Case I: BA extended and CD extended intersect at Q.

Refer to the diagram on the below. Since $ABCD$ is cyclic, $QA \cdot QB = QC \cdot QD$, i.e., the power of Q with respect to the circumcircles of ΔABP and ΔCDP are the same. Hence, QP must be tangent to both circles.



It is easy to see that $\Delta QAP \sim \Delta QPB$.

Hence, $\frac{AQ}{AP} = \frac{QP}{BP}$. (1)

Similarly, $\Delta QDP \sim \Delta QPC$ and we have $\frac{DQ}{DP} = \frac{QP}{CP}$. (2)

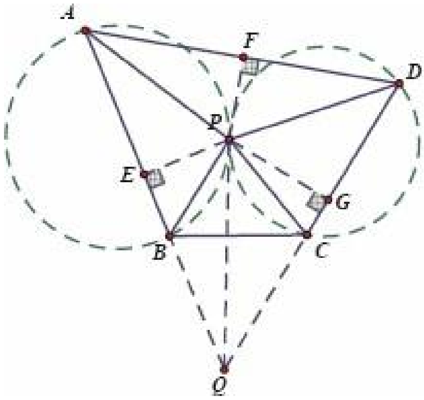
Since $\angle AEP = \angle AFP = 90°$, A, E, P, F are concyclic where AP is a diameter. By Sine Rule, $\frac{EF}{\sin \angle BAD} = AP$, i.e., $EF = AP \sin \angle BAD$.

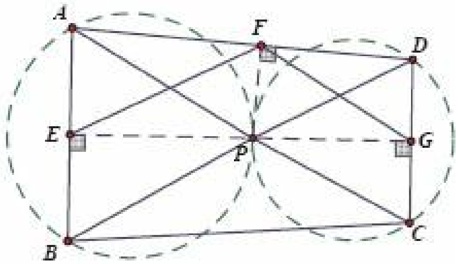
Similarly, $FG = DP \sin \angle ADC$.

It follows that $\frac{EF}{FG} = \frac{AP \sin \angle BAD}{DP \sin \angle ADC} = \frac{AP}{DP} \cdot \frac{\sin \angle QAD}{\sin \angle QDA} = \frac{AP}{DP} \cdot \frac{DQ}{AQ}$.

By (1) and (2), we have $\frac{EF}{FG} = \frac{BP}{PQ} \cdot \frac{PQ}{CP} = \frac{BP}{CP}$.

Case II: AB extended and DC extended intersect at Q.
Refer to the left diagram below. A similar argument applies and we still have $\frac{EF}{FG} = \frac{BP}{CP}$.





Case III: *AB* // *CD*

Refer to the right diagram above. We see that *ABCD* is an isosceles trapezium (Exercise 3.1) and *E*, *P*, *G* are collinear (Example 1.1.11). We still have the circumcircles of Δ*ABP* and Δ*CDP* tangent at *P*. Now the radical axes, two of which are *AB*, *CD*, must be parallel (Theorem 4.3.2). Hence, *AB*, *CD* are perpendicular to the line connecting the circumcenters of Δ*ABP* and Δ*CDP*. It follows that *P* lies on the perpendicular bisectors of *AB* and *CD*, which implies *AP* = *BP* and *CP* = *DP*.
Since *A*, *E*, *P*, *F* are concyclic where *AP* is a diameter, we still have *EF* = *AP* sin ∠*BAD* and similarly, *FG* = *DP* sin ∠*ADC*.

Since *AB* // *CD*, we have sin ∠*BAD* = sin ∠*ADC* because ∠*BAD* and ∠*ADC* are supplementary. Now $\frac{EF}{FG}=\frac{AP}{DP}=\frac{BP}{CP}$.

In conclusion, $\frac{EF}{FG}=\frac{BP}{CP}$ holds in all cases.

Now ∠*BPC* = ∠*BAP* + ∠*CDP* = ∠*EFP* + ∠*GFP* (angles in the same arc) = ∠*EFG*. We conclude that Δ*FEG* ~ Δ*PBC*.

□

5.4 Exercises

1. Given an acute angled triangle Δ*ABC* and its circumcenter *O*, *BD*, *CE* are heights. Show that *AO* ⊥ *DE*.

2. Given a semicircle centered at *O* whose diameter is *AB*, draw *OP* ⊥ *AB*, intersecting the semicircle at *P*. Let *M* be the midpoint of *AP*. Draw *PH* ⊥ *BM* at *H*. Show that $PH^2 = AH \cdot OH$.

3. (IND 94) Let *I* be the incenter of Δ*ABC* and the incircle of Δ*ABC* touches

BC, AC at D, E respectively. If BI extended and DE extended intersect at P, show that $AP \perp BP$.

4. (AUT 09) Given an acute angled triangle ΔABC where D, E, F are the midpoints of BC, AC, AB respectively and AP, BQ, CR are heights. Let X, Y, Z be the midpoints of QR, PR, PQ respectively. Show that DX, EY, FZ are concurrent.

5. Given a non-isosceles acute angled triangle ΔABC and its circumcircle $\odot O$, H is the orthocenter of ΔABC and M, N are the midpoints of AB, BC respectively. If MH extended and NH extended intersect $\odot O$ at P, Q respectively, show that P, Q, M, N are concyclic.

6. Given a right angled triangle ΔABC where $\angle A = 90°$, $AD \perp BC$ at D. Let the radii of the incircles of ΔABC, ΔABD, ΔACD be r, r_1, r_2 respectively. Show that $r + r_1 + r_2 = AD$.

7. Given a rectangle $ABCD$ where $AB = 1$ and $BC = 2$, P, Q are on BD, BC respectively. Find the smallest possible value of $CP + PQ$.

8. Given an acute angled triangle ΔABC and its orthocenter H, M is the midpoint of BC. Draw a line ℓ passing through H and perpendicular to MH, intersecting AB, AC at P, Q respectively. Show that H is the midpoint of PQ.

9. Let P be a point outside $\odot O$ and PA, PB touch $\odot O$ at A, B respectively. C is a point on AB and the circumcircle of ΔBCP intersects $\odot O$ at B and D. Let Q be a point on PA extended such that $OP = OQ$. Show that $AD // CQ$.

10. Given an acute angled triangle ΔABC and its circumcircle, AD, BE are heights. X lies on the minor arc $\overset{\frown}{AC}$. If the lines BX and AD intersect at P, and the lines AX and BE intersect at Q, show that DE passes through the midpoint of the line segment PQ.

11. Given a right angled triangle ΔABC where $\angle A = 90°$ and its circumcircle Γ, P is a point on Γ and $PH \perp BC$ at H. D, E are points on Γ such that $PD = PE = PH$. Show that DE bisects PH.

12. Let AB be a diameter of $\odot O$. P, Q are points outside the circle such that PA intersects $\odot O$ at C, PB extended intersects $\odot O$ at D and QC, QD touch $\odot O$ at C, D respectively. If AD extended and PQ extended intersect at E, show that B, C, E are collinear.

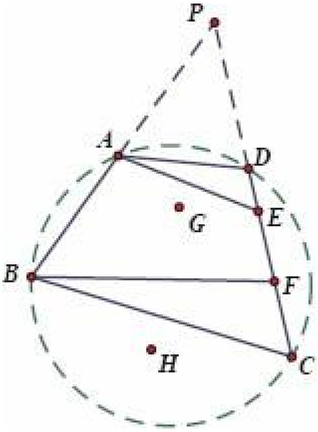
13 (CHN 12) Let Γ be the circumcircle of ΔABC and I be the incenter of ΔABC. Let AI extended and BI extended intersect Γ at D, E respectively. Draw a line ℓ_1 passing through I such that $\ell_1 // AB$. Draw a line ℓ_2 tangent to Γ at C. If ℓ_1, ℓ_2 intersect at F, show that D, E, F are collinear.

14. In ΔABC, ∠A = 90° . D, E are on AC, AB respectively such that BD, CE bisect ∠B, ∠C respectively. Draw $AP \perp DE$, intersecting BC at P. Show that $AB - AC = BP - CP$.

15. Given a parallelogram $ABCD$, the circumcircle of ΔABD intersects AC extended at E. P is a point on BD such that ∠BCP = ∠ACD. Show that ∠AED = ∠BEP.

16. Let CD be a diameter of ⊙O. Points A, B on ⊙O are on opposite sides of CD. PC is tangent to ⊙O at C, intersecting the line AB at P. If the lines BD and OP intersect at E, show that $AC \perp CE$.

17. Refer to the diagram on the below. Given a cyclic quadrilateral $ABCD$, where BA extended and CD extended intersect at P, E, F lie on CD. Let G, H denote the circumcenters of ΔADE and ΔBCF respectively. Show that if A, B, F, E are concyclic, then P, G, H are collinear.



Chapter 6

Geometry Problems in Competitions

We have included a number of geometry problems from competitions in the previous chapters as examples. One may see that those problems are generally much harder than the standard exercises: simply applying a known theorem will not be an effective strategy. It could be difficult to even relate the conclusion to the conditions given. Indeed, this is a major obstacle encountered by the beginners: how to *start* problem solving? On the other hand, reading the solutions provided does not seem to be inspiring. Those solutions are usually written in an elegant and splendid manner, but do not show the beginners *how* one can think of such a solution.

One definitely finds it useful to be familiar with the basic skills and commonly used techniques illustrated in the previous chapters. Besides, we will introduce a few strategies in this chapter to tackle challenging geometry problems, while elaborating these strategies with examples from various competitions in the past years. Our focus is to seek clues and insights for each problem and hence, carry out the strategy which gradually leads to the solution.

6.1 Reverse Engineering

Not all competition questions are unreasonably difficult. Indeed, for those (relatively) easy questions, one simple but effective strategy is reverse engineering. This includes the following:

- Expect what the last step of the solution *could* be.

  For example, if we are to show concyclicity, it could be concluded by equal angles, supplementary angles or line segments which compose of the Tangent Secant Theorem or the Intersecting Chords Theorem. If we are to show collinearity, it could be concluded by either Menelaus' Theorem or supplementary angles. If we are to show equal line segments, it could be concluded by isosceles, congruent or similar triangles.

Knowing the sketch of (the last part of) the proof gives inspiration on what intermediate steps one may expect and attempt to show.

- Discover what **should** be true by assuming the conclusion is true.

  Of course, the conclusion to be shown should be true. Hence, by assuming this extra *condition*, we may discover what **should** be true (but is yet to be shown). For example, if we are to show equal angles and we assume they are, we may find a pair of triangles which **should** be similar. Now showing the similar triangles (say by line segments in ratio) leads to the conclusion!

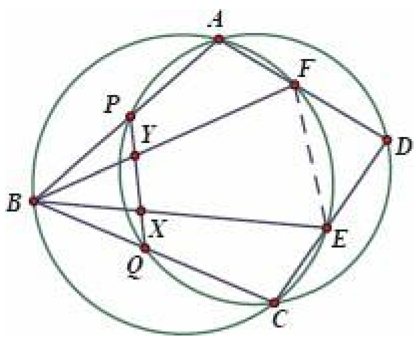
- Simplify the conclusion ("It suffices to show…").

  One shall always attempt to *link* the conclusion to the conditions given. Writing down "it suffices to show…" could transform the conclusion, moving it *towards* the given conditions.

  Unfortunately, there is often more than one way to approach the conclusion or the intermediate steps, while most approaches will **not** lead to a complete proof. Be resilient and do not give up easily! It is common for even the most experienced contestants to have a few failed attempts before reaching a valid proof.

Example 6.1.1 (HRV 09) Given a quadrilateral *ABCD*, the circumcircle of ΔABC intersects *CD*, *AD* at *E*, *F* respectively, and the circumcircle ΔACD intersects *AB*, *BC* at *P*, *Q* respectively. If *BE*, *BF* intersect of *PQ* at *X*, *Y* respectively, show that *E*, *F*, *Y*, *X* are concyclic.

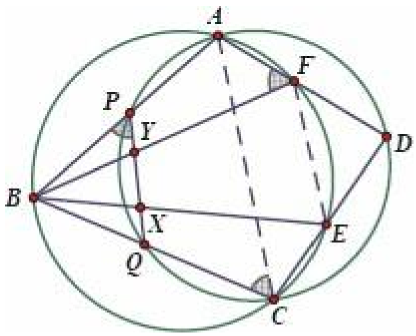
Insight. We are not given much information besides the two circles. Hence, it is natural to expect a proof by the angle properties. Refer to the diagram on the below.



Since we do not know much about $\angle EXF$ or $\angle EYX$, can we show that $\angle FYP = \angle FEX$? $\angle FEC$ is on the circumference of a circle, but $\angle FYP$ is not. However, one may write $\angle FYP = \angle PBY + \angle BPY$.

***Proof*.** Refer to the diagram on the below. Since *A, B, E, F* are concyclic, we have $\angle BEF = 180° - \angle BAF = \angle ABF + \angle AFB$.
Notice that $\angle AFB = \angle ACB = \angle BPY$ (because *A, C, Q, P* are concyclic). Hence, $\angle BEF = \angle ABF + \angle BPY = \angle FYP$. It follows that *E, F, Y, X* are concyclic.

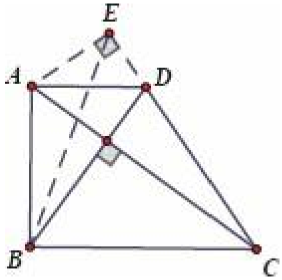
□



Note: There is more than one way to solve this problem. For example, one sees that $\angle FYP = \angle PBY + \angle BPY = \angle ACF + \angle ACB = \angle BCF = \angle BEF$, which also leads to the conclusion. Indeed, it is an effective strategy to apply reverse engineering for this problem, i.e., repeatedly simplifying the conclusion by writing down "it suffices to show…" which eventually leads to a clear fact (about angles) and completes the proof.

Example 6.1.2 (SVN 08) *ABCD* is a trapezium where $BC \parallel AD$ and $AB \perp BC$. It is also given that $AC \perp BD$. Draw $AE \perp CD$, intersecting *CD* extended at *E*. Show that $\frac{BE}{CE} = \frac{AD \cdot BD}{BD^2 - AD^2}$.

***Insight*.** Refer to the diagram on the below. One immediately sees $BD^2 - AD^2 = AB^2$ and hence, we are to show $\frac{BE}{CE} = \frac{AD \cdot BD}{AB^2}$.



Can we simplify $\frac{AD \cdot BD}{AB^2}$? If yes, the problem may be solved by similar triangles.

How is AB related to AD and BD? Can you see that $AB^2 = AD \cdot BC$?

Proof. It is easy to see $\Delta ABD \sim \Delta BCA$. Hence, we have $\frac{AB}{BC} = \frac{AD}{AB}$, or $AB^2 = AD \cdot BC$. It follows that $\frac{AD \cdot BD}{BD^2 - AD^2} = \frac{AD \cdot BD}{AB^2} = \frac{AD \cdot BD}{AD \cdot BC} = \frac{BD}{BC}$. Now it suffices to show $\frac{BE}{CE} = \frac{BD}{BC}$, or $\Delta BCD \sim \Delta ECB$.

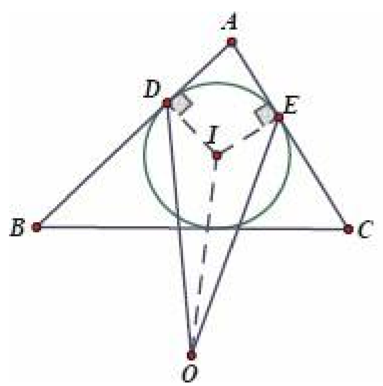
Since $\angle AEC = \angle ABC = 90°$, we have A, B, C, E concyclic and hence, $\angle CBD = \angle BAC = \angle CEB$. Now $\Delta BCD \sim \Delta ECB$ and the conclusion follows.

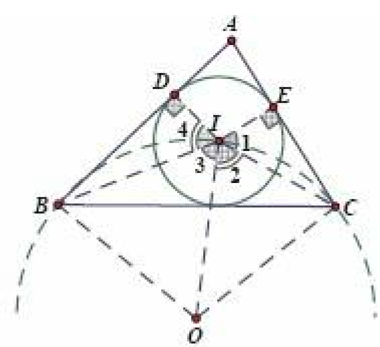
□

Note: If one writes $AB^2 = BD \cdot BF$ where AC and BD intersect at F, it may not be easy to show $\frac{BE}{CE} = \frac{AD}{BF}$ because it is not clear how AD is related to BE or CE. Hence, AD should be cancelled out, i.e., we shall write $AB^2 = AD \cdot *$. Now it is easy to see that * is BC.

Example 6.1.3 (CGMO 12) Let I be the incenter of ΔABC whose incircle touches AB, AC at D, E respectively. If O is the circumcenter of of ΔBCI, show that $\angle ODB = \angle OEC$.

Insight. Refer to the left diagram below. Even though BD, CE are tangent to the incircles, it is not clear which angle on the circumference is equal to $\angle ODB$ or $\angle OEC$, as we do not know where OD, OE intersect $\odot I$. How about the supplement of these angles? Can we show $\angle ADO = \angle AEO$? At least we know $\angle ADI = \angle AEI = 90°$. Can we show $\angle ODI = \angle OEI$?





Since $DI = EI$, we **should** have $\Delta ODI \cong \Delta OEI$. How can we show these triangles are congruent? Can we show $\angle OID = \angle OIE$?

Proof. Refer to the right diagram above. We write $\angle OIE = \angle 1 + \angle 2$ and

$\angle OID = \angle 3 + \angle 4$.

Notice that $\angle 1 = 90° - \angle ECI = 90° - \frac{1}{2}\angle ACB$ and since $OC = OI$,

$\angle 2 = 90° - \frac{1}{2}\angle COI$. Notice that $\angle COI = 2\angle CBI$ (Theorem 3.1.1).

Hence, $\angle COI = \angle ABC$ and we have $\angle 2 = 90° - \frac{1}{2}\angle ABC$.

Similarly, $\angle 3 = 90° - \frac{1}{2}\angle ACB$ and $\angle 4 = 90° - \frac{1}{2}\angle ABC$.

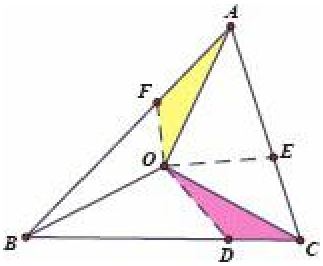
It follows that $\angle 1 + \angle 2 = \angle 3 + \angle 4$, i.e., $\angle OID = \angle OIE$. This implies $\angle ODI \cong \angle OEI$ (S.A.S.). Now we have $\angle ODI = \angle OEI$ and hence the conclusion.

□

Note: One familiar with the basic facts about the incenter and the circumcircle easily sees that *AI* extended intersects the circumcircle of Δ*ABC* at *O*, the circumcenter of Δ*BIC* (Example 3.4.2 and Exercise 3.14). Since *O* lies o n *AI*, the perpendicular bisector of *DE*, the conclusion follows immediately.

Example 6.1.4 (APMO 13) Given an acute angled triangle Δ*ABC* and its circumcenter *O, AD, BE, CF* are heights. Show that the line segments *OA, OF, OB, OD, OC, OE*dissect Δ*ABC* into three pairs of triangles that have equal areas.

Insight. First, we shall decide which of the triangles **could** be of equal area. Refer to the diagram on the below. Since *F* is not the midpoint of *AB*, $[\Delta AOF] \neq [\Delta BOF]$. Observe that we shall **not** have $[\Delta AOF] = [\Delta AOE]$ (Consider the case when $\angle C$ is almost 90°.)



Nor shall we have [Δ*AOF*] = [Δ*COE*]. Otherwise the triangles cannot be paired up in a *symmetric* manner. It seems that we should show $[\Delta AOF] = [\Delta COD]$, $[\Delta AOE] = [\Delta BOD]$ and $[\Delta BOF] = [\Delta COE]$.

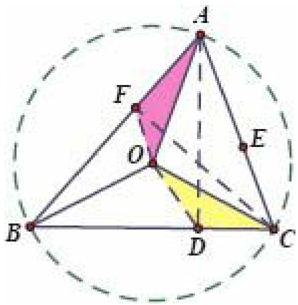
Apparently, these triangles are not congruent. Notice that

$$[\Delta AOF]=\frac{1}{2}AO\cdot AF\sin\angle OAF \text{ and } [\Delta COD]=\frac{1}{2}CO\cdot CD\sin\angle OCD.$$

Since $AO = CO$, it suffices to show $AF\sin\angle OAF = CD\sin\angle OCD$. This should not be difficult since we have the right angled triangles (heights) and the circumcircle.

Proof. Refer to the diagram on the below. We have $AF = AC\cos\angle A$ and $CD = AC\cos\angle C$.

Hence, $\frac{AF}{CD}=\frac{\cos\angle A}{\cos\angle C}$. (1)



Notice that $\angle OAF = 90° - \frac{1}{2}\angle AOB = 90° - \angle C.$

Similarly, $\angle OCD = 90° - \angle A$.

$$\text{Now } \frac{[\Delta AOF]}{[\Delta COD]}=\frac{\frac{1}{2}AF\cdot AO\sin\angle OAF}{\frac{1}{2}CD\cdot CO\sin\angle OCD}=\frac{AF}{CD}\cdot\frac{\sin(90°-\angle C)}{\sin(90°-\angle A)}$$

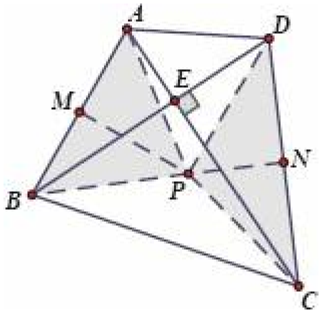
$=\frac{AF}{CD}\cdot\frac{\cos\angle C}{\cos\angle A}=1$ by (1).

Similarly, $[\Delta AOE]=[\Delta BOD]$ and $[\Delta BOF]=[\Delta COE]$.

□

Example 6.1.5 (IMO 98) In a cyclic quadrilateral $ABCD$, $AC\perp BD$ and AB, CD are not parallel. If the perpendicular bisectors of AB and CD intersect at P, show that $[\Delta ABP] = [\Delta CDP]$.

Insight. One notices that ΔABP and ΔCDP are isosceles triangles. Hence, we are to show



$$\frac{[\Delta ABP]}{[\Delta CDP]}=\frac{\frac{1}{2}AP^2\sin\angle APB}{\frac{1}{2}CP^2\sin\angle CPD}=1.$$

How are *AP* and *CP* related? Since *ABCD* is cyclic, say inscribed inside the circle Γ, the center of Γ must lie on the perpendicular bisectors of *AB, CD*. Indeed, *P* is the center of Γ and we have *AP = CP*. Refer to the diagram above.

Now it suffices to show sin∠*APB* = sin∠*CPD*. It seems from the diagram ∠*APB* ≠ ∠*CPD*. Can we show ∠*APB* = 180° – ∠*CPD* instead?

(**Hint**: Can you see ∠*APB* = 2∠*ACB* ?)

Proof. Since *ABCD* is cyclic, one sees that *P* is the center of the circumcircle of *ABCD*. Hence, *PA* = *PB* = *PC* = *PD*.

Since $[\Delta ABP]=\frac{1}{2}PA^2\sin\angle APB$ and $[\Delta CDP]=\frac{1}{2}PC^2\sin\angle CPD$, it suffices to show sin∠*APB* = sin∠*CPD*.

We claim that ∠*APB* + ∠*CPD* = 180°. In fact, since *P* is the centre of the circumcircle of *ABCD*, ∠*APB* = 2∠*ACB* (Theorem 3.1.1).

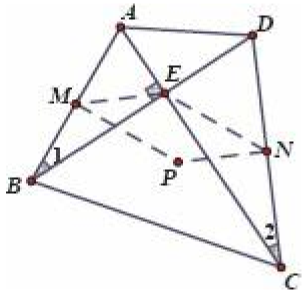
Similarly, ∠*CPD* = 2∠*CBD*. Since ∠*ACB* + ∠*CBD* = 90°, we must have ∠*APB* + ∠*CPD* = 180°. This completes the proof.

□

Note: One may also solve the problem by considering

$$[\Delta ABP]=\frac{1}{2}AB\cdot PM \text{ and } [\Delta CDP]=\frac{1}{2}CD\cdot PN.$$

It suffices to show $\frac{AB}{PN}=\frac{CD}{PM}$.



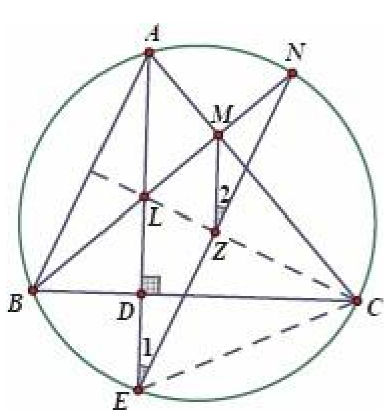
Notice that in the right angled triangle ΔABE, $AB = 2EM$ because M is the midpoint of AB.

Similarly, $CD = 2EN$. Now it suffices to show $\frac{EM}{PN} = \frac{EN}{PM}$. (1)

In fact, we claim that $EMPN$ is a parallelogram. Refer to the diagram above on the right. We have $\angle BEM = \angle 1 = \angle 2 = \angle CEN$. Now
$\angle EMP = 90° - \angle AME = 90° - 2\angle 1$ (2)
$\angle MEN = 90° + \angle BEM + \angle CEN = 90° + 2\angle 1$ (3)
(2) and (3) give $\angle EMP + \angle MEN = 180°$ and hence, $EM // PN$.
Similarly, $EN // PM$ and $EMPN$ is a parallelogram. This implies (1) and the conclusion follows.

Example 6.1.6 (HEL 11) In an acute angled triangle ΔABC, $AB < AC$, $AD \perp BC$ at D and AD extended intersects the circumcircle of ΔABC at E. The perpendicular bisector of AB intersects AD at L. BL extended intersects AC at M and intersects the circumcircle of ΔABC at N. EN and the perpendicular bisector of AB intersect at Z. Show that if $AC = BC$, then $MZ \perp BC$.

Insight. Refer to the diagram below. Since $AD \perp BC$, we **should** have $MZ // AD$. Can we show $\angle 1 = \angle 2$? We **should** have $\angle 2 = \angle MCN$ and hence, C, N, M, Z **should** be concyclic.

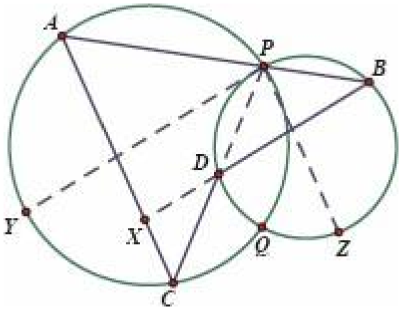


Notice that there are many equal angles in the diagram. In fact, the isosceles triangle ΔABC is symmetric about the perpendicular bisector of AB. One may also notice that L is the orthocenter of ΔABC. It should not be difficult to show the concyclicity by angle-chasing.

Proof. Since $AC = BC$, it is easy to see that ΔABC is symmetric about the perpendicular bisector of AB. Hence, C, Z, L are collinear, which gives the angle bisector of $\angle ACB$ (and the perpendicular bisector of AB). In particular, L is the orthocenter of ΔABC.
Recall that E and L are symmetric about BC (Example 3.4.3). It follows that $\angle N = \angle BCE = \angle BCL = \angle ACL$. Hence, C, N, M, Z are concyclic. Now $\angle 2 = \angle MCN = \angle 1$, which implies $AE // MZ$, i.e., $MZ \perp BC$.

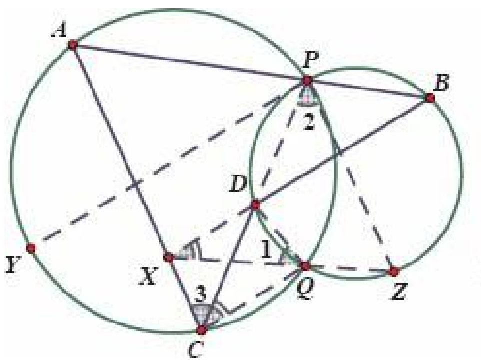
□

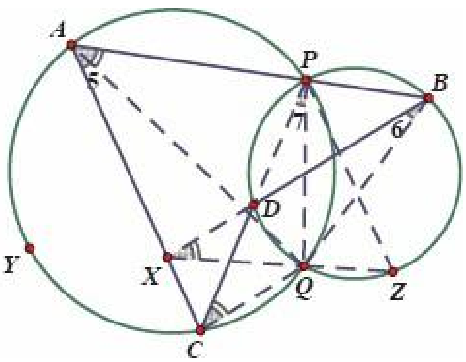
Example 6.1.7 (USA 07) Refer to the diagram on the below. Γ_1, Γ_2 are circles intersecting at P, Q. AC, BD are chords in Γ_1, Γ_2 respectively such that AB intersects CD extended at P. AC intersects BD extended at X Let Y, Z be on Γ_1, Γ_2 respectively such that $PY // BD$ and $PZ // AC$. Show that Y, X, Q, Z are collinear.



Insight. It seems that Y and Z are constructed in a symmetric manner. If we can show that X, Q, Z are collinear, perhaps a similar argument applies for X, Q, Y. Refer to the following diagram.

We are to show $\angle l = 180° - \angle DQZ = \angle 2$. Since $PZ // AC$, we have $\angle 2 = \angle 3$ and hence, it suffices to show $\angle 1 = \angle 3$. Hence, C, Q, D, X **should** be concyclic. Can we show $\angle DCQ = \angle DXQ$?





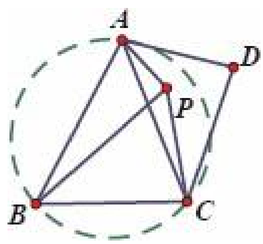
Refer to the right diagram above. Clearly, $\angle DCQ = \angle PAQ$ and hence, we **should** have $\angle BAQ = \angle BXQ$ and A, B, Q, X **should** be concyclic. Hence, it suffices to show $\angle 5 = \angle 6$.
Can you see that $\angle 5 = \angle 7 = \angle 6$ by concyclicity? We leave it to the reader to complete the proof. (**Hint**: One may conclude that Q, X, Y are collinear by observing that $\angle CQX = \angle CQY$, where $\angle CQX = \angle CDX = \angle CPY = \angle CQY$ by concyclicity and $PY // BD$.)

6.2 Recognizing a Relevant Theorem

Occasionally, one may encounter a geometry problem in the competition where the construction seems closely related to a particular theorem (or a well-known fact). It might be a wise strategy to apply the theorem and find the missing links during the process. If you are successful, there is a high chance that the proof is almost complete.

Example 6.2.1 (CHN 08) Given a convex quadrilateral $ABCD$ where $\angle B + \angle D < 180°$, P is an arbitrary point in ΔACD and we define $f(P) = PA \cdot BC + PD$ $AC + PC \cdot AB$ Show that when $f(P)$ attains the minimal value, B, P, D are collinear.

Insight. One should recognize that $f(P)$ is closely related to Ptolemy's Theorem. In fact, we have $PA \cdot BC + PC \cdot AB \geq PB \cdot AC$ and the equality holds if and only if P lies on the circumcircle of ΔABC. Refer to the diagram on the below. Now it is easy to complete the proof.

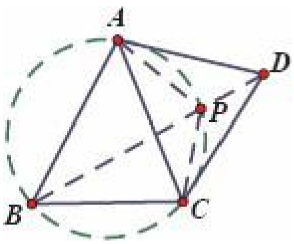


Proof. Notice that $f(P) = PA \cdot BC + PD \cdot AC + PC \cdot AB \geq PB \cdot AC + PD \cdot AC$ by

Ptolemy's Theorem (1)
$= (PB + PD) \cdot AC \geq BD \cdot AC$ by Triangle Inequality. (2)
Hence, the minimal possible value of $f(P)$ is $BD \cdot AC$. This is only attainable if the equality holds in both (1) and (2), i.e., we must have that P lies on the circumcircle of ΔABC and P lies on BD (i.e., B, P, D are collinear). This complete the proof.

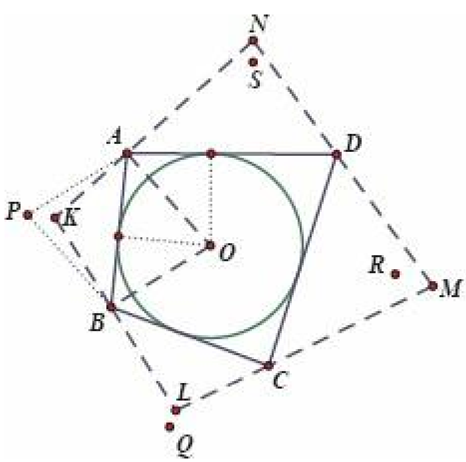
□

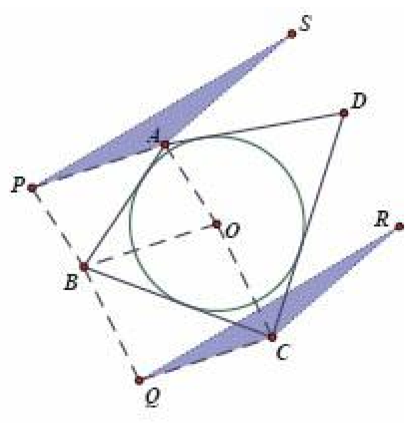
Note: Refer to the diagram on the below. It is easy to see that D must be outside the circumcircle of $\angle ABC$ since $\angle B + \angle D < 180°$. Hence, P must lie between B and D. Indeed, P is the intersection of BD and the circumcircle of ΔABC.



Example 6.2.2 (RUS 04) Let ⊙O be inscribed inside a quadrilateral $ABCD$. It is given that the external angle bisectors of $\angle A$, $\angle B$ intersect at K, the external angle bisectors of $\angle B$, $\angle C$ intersect at L, the external angle bisectors of $\angle C$, $\angle D$ intersect at M, and the external angle bisectors of $\angle D$, $\angle A$ intersect at N. If the orthocenters of ΔABK, ΔBCL, ΔCDM, ΔDAN are P, Q, R, S respectively, show that $PQRS$ is a parallelogram.

Insight. Refer to the following left diagram. How are the external angle bisectors related to the orthocenter (right angles)? Recall that the angle bisectors of neighboring supplementary angles are perpendicular. Can you see that $OA \perp AK$? Can you see that OA // BP and similarly, OB //AP ? We have a parallelogram $AOBP$.





Proof. It is easy to see that *OA* bisects $\angle BAD$. Hence, $OA \perp AK$ because they bisect neighboring angles which are supplementary. We also have $BP \perp AK$ because *P* is the orthocenter of ΔABK.

Hence, $OA \,//\, BP$ and similarly, $OB \,//\, AP$. It follows that *AOBP* is a parallelogram. Similarly, *BOCQ, CODR* and *DOAS* are parallelograms. Now $AP = OB = CQ$ and $AP \,//\, OB \,//\, CQ$. Similarly, $AS = CR$ and $AS \,//\, CQ$. It follows that $\Delta APS \cong \Delta CQR$ (S.A.S.). Refer to the right diagram above. We conclude that $PS = QR$, $PS \,//\, QR$ and hence, *PQRS* is a parallelogram.

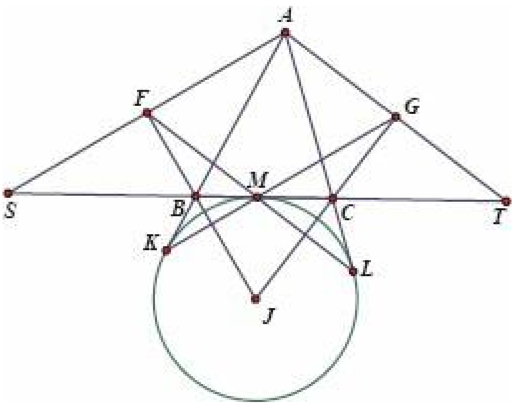
□

Note: One may recall that $AB + CD = BC + AD$ since $\odot O$ is inscribed inside *ABCD*. However, this is not related to the conclusion.

Example 6.2.3 (IMO 12) Let *J* be the ex-center of ΔABC opposite the vertex *A*. This ex-circle (i.e., the circle centered at *J* and tangent to *BC, AB* extended and *AC* extended) is tangent to *BC* at *M*, and is tangent to the lines *AB, AC* at *K, L* respectively. Let the lines *LM, BJ* meet at *F* and the lines *KM, CJ* meet at *G. If AF* extended and *AG* extended meet the line *BC* at *S*, *T* respectively, show that *M* is the midpoint of *ST*.

Insight. Refer to the following diagram. We are to show $SM = TM$. Notice that there are many lines intersecting each other and the ex-circle gives us many equal line segments. Hence, we may apply Menelaus' Theorem involving *SM* and *TM*.

Apply Menelaus' Theorem to ΔASC intersected by the line *FL* and we obtain



$$\frac{SM}{CM}\cdot\frac{CL}{AL}\cdot\frac{AF}{SF}=1.$$

Since $CM = CL$, we have

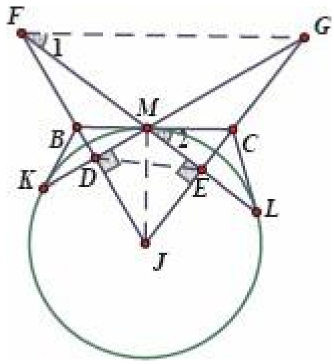
$$SM=\frac{SF\cdot AL}{AF}.$$

Similarly, apply Menelaus' Theorem to ΔATB intersected by the line GK and we have $TM = \dfrac{TG \cdot AK}{AG}$.

We are to show $SM = TM$, i.e., $\dfrac{SF \cdot AL}{AF} = \dfrac{TG \cdot AK}{AG}$.

Clearly, $AK = AL$ Hence, it suffices to show that $\dfrac{SF}{AF} = \dfrac{TG}{AG}$, i.e., **should** have $FG \parallel BC$.

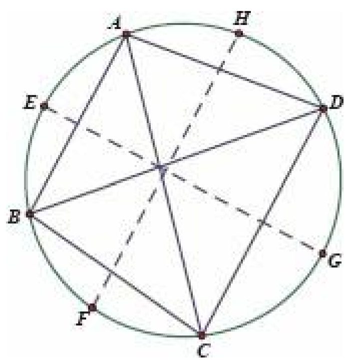
Notice that we can simplify the diagram significantly because *A, S, T* can be neglected. Refer to the diagram on the below. It suffices to show that $\angle 1 = \angle 2$. Since $BJ \perp MK$ and $CJ \perp ML$, one sees a number of concyclicity and hence, many equal angles.



Indeed, we have $\angle 1 = \angle MDE$ (since *D, E, G, F* are concyclic)

$= \angle MJE$ (since *D, J, E, M* are concyclic)
$= 90° - \angle EMJ = \angle 2$ (since $MJ \perp$ BC). This completes the proof.

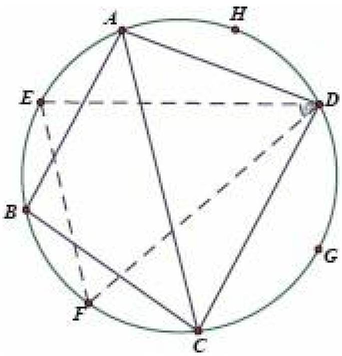
Example 6.2.4 (IND 11) Refer to the diagram on the below. A quadrilateral *ABCD* is inscribed inside a circle. Let *E, F, G, H* be the midpoints of arcs $\overset{\frown}{AB}, \overset{\frown}{BC}, \overset{\frown}{CD}, \overset{\frown}{DA}$ respectively.

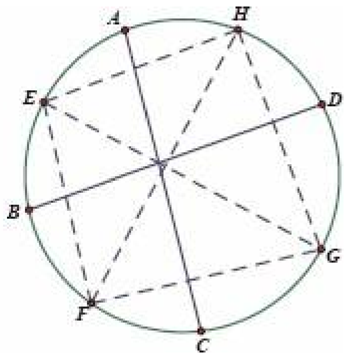


It is known that $AC \cdot BD = EG \cdot FH$ Show that AC, BD, EG and FH are concurrent.

Insight. Apparently there are very few clues. In particular, we do not know how $AC \cdot BD = EG \cdot FH$can be applied. How are these line segments related?

While AC may not be related to EG, it is not difficult to see that AC is related t o EF, because $\angle EDF = \frac{1}{2}\angle ADC$! (Can you see it?) Refer to the left diagram below.





Similarly, AC is related to HG and BD is related to EH, FG (Can you see that $\angle GBH = 90° - \angle EDF$?) Refer to the right diagram above. If we can replace AC and BD by EF, FG, GH and EH, the condition given becomes a relationship between the sides of a cyclic quadrilateral and its diagonals. Is it reminiscent of Ptolemy's Theorem?

Proof. It is easy to see that $\angle EDB + \angle BDF = \frac{1}{2}\angle ADB + \frac{1}{2}\angle BDC$, i.e., $\angle EDF = \frac{1}{2}\angle ADC$. By Sine Rule, $\frac{AC}{\sin \angle ADC} = 2R = \frac{EF}{\sin \angle EDF}$, where R is the radius of the circle. Let $\angle EDF = \alpha$.

We have $AC = \frac{EF}{\sin\alpha}\sin 2\alpha = \frac{EF}{\sin\alpha} 2\sin\alpha\cos\alpha = 2EF\cos\alpha$. (1)

Similarly, $AC = 2HG \cos\angle GBH = 2HG \cos(90° - \alpha) = 2HG \sin\alpha$. (2)
Let $\angle FAG = \beta$. We also have $BD = 2FG\cos\beta = 2EH \sin\beta$. Ptolemy's Theorem states $EF \cdot HG + FG \cdot EH = EG \cdot FH$. (3)
By (1) and (2),

$$EF \cdot HG = \frac{AC^2}{4\sin\alpha\cos\alpha} = \frac{AC}{2}\cdot\frac{AC}{\sin 2\alpha} = \frac{AC}{2}\cdot 2R = AC \cdot R.$$

Similarly, $FG \cdot EH = BD \cdot R$.
Now (3) gives $EF \cdot HG + FG \cdot EH = (AC + BD) \cdot R = EG \cdot FH$.

Since $EG \cdot FH = AC \cdot BD$, we have $(AC + BD) \cdot 2R = 2AC \cdot BD$. (4)
Notice that $2R$ is the diameter, i.e., $AC, BD \leq 2R$.

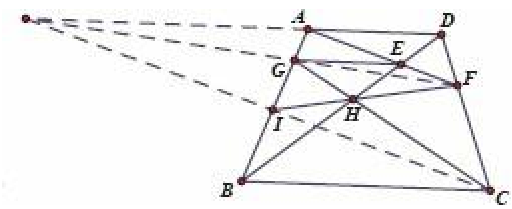
It follows that $(AC + BD) \cdot 2R = AC \cdot 2R + BD \cdot 2R \geq AC \cdot BD + BD \cdot AC = 2AC \cdot BD$, where the equality holds by (4). This is only possible if $AC = BD = 2R$, i.e., AC, BD are both diameters of the circle.
Since $AC \cdot BD = EG \cdot FH$, EG, FH are also diameters. In conclusion, AC, BD, EG, EH are concurrent at the center of the circle.

□

Note: In (1), we applied the double angle formula $\sin 2\alpha = 2 \sin\alpha \cos\alpha$, which could be found in most pre-calculus textbooks.

Example 6.2.5 (UKR 11) Given a trapezium *ABCD, AD // BC* and *F* is a point on *CD. AF* and *BD* intersect at *E*. Draw *EG // AD*, intersecting *AB* at *G*. *BD* and *CG* intersect at *H. AB* and *FH* extended intersect at *I*. Show that the lines *AD, CI, FG* are concurrent.

Insight. Refer to the diagram on the below. It seems not easy to show the lines *AD, CI, FG* are concurrent. Notice that the intersection of these lines is far from the trapezium *ABCD*.

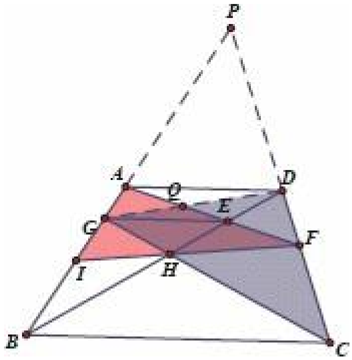


Since *AD // EG // BC*, the Intercept Theorem gives many equations of line segments in the trapezium *ABCD*. Is it possible for us to derive the conclusion from these line segments instead of the extensions of *AD, CI, FG*? Recall Desargues' Theorem. Can we find two triangles whose vertices are *A, D, C, I, F, G*, while the lines connecting corresponding vertices are *AD, CI, FG* respectively?

It is not a difficult task. In fact, since *A, G, I* and *C, D, F* are collinear, we do not have many choices left, one of which is ΔAFI and ΔDGC. Can we show that these two triangles satisfy the condition for Desargues' Theorem, i.e., say the lines *AB, CD* intersect at *P* and *AF, DG* intersect at *Q*, can we show that *P, Q, H* are collinear? Refer to the diagram below.

Proof. Let the lines *AB*, *CD* intersect at P and *AF*, *DG* intersect at *Q*. We claim that P, Q, H are collinear. By Menelaus' Theorem, it suffices to show that $\frac{AQ}{EQ}\cdot\frac{EH}{BH}\cdot\frac{BP}{AP}=1$.

Since $EG \,//\, AD \,//\, BC$, we have



$$\frac{AQ}{EQ}=\frac{AD}{EG},\ \frac{EH}{BH}=\frac{EG}{BC}\ \text{and}\ \frac{BP}{AP}=\frac{BC}{AD}.$$

It follows that $\frac{AQ}{EQ}\cdot\frac{EH}{BH}\cdot\frac{BP}{AP}=\frac{AD}{EG}\cdot\frac{EG}{BC}\cdot\frac{BC}{AD}=1$.

Since P, Q, H are collinear, the conclusion follows by applying Desargues' Theorem to ΔAFI and ΔDGC.

□

6.3 Unusual and Unused Conditions

A typical geometry problem in competitions comes with a few given conditions. Besides those more standard conditions (parallel and perpendicular lines, midpoints, angle bisectors, circles and tangency), there may be unusual conditions which easily catch the attention of the contestants. For example:

- Angles or line segments which are far apart but equal
- 30°, 45° or 60° angles
- Points constructed in an unusual manner
- Equations of line segments or angles, the geometric meanings of which apparently not clear
- Points, lines or circles which coincide unexpectedly

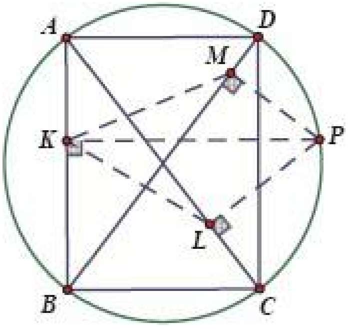
One naturally expects such conditions to play a critical role when solving the geometry problem. Hence, it is worthwhile to spend time and effort focusing on these conditions, which may lead to an important intermediate step.

On the other hand, it is also common that one cannot find any clue after exploring the unusual condition, or even cannot see any *sense* about it. Do not be frustrated! It could be a wise strategy to leave it aside and focus on other conditions, writing down intermediate steps which could be derived. We shall attempt to link those steps to the conclusion and expect to be stuck during the process (because we still have unused conditions). Now you may find the unused condition handy: it may be exactly the missing link needed!

Geometry problems in competitions are generally well constructed and the conditions given are exactly sufficient (because unnecessary extra conditions may cause inconsistency). Hence, if all the conditions given have been applied and a chain of derivations is constructed, most probably you are very close to the complete proof.

Example 6.3.1 (CZE-SVK 09) Given a rectangle *ABCD* inscribed inside $\odot O$, *P* is a point on the minor arc $\overset{\frown}{CD}$. Let *K, L, M* be the feet of the perpendiculars from *P* to *AB, AC, BD* respectively. Show that $\angle LKM = 45°$ if and only if *ABCD* is a square.

Insight. One immediately notices that $\angle LKM = 45°$ is an unusual condition. How is it related to *ABCD*, in which case a square? Refer to the diagram on the below. If *ABCD* is a square, we must have $\angle CAD = \angle CBD = 45°$.



It seems we shall apply the angle properties around the circles. (Notice that the right angles give a number of concyclicity.)

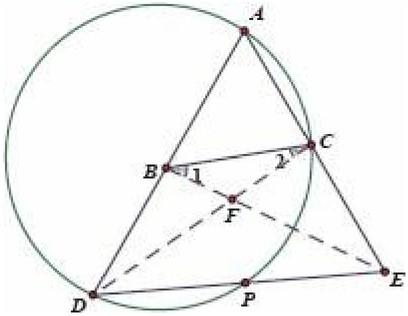
Proof. Since $\angle BKP = \angle BMP = 90°$, *B, K, M, P* are concyclic and $\angle PKM = \angle PBM$. Similarly, *A, K, L, P* are concyclic, which implies $\angle PKL = \angle PAL = \angle PBC$ (angles in the same arc).
Now $\angle LKM = \angle PKM + \angle PKL = \angle PBM + \angle PBC = \angle CBD$. Hence, $\angle LKM = 45°$ if and only if $\angle CBD = 45°$, i.e., *ABCD* is a square.

□

Note: One who attempts to show $\angle LKM = \angle CBD$ by concyclicity directly may find it difficult because we do not know much about the intersection of the lines *KL* and *BC*

Example 6.3.2 (IRN 11) Given ΔABC where $\angle A = 60°$, *D, E* are on *AB, AC* extended respectively such that $BD = CE = BC$If the circumcircle of ΔACD intersects *DE* at *D* and *P*, show that *P* lies on the angle bisector of ΔBAC.

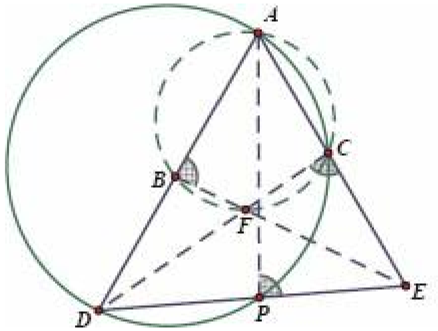
Insight. Refer to the diagram on the below. One immediately notice that $\angle A = 60°$ is an unusual condition. Moreover, it is not easy to draw $BD = BC = CE$ when given ΔADE. How could we apply these conditions?

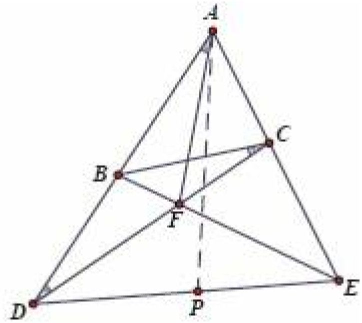


Notice that $BD = BC = CE$ gives two isosceles triangles ΔBCD and ΔCBE, where $\angle l$ and $\angle 2$ are related to $\angle A$.
In fact, $\angle ABC = 2\angle 2$ and $\angle ACB = 2\angle 1$. Since $\angle A = 60°$, we must have $\angle ABC + \angle ACB = 120°$ and hence, $\angle 1 + \angle 2 = 60°$.
Let *BE* and *CD* intersect at *F*. One sees that $\angle BFD = \angle 1 + \angle 2 = 60° = A$ and hence, *A, B, F, C* are concyclic.

Notice that the diagram should be *symmetric:* since *P* **should** lie on the angle bisector of $\angle BAC$, if *A, C, P, D* are concyclic, then *A, B, P, E* **should** be concyclic as well. Can you show it?

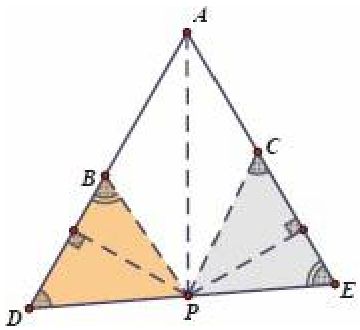
Refer to the left diagram below. Since *A, B, F, C* are concyclic, we have $\angle ABF = \angle ECF$. Since *A, C, P, D* are concyclic, we must have $\angle ECF = 180° - \angle ACD = 180° - \angle APD = \angle APE$. It follows that $\angle ABF = \angle APE$ and hence, *A, B, P, E* are concyclic.

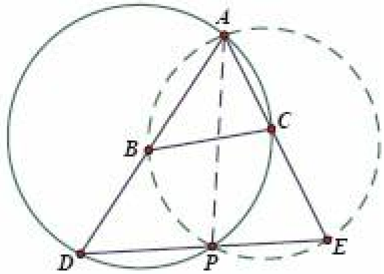




We are to show AP bisects $\angle A$, i.e., $\angle BAP = \angle CAP$. Since $\angle BAP = \angle BEP$ and $\angle CAP = \angle CDP$ by concyclicity, it suffices to show $DF = EF$. Refer to the right diagram above.
Can you see that $DF = EF = AF$, i.e., F is the circumcenter of ΔADE ? (**Hint**: Can you see $\angle BAF = \angle BCF = \angle BDF$?) We leave it to the reader to complete the proof.

Note: Upon showing the concyclicity of A, B, F, C and A, B, P, E, there are many ways to show that AP is the angle bisector. For example, can you see that P is of the same distance from the lines AB and AC? Refer to the following left diagram. Can you see $\Delta BDP \cong \Delta ECP$ (A.A.S.)? Since BD and CE are corresponding sides, the heights from P to BD, CE respectively must be the same.





Alternatively, one may also show the conclusion by the Angle Bisector Theorem, i.e., we are to show $\frac{AD}{AE} = \frac{PD}{PE}$.

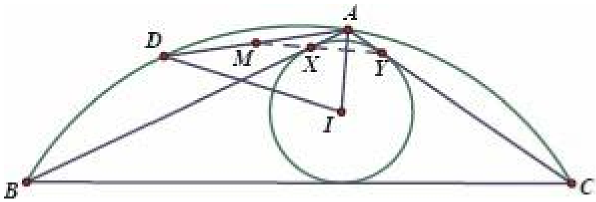
Refer to the right diagram above. By the Tangent Secant Theorem, $AD \cdot BD = PD \cdot DE$ and $AE \cdot CE = PE \cdot DE$.

Since $BD = CE$, we have $\frac{AD}{AE} = \frac{AD \cdot BD}{AE \cdot CE} = \frac{PD \cdot DE}{PE \cdot DE} = \frac{PD}{PE}$, which completes the proof.

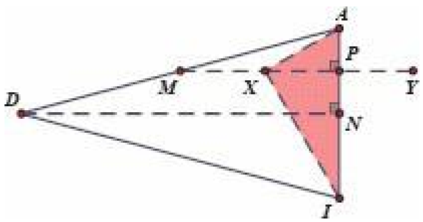
Example 6.3.3 (RUS 08) Given $\odot I$ inscribed inside ΔABC, AB, AC touch $\odot I$

at *X, Y* respectively. Let *CI* extended intersect the circumcircle of ΔABC at *D*. If the line *XY* passes through the midpoint of *AD*, find $\angle BAC$.

Insight. One immediately notices that the line *XY* passing through the midpoint of *AD* is an unusual condition, without which ΔABC, its incenter *I* and *CD* give a standard diagram (Example 3.4.2). Refer to the diagram below.



We have $AD = DI$, i.e., ΔAID is an isosceles triangle. It is easy to see that $XY \perp AI$. Let *M* be the midpoint of *AD*. How could you apply the condition that *X, Y, M* are collinear?



Can you see where *AI* and *XY* intersect? Refer to the diagram above.

Let *N* be the midpoint of *AI*. Clearly, $DN \perp AI$ and hence, $DN // XY$.

Let *AI* intersect *XY* at *P*. Since *X, Y, M* are collinear, *P* is the midpoint of *AN*. It follows that $AP = \frac{1}{4}AI$.

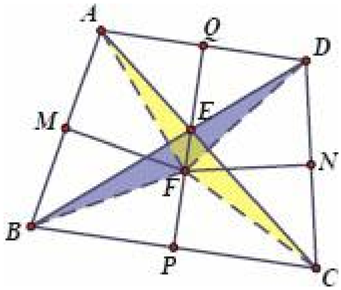
What can you say about the right angled triangle ΔAXI? Can you see that $\frac{AP}{PX} = \frac{PX}{PI}$ and hence, $\left(\frac{AP}{PX}\right)^2 = \frac{AP}{PX} \cdot \frac{PX}{PI} = \frac{AP}{PI} = \frac{1}{3}$?

Now it is easy to see that $\angle XAI = 60°$ and hence, $\angle BAC = 120°$. We leave the details to the reader.

Note: If one draws an acute angled triangle ΔABC, the line *XY* will not even intersect the line segment *AD*. By constructing the diagram carefully, one should realize that $\angle BAC$ is obtuse.

Example 6.3.4 (CGMO 11) Let *ABCD* be a quadrilateral where *AC, BD* intersect at *E*. Let *M, N* be the midpoints of *AB, CD* respectively and the perpendicular bisectors of AB, CD intersect at F. If the line EF intersects BC, AD at P, Q respectively and it is given that $FM \cdot CD = FN \cdot AB$ and $BP \cdot DQ = CP \cdot AQ$, show that $PQ \perp BC$.

Insight. One immediately notices the unusual conditions $FM \cdot CD = FN \cdot AB$ and $BP \cdot DQ = CP \cdot AQ$, but apparently, they refer to different properties.



$FM \cdot CD = FN \cdot AB$ implies $\frac{FM}{FN} = \frac{AB}{CD}$. Refer to the previous diagram. What can you conclude about the (isosceles) triangles ΔABF and ΔCDF?

Can you see that ΔBDF and ΔACF are congruent? What can you conclude upon obtaining the equal angles, say $\angle CAF = \angle DBF$?

$BP \cdot DQ = CP \cdot AQ$ gives $\frac{BP}{CP} = \frac{AQ}{DQ}$. Is it reminiscent of the Angle Bisector Theorem? Where is the angle bisector?

Indeed, the diagram is symmetric. If one sees that F is the center of the circle where $ABCD$ is inscribed, the proof is almost complete.

Proof. Notice that $FM \cdot CD = FN \cdot AB$ implies $\frac{FM}{FN} = \frac{AB}{CD}$. Since M, N are the midpoints of AB, CD respectively, it is easy to see that the isosceles triangles ΔABF and ΔCDF are similar.

In particular, we have $\angle AFB = \angle CFD$, which implies $\angle AFC = \angle BFD$. Since $AF = BF$ and $CF = DF$, we have $\Delta BDF \cong \Delta ACF$ (S.A.S.).
It follows that $\angle CAF = \angle DBF$ and hence, A, B, F, E are concyclic. Similarly, $\angle BDF = \angle ACF$ and C, D, E, F are concyclic.

We have $\angle BEF = \angle BAF$ and $\angle CEF = \angle CDF$ by concyclicity. Notice that $\angle BAF = \angle CDF$ (because $\Delta ABF \sim \Delta DCF$). Now $\angle BEF = \angle CEF$, i.e., EP bisects $\angle BEC$. By the Angle Bisector Theorem, $\frac{BE}{CE} = \frac{BP}{CP}$. (1)

Similarly, EQ bisects $\angle AED$ $\frac{AE}{DE}=\frac{AQ}{DQ}$. (2)

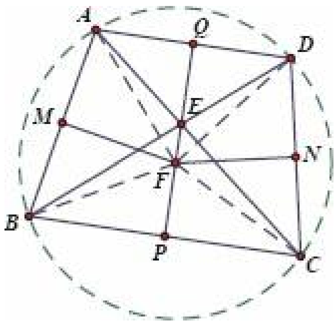
We are given $BP \cdot DQ = CP \cdot AQ$, *i.e.*, $\frac{BP}{CP}=\frac{AQ}{DQ}$. (3)

(1), (2) and (3) imply that $\frac{BE}{CE}=\frac{AE}{DE}$, i.e., $BE \cdot DE = AE \cdot CE$.

By the Intersecting Chords Theorem, *ABCD* is cyclic. Clearly, it is inscribed in a circle centered at *F*.

Now $AF = BF = CF = DF$ and hence, $AB = CD$. Refer to the diagram on the below. It is easy to see that $\Delta ABF \cong \Delta DCF$ (S.A.S.). Hence, $AB = CD$ and *ABCD* is an isosceles trapezium where $AD // BC$. It follows that $PQ \perp BC$.

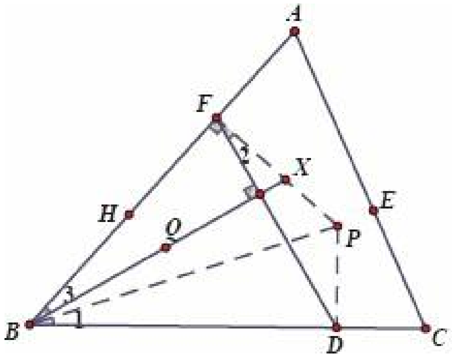
□

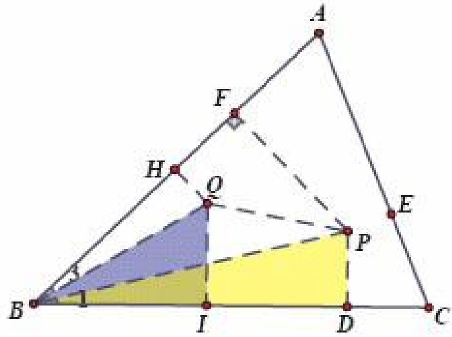


Note: Since the diagram is symmetric, we **should** have $PQ \perp AD$ as well, i.e., $AD // BC$. Upon showing that *EF* bisects $\angle BEC$, one naturally expects that *ABCD* is an isosceles trapezium.

Example 6.3.5 (BGR 11) Let *P* be a point inside the acute angled triangle ΔABC. *D, E, F* are the feet of the perpendiculars from *P* to *BC, AC, AB* respectively. *Q* is a point inside ΔABC such that $AQ \perp EF$ and $BQ \perp DF$. Draw $QH \perp AB$ at *H*. Show that *D, E, F, H* are concyclic.

Insight. We are given a lot of right angles. In particular, one notices that the construction of *Q* is unusual. What can we obtain from *Q*?
Refer to the left diagram below. By applying the properties of right angles, can you see that $\angle 1 = \angle 2 = \angle 3$?

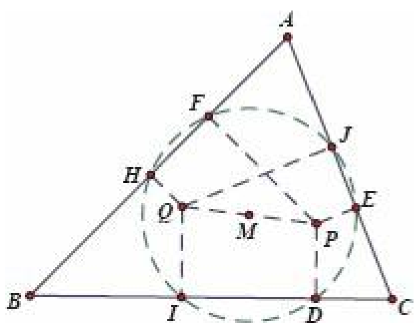




If we draw $QI \perp BC$ at I, can you see that the right angled triangles ΔPBD, ΔPBF, ΔQBH, ΔQBI are closely related? (**Hint**: Can you see similar triangles?) Refer to the right diagram above. How are BI, BH, BD, BF related to BP and BQ, say via similar triangles? How are HF and DI related to PQ?

Proof. Since $PD \perp BC$ and $PF \perp AB$, B, D, P, F are concyclic and hence, $\angle 1 = \angle 2$. Let BQ extended intersect BF at X. We have $\angle 2 = \angle 3$ in the right angled triangle ΔBFX. Hence, $\angle 1 = \angle 3$ and it follows that $\angle CBQ = \angle ABP$.
Draw $QI \perp BC$ at I. We have $BI \cdot BD = BQ \cos\angle CBQ \cdot BP \cos\angle 1$ and $BH \cdot BF = BQ \cos\angle 3 \cdot BP \cos\angle ABP$. It follows that $BI \cdot BD = BH \cdot BF$ Now H, F, D, I are concyclic by the Tangent Secant Theorem.

Let M be the midpoint of PQ. It is easy to see that the perpendicular bisectors of both HF and DI pass through M. Let $\odot M$ denote the circle centered at M with radius DM. Clearly, H, F, D, I lie on $\odot M$. Refer to the diagram on the below.

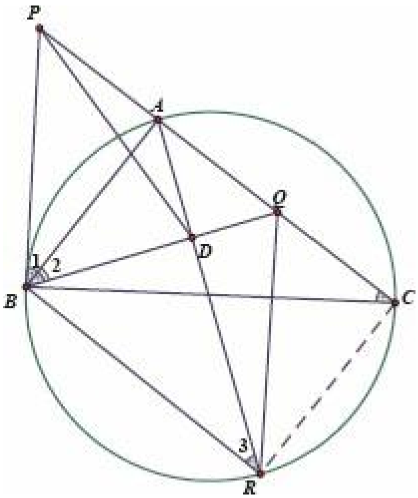


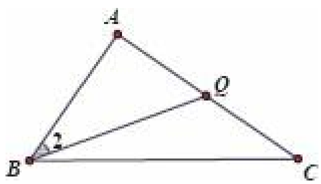
Similarly, if we draw $QJ \perp AC$ at J, one sees that E, J, I, D also lie on $\odot M$, i.e., D, E, F, H, I, J are concyclic. This completes the proof.

□

Example 6.3.6 (IMO 13) Given ΔABC with $\angle B > \angle C$, Q is on AC and P is on CA extended such that $\angle ABP = \angle ABQ = \angle C$. D is a point on BQ such that $PB = PD$. AD extended intersect the circumcircle of ΔABC at R. Show that $QB = QR$.

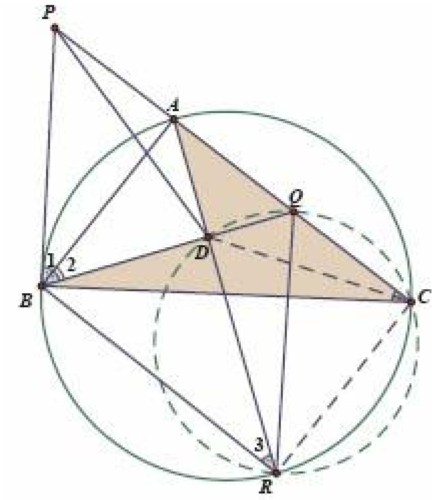
Insight. Refer to the diagram on the below. One immediately notices the condition $\angle 1 = \angle 2 = \angle ACB$. We also have $\angle ACB = \angle 3$ (angles in the same arc).





Recall a basic result of similar triangles as shown in the diagram above. Since $\angle 2 = \angle C$, we must have $\Delta ABQ \sim \Delta ACB$. Since $\angle 2 = \angle 3$, we also have $\Delta ABD \sim \Delta ARB$. It follows that $\angle ABR = \angle ADB = \angle QDR$. Since $\angle ABR + \angle ACR = 180°$, we must have $\angle QDR + \angle ACR = 180°$, which implies C, Q, D, R are concyclic.

Refer to the diagram on the below. We are to show $QB = QR$ Of course, the most straightforward method is to show that $\angle QBR = \angle QRB$. Since we have two circles and a few pairs of similar triangles, perhaps we shall seek more equal angles.



We can write:

$$\angle QRB = \angle 3 + \angle QRD = \angle ACB + \angle QCD$$

$$\angle QBR = \angle CBR + \angle CBQ = \angle CAR + \angle CBQ.$$

Now it suffices to show that $\angle ACB + \angle QCD = \angle CAR + \angle CBQ$. (1)

Notice that all these angles are related to the shaded region in the diagram. In particular, $\angle ACB + \angle CAR + \angle CBQ = \angle ADB$ (exterior angles of ΔACD and ΔBCD). How is this related to (1)? If one cannot see the clue, substitute $\angle CAR + \angle CBQ = \angle ADB - \angle ACB$ into (1)! Now it suffices to show that $2\angle ACB + \angle QCD = \angle ADB$. (2)

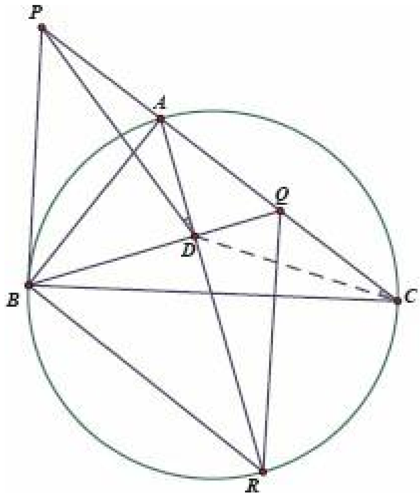
How can we show (2)? Notice that this is not true for an arbitrary (concave) quadrilateral *ABCD*. Which are the conditions given we have **not** used yet? We have not used:

- $PB = PD$
- PB is a tangent (i.e., $\angle 1 = \angle C$).

Could these two conditions help us?

Since $PB = PD$, we immediately have $\angle PDB = \angle PBD = 2\angle ACB$. This is awesome! Now (2) becomes $\angle PDB + \angle QCD = \angle ADB$ and it suffices to show that $\angle QCD = \angle ADB - \angle PDB = \angle PDA$.

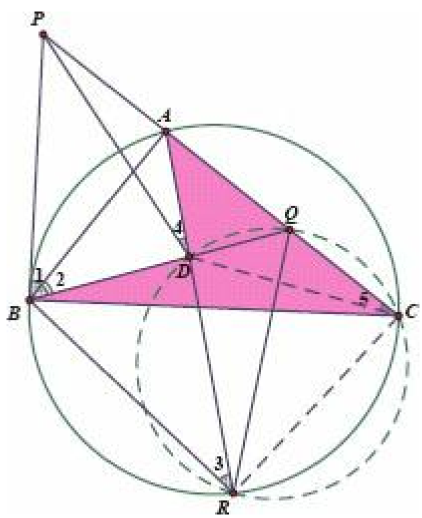
Refer to the diagram on the below. We could reach the conclusion by showing $\Delta PAD \sim \Delta PDC$. In fact, these two triangles **should** be similar.



Can we show $\frac{PA}{PD} = \frac{PD}{PC}$, or equivalently, $PA \cdot PC = PD^2$?

Notice that we have $PB = PD$ and $PB^2 = PA \cdot PC$ since PB is a tangent. Now both unused conditions have made their contributions, which complete the proof.

Proof. Refer to the diagram on the below. It is given that $\angle 1 = \angle 2 = \angle ACB = \angle 3$. Hence, $\Delta ABD \sim \Delta ARB$ and we have $\angle ADB = \angle ABR = 180° - \angle ACR$. It follows that $\angle BDR = \angle ACR$, which implies that C, Q, D, R are concyclic.
Since $\angle 1 = \angle ACB$, PB is tangent to the circumcircle of ΔABC. Given $PB = PD$, we have $PD^2 = PB^2 = PA \cdot PC$.



This implies $\frac{PA}{PD} = \frac{PD}{PC}$ and hence, $\Delta PAD \sim \Delta PDC$. We have $\angle 4 = \angle 5$.

Now $\angle QBR = \angle CBR + \angle CBQ = \angle CAR + \angle CBQ$
$= \angle ADB - \angle ACB$ (exterior angles of ΔACD and ΔBCD)

$$= \angle PDB + \angle 4 - \angle ACB = \angle PBD + \angle 4 - \angle ACB \text{ (since } PB = PD)$$

$$= 2\angle ACB + \angle 4 - \angle ACB = \angle ACB + \angle 4 = \angle ACB + \angle 5$$

$$= \angle 3 + \angle ARQ = \angle QRB, \text{ which implies } PD = PB.$$

Note:

(1) The last section of angle-chasing is a concise argument and one needs to be very familiar with basic properties of angles, especially in circles. In fact, such angle-chasing is commonly seen in geometry problems and is considered a fundamental technique. Nevertheless, we should point out that such a compact argument presented is only for mathematical elegance. In fact, it is not inspiring as the reader following the argument may not see **how** to search for clues and reach the conclusion. This is exactly why we spend a few more pages in explaining the insight.

(2) One may find an alternative solution starting from the Angle Bisector Theorem: Since *BA* bisects of $\angle PBD$, we have $\frac{QB}{QA} = \frac{PB}{PA}$. Since we are to show $QB = QR$, it suffices to show that $\frac{QR}{QA} = \frac{PB}{PA} = \frac{PD}{PA}$ because $PB = PD$. Upon showing *C, Q, D, R* concyclic, it is easy to see that $\Delta ACD \sim \Delta ARQ$ and hence, $\frac{QR}{QA} = \frac{CD}{AD}$. Now it suffices to show $\frac{PD}{PA} = \frac{CD}{AD}$, but this is because $\Delta PCD \sim \Delta PDA$.

6.4 Seeking Clues from the Diagram

A well-constructed diagram could be very helpful in problem-solving, especially for those more challenging problems in competition where the insight is not clear. Although referring to the diagram is **not** a valid proof, it may give us hints on what **could** be correct. One should always construct a diagram according to the description in the problem without any loss of generality. For example, given a triangle ΔABC where *P* is an arbitrary point on *BC*, one should avoid drawing an isosceles or right angled triangle, and choose *P* to be distinct from the midpoint of *BC* and the feet of the perpendicular from *A*. If one constructs a **general** diagram and observes any geometric fact from the diagram, for example, a right angle, collinearity or concyclicity, it **may** be true! One may attempt to show it, or assume it is true and seek intermediate steps which could be deduced.

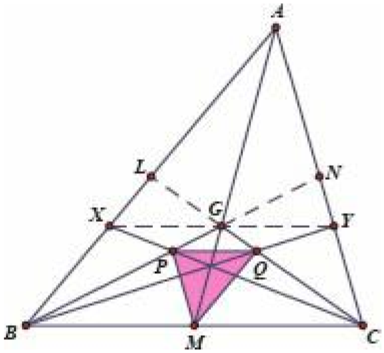
Drawing a (reasonably) **accurate** diagram may help us substantially in seeking clues. Note that if circumcircles, incircles, tangent lines or equal angles are given, one should **not** construct these geometric objects casually. For example, when given a circle inscribed in a triangle, it is recommended that one draws the triangle and its angle bisectors to locate the incenter, constructs the incircle with a compass, and introduces heights to find the points of tangency. A poorly constructed or distorted diagram may be misleading and heavily distract one from acquiring the insight.

One should also learn to **simplify** the diagram, erasing lines, points and circles during problem-solving when necessary. Indeed, when the diagram is complicated, one may fail to recognize even the most elementary geometric facts (for example, radii of a circle which are the same, equal tangent segments, perpendicular bisectors which give isosceles triangles, etc.). In particular, if circumcenters or orthocenters are given, one should only draw explicitly the circles and altitudes which are necessary. Otherwise, the diagram may become unreadable!

When exploring a part of the diagram which demonstrates a specific geometric structure, one may consider drawing a **separate** diagram focused on that part. In a much simpler setting, one may find it easier to seek clues or recognize a well-known result. Refer to Example 6.3.3 for an illustration on this strategy.

Example 6.4.1 (APMO 91) Let G be the centroid of ΔABC. Draw a line XY // BC passing through G, intersecting AB, AC at X, Y respectively. BG and CX intersect *at* P. CG and BY intersect at Q. If M is the midpoint of BC, show that $\Delta ABC \sim \Delta MQP$.

Insight. Refer to the diagram on the below. It *seems* that the corresponding sides of ΔABC and ΔMQP are parallel. Can we show it, say PQ // BC? Since we are given the centroid and a parallel line, we can find the ratio of the line segments easily.



***Proof*.** Let L, N be the midpoints of AB, AC respectively. Since $XY \parallel BC$ and G is the centroid, it is easy to see that $GX = GY$.

Hence, $\frac{PX}{CP} = \frac{GX}{BC} = \frac{GY}{BC} = \frac{QY}{BQ}$ and by the Intercept Theorem, $PQ \parallel BC$.

One also sees that $\frac{GY}{CM} = \frac{AG}{AM} = \frac{2}{3}$. Hence, $\frac{GQ}{CQ} = \frac{GY}{BC} = \frac{1}{3}$.

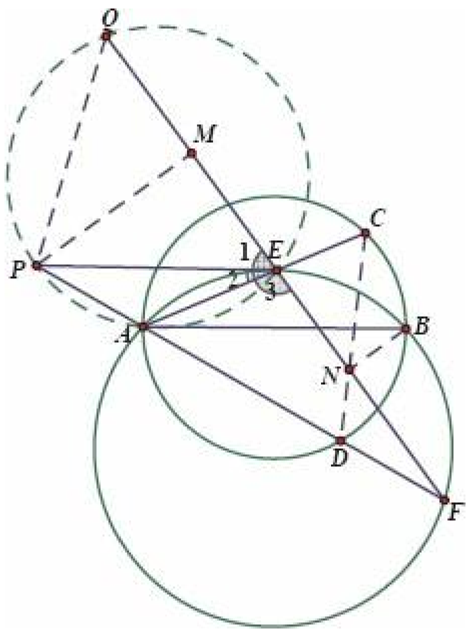
Since $\frac{CG}{CL} = \frac{2}{3}$, we have $\frac{CQ}{CL} = \frac{CQ}{CG} \cdot \frac{CG}{CL} = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$, i.e., Q is the midpoint of CL. Hence, $MQ \parallel AB$ by the Midpoint Theorem.

Similarly, $MP \parallel AC$ and the conclusion follows.

□

Example 6.4.2 (TUR 10) Given a circle Γ_1 where AB is a diameter, C, D lie on Γ_1 and are on different sides of the line AB. Draw a circle Γ_2 passing through A, B, intersecting AC at E and AD extended at F. Let P be a point on DA extended such that PE is tangent to Γ_2 at E. Let Q be a point (different from E) on the circumcircle of ΔAEP such that $PE = PQ$ Let M be the midpoint of EQ. If CD and EF intersect at N, show that $PM \parallel BN$.

***Insight*.** One may notice that constructing such a diagram following the instructions given is not a simple task. However, it could be rewarding. Refer to the diagram on the right. It *seems* that E, F, Q are collinear. Is it true?



Since $PE = PQ$ and M is the midpoint of EQ, we immediately have $PM \perp EQ$. *If E, F, Q* are indeed collinear, we should have $BN \perp EF$.
What can we say about BN and EF? Can you see $BC \perp AC$ and $BD \perp AD$?
Can you see $BN \perp EF$ while CD is a Simson's Line of ΔAEF ?

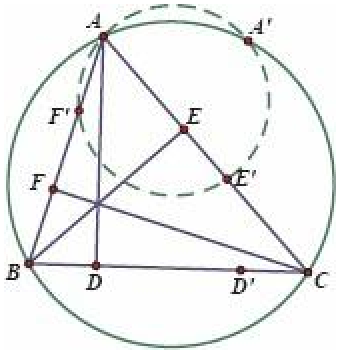
We are to show $PM // BN$. Hence, we **should** have *E, F, Q* collinear. Can we show that $\angle 1 + \angle 2\ \angle 3 = 180°$?

Proof. Since $PE = PQ$, we have $PM \perp EQ$. Since AB is a diameter of Γ_1, *C, D* are the feet of the perpendiculars from B to AE, AF respectively. Hence, CD is the Simson's Line of ΔAEF with respect to B. It follows that $BN \perp EF$. Now it suffices to show that *E, F, Q* are collinear.
Notice that $\angle 1 = \angle Q = \angle EAF$ (Corollary 3.1.5) and $\angle 2 = \angle F$ (Theorem 3.2.10). Now $\angle 1 + \angle 2 + \angle 3 = \angle EAF + \angle F + \angle 3 = 180°$. This completes the proof.

□

Example 6.4.3 (VNM 09) Let Γ be the circumcircle of an acute angled triangle ΔABC, where *D, E, F* are the feet of the altitudes from *A, B, C* respectively. Let D', E', F' be the points of reflection of *D, E, F* about the midpoints of *BC, AC, AB* respectively. The circumcircles of $\Delta AE'F'$, $\Delta BD'F'$, $\Delta CD'E'$ meet Γ again at A', B', C' respectively. Show that $A'D$, $B'E$ and $C'F$ are concurrent.

Insight. Apparently, the construction of the diagram is complicated. For example, we draw the diagram on the right to locate A'. (Notice that we have already omitted the midpoints of *AB, BC, CA*.) If we continue to construct *B*' and *C*', the diagram might be unreadable! Perhaps we should examine the property of *A*' first.

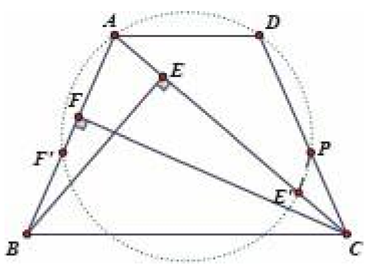


It *seems* from the diagram that *A*' is symmetric to *A* with respect to the perpendicular bisector of *BC*, i.e., $A'D' \perp BC$. If we can show that *A′* is indeed symmetric to *A*, the diagram could be significantly simplified.

Proof. We first show the following lemma.

Let ABCD be an isosceles trapeium where AD // BC and AB = CD. Draw E$\perp$ *AC at E and* $CF \perp AB$ *at F. Let E', F' be on AC, AB respectively such that AE = C and AF = BF'. We have A, D, E', F' concyclic.*

Refer to the diagram on the below. Let *P* be the reflection of *F′* about the perpendicular bisector of *BC*. Since *ABCD* is an isosceles trapezium, we have *CP = BF'= AF*.



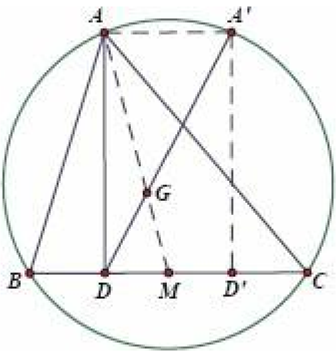
Notice that $\dfrac{AF}{AC} = \cos\angle BAC = \dfrac{AE}{AB}$.

Since $AE = CE'$ and $AB = CD$, we must have $\dfrac{CP}{CA} = \dfrac{AF}{CA} = \dfrac{AE}{AB} = \dfrac{CE'}{CD}$, i.e, $\dfrac{CP}{CA} = \dfrac{CE'}{CD}$. Now $CP \cdot CD = CA \cdot CE'$, which implies *A, D, P, E*' are concyclic.

Since *ADPF*' is an isosceles trapezium, we conclude that *A, D, P, E*', *F*' are concyclic.

We apply this lemma to the original problem. Let X be the point symmetric to A with respect to the perpendicular bisector of BC. Now $ABCX$ is an isosceles trapezium and by the lemma, A, X, E, F are concyclic. This implies A' and X coincide (since the circumcircle of $\Delta AE'F'$ intersect Γ only at A and A'). In particular, $ADD'A'$ is a rectangle.

We are to show that $A'D$, $B'E$, $C'F$ are concurrent. Let us examine the property of $A'D$. Refer to the diagram on the right where M is the midpoint of BC.



Since $ADD'A'$ is a rectangle, we have $DM = \frac{1}{2}AA'$.

It follows that $\frac{MG}{AG} = \frac{DM}{AA'} = \frac{1}{2}$ and hence, G is the centroid of ΔABC.

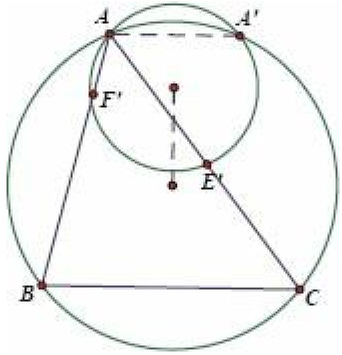
We conclude that $A'D$ passes through the centroid of ΔABC. Similarly, $B'E$ and $C'F$ must pass through the centroid of ΔABC as well. This completes the proof.

□

Note:

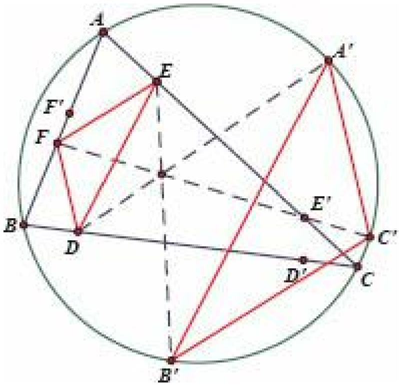
(1) One may also use the power of a point to show that A' is symmetric to A with respect to the perpendicular bisector of BC. In particular, one may show that $BF' \cdot BA = CE' \cdot CA$ (because $\frac{BF'}{CE'} = \frac{AF}{AE} = \frac{AC}{AB}$) and hence, B and C have the same power with respect to the circumcircle of $\Delta AE'F'$.

Refer to the diagram on the below. It follows that the circumcenter of $\Delta AE'F'$ is equidistant to B and C and hence, lies on the perpendicular bisector of BC. Now the line passing through the circumcenters of $\Delta AE'F'$ and ΔABC is perpendicular to BC.



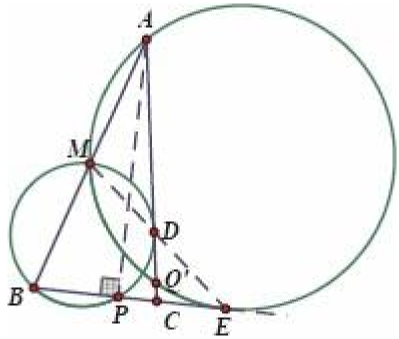
This line must be perpendicular to AA' as well (Theorem 3.1.20). We conclude that $AA' // BC$.

(2) One may also observe the diagram and attempt to show $DE // A'B'$. Give the reflections A' and B', this is not difficult (Exercise 3.13). Similarly, we have $EF // B'C'$ and $DF // A'C'$. Refer to the diagram on the below. Now $A'D$, $B'E$, $C'F$ are concurrent by Theorem 2.5.11.



Example 6.4.4 (BLR 11) Given an acute angled triangle ΔABC, M is the midpoint of AB. Let P, Q be the feet of the perpendiculars from A to BC and from B to AC respectively. If the circumcircle of ΔBMP is tangent to the line segment AC, show that the circumcircle of ΔAMQ is tangent to the line BC.

Insight. Refer to the diagram on the below. It is not easy to show a circle tangent to a line. However, notice that the circle passing through A, M and tangent to the line BC is unique. Hence, we may draw this circle and show that it intersects AC exactly at Q.

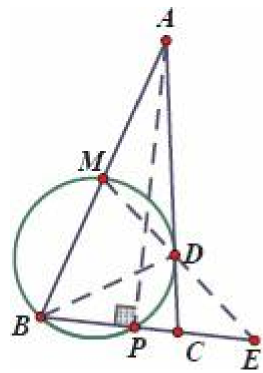


Let AC touch the circumcircle of ΔBMP at D. **If** BC extended touches the circumcircle of ΔAMQ at E, we would have $BE^2 = BM \cdot BA$. Since M is the midpoint, we **should** have $BE^2 = BM \cdot BA = AM \cdot AB = AD^2$, i.e., $BE = AD$.

It *seems* from the diagram that M, D, E are collinear. Can we show it? If M, D, E are indeed collinear, we **should** have, by Menelaus' Theorem, that $\frac{AM}{BM} \cdot \frac{BE}{CE} \cdot \frac{CD}{AD} = 1$, which implies $CD = CE$ (Notice that we have utilized the condition $AM = BM$ once more, even though it is not clear at first glance how this condition could be applied.)

Can we show $CD = CE$, say by showing $\angle CED = \angle CDE$? Notice that $\angle CDE = \angle ADM = \angle ABD$.

Proof. Refer to the diagram on the below. Let the circumcircle of ΔBMP touch AC at D and MD extended intersect BC extended at E. We claim that $\angle CED = \angle CDE$. Notice that:



$\angle CDE = \angle ADM = \angle ABD$. (1)

$\angle CED = 180° - \angle BME - \angle ABE$. (2)

Since PM is the median on the hypotenuse of the right angled triangle ΔABP, we must have $\angle ABE = \angle BPM = \angle BDM$.

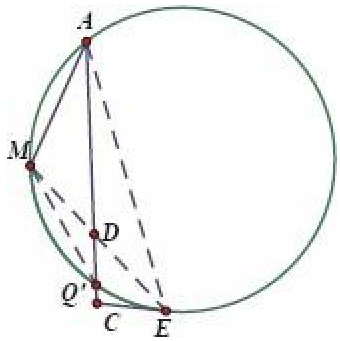
By (2), $\angle CED = 180° - \angle BME - \angle BDM = \angle ABD$. (3)

(1) and (3) imply $\angle CED = \angle CDE$ and hence, $CD = CE$.

By Menelaus' Theorem, $\frac{AM}{BM} \cdot \frac{BE}{CE} \cdot \frac{CD}{AD} = 1$. Since $AM = BM$ and $CD = CE$, we must have $BE = AD$. It follows that $BE^2 = AD^2 = AB \cdot AM = AB \cdot BM$. By the Tangent Secant Theorem, BE touches the circumcircle of ΔAME at E.

Let the circumcircle of ΔAME intersect AC at Q'. We claim that $BQ' \perp AC$. Since $AM = BM$, it suffices to show $AM = MQ$ (Example 1.1.8), or equivalently, $\angle AQ'M = \angle MAQ'$.

Refer to the diagram on the below. Notice that



$\angle AQ'M = \angle ADM - \angle Q'ME$.

Since $\angle ADM = \angle CDE = \angle CED = \angle MAE$ and $\angle Q'ME = \angle Q'AE$, we have

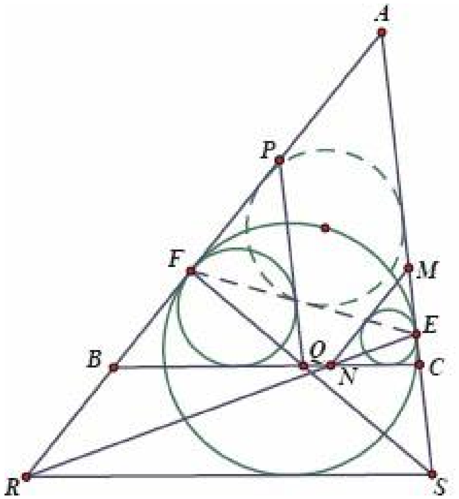
$\angle AQ'M = \angle MAE - \angle Q'AE = \angle MAQ'$.

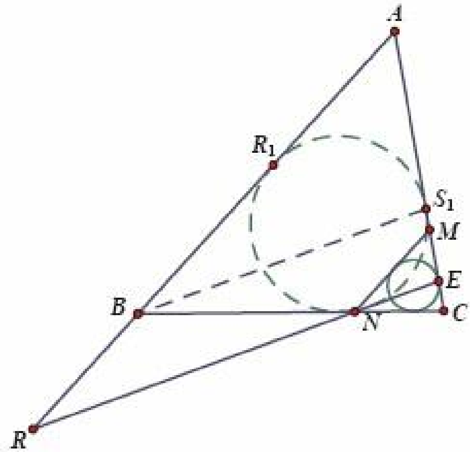
This completes the proof.

□

Example 6.4.5 (USA 10) Given ΔABC, M, N are on AC, BC respectively such that $MN \,//\, BC$, and P, Q are on AB, BC respectively such that $PQ \,//\, AC$. Given that the incircle of ΔCMN touches AC at E and the incircle of ΔBPQ touches AB at F, the lines EN, AB intersect at R and the lines FQ, AC intersect at S. Show that if $AE = AF$, then the incenter of ΔAEF lies on the incircle of ΔARS.

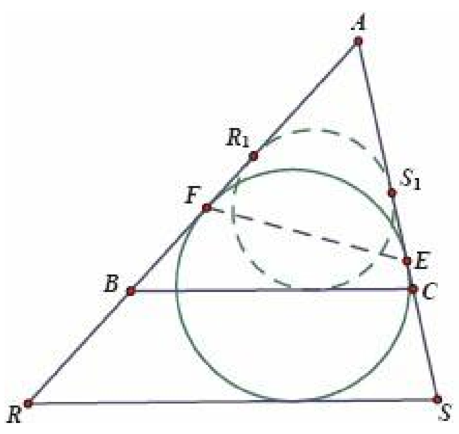
Insight. First, we draw the diagram according to the description. Refer to the left diagram below. There are many circles and lines and it becomes difficult to seek clues. Since the incircles of ΔBPQ and ΔCMN are constructed similarly, we may focus on one of them.

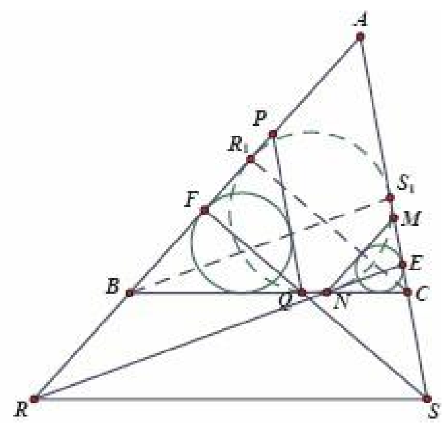




Refer to the right diagram above. It is easy to see that $\Delta ABC \sim \Delta MNC$. Hence, if we draw the incircle of ΔABC, which touches AB, AC at R_1,S_1 respectively, then S_1 and E are corresponding points in ΔABC and ΔMNC respectively. It follows that $BS_1 \,//\, EN$. Similarly, we have $CR_1 \,//\, FQ$. It *seems* from the left diagram above that $BC \,//\, RS$. Can you prove it by the Intercept Theorem? (Notice that $AE = AF$ and $AR_1 = AS_1$.)

Now $\Delta ABC \sim \Delta ARS$ and hence, the incircle of ΔABC corresponds to the incircle of $AARS$. Since R_1 and F (and similarly S_1 and E) are corresponding points of the similar triangles ΔABC and ΔARS, the incircle of ΔARS touches AR, AS at E, F respectively! Refer to the left diagram below. Notice that we have removed the unnecessary lines and points.





Proof. Let the incircle of ΔABC touch AB, AC at R_1, S_1 respectively. Since $MN \,//\, AB$, $\Delta ABC \sim \Delta MNC$. Notice that S_1 and E are corresponding points in the similar triangles ΔABC and ΔMNC. We conclude that $BS_1 \,//\, ER$.

It follows that $\frac{AB}{AR} = \frac{AS_1}{AE}$. Similarly, we must have $CR_1 \quad // FS$ and $\frac{AC}{AS} = \frac{AR_1}{AF}$. Since $AR_1 = AS_1$ and $AE = AF$, we must have $\frac{AB}{AR} = \frac{AC}{AS}$. By the Intercept Theorem, $BC // RS$. Refer to the right diagram above.

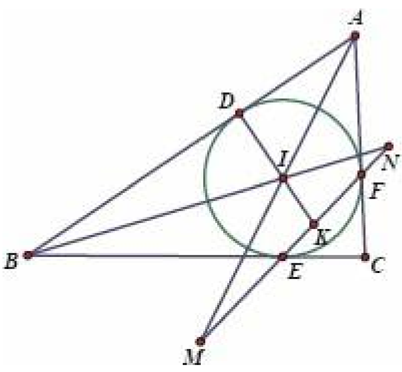
Now we have $\Delta ARS \sim \Delta ABC$. We are to show the incenter of ΔAEF lies on the incircle of ΔARS. Notice that R_1 and F are corresponding points in the similar triangles ΔABC and ΔARS, because $\frac{AR_1}{AF} = \frac{AC}{AS} = \frac{AB}{AC}$. A similar argument applies for S_1 and E as well. Now it suffices to show that the incenter of ΔAR_1S_1 lies on the incircle of ΔABC.

Since AR_1, AS_1 are tangent to the incircle of ΔABC, called $\odot I$, the incenter of ΔAR_1S_1 is exactly the intersection of AI and $\odot I$, i.e., the midpoint of the arc $\overset{\frown}{R_1S_1}$ (Exercise 3.5). This completes the proof.

□

Note: We used correspondence between similar triangles extensively in the proof above. One not familiar with these properties could always use similar triangles to argue instead, although it will make the proof unnecessarily lengthy.

Example 6.4.6 (CHN 12) Refer to the diagram below. I is the incenter of ΔABC, whose incircle $\odot I$ touches AB, BC, CA at D, E, F respectively. If the line EF intersects the lines AI, BI, DI at M, N, K respectively, show that $DM \cdot KE = DN \cdot KF$.

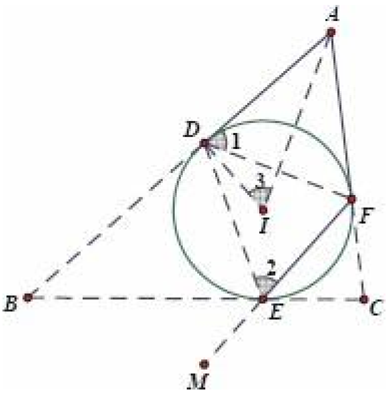


Insight. Since there is a circle in the diagram, the conclusion reminds us of the Tangent Secant Theorem. However, it seems DM, KE are not part of a secant line of $\odot I$.

Can we show $\frac{DM}{DN} = \frac{KF}{KE}$ instead?

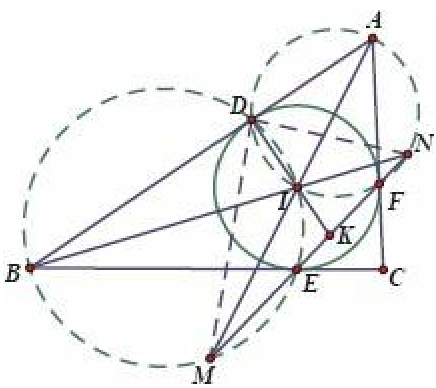
It seems not easy either because we do not see similar triangles immediately which relate *DM*, *DN*, *KE* and *KF*.

Where does the difficulty come from? We do not know the properties of the line *MN* (including *E*, *F* and *K*). Perhaps we should first study the properties of this line and the points on it. Let us focus on one side of the triangle and its incircle. Refer to the diagram on the below. We have erased the unnecessary lines and points.



Now it is clear that $\angle 1 = \angle 2$ because *AD* is tangent to $\odot I$. We also have $\angle 1 = \angle 3$ because $AD \perp DI$ and $AI \perp DF$. Hence, $\angle 2 = \angle 3$, which implies $\angle DEM = \angle DIM$ (since *A*, *I*, *M* are collinear). It follows that *D, I, E, M* are concyclic.

Similarly, we also have *D, I, F, N* concyclic. Refer to the diagram on the below. Notice that the three circles give $KE \cdot MK = KI \cdot DK = KF \cdot NK$ (Tangent Secant Theorem).

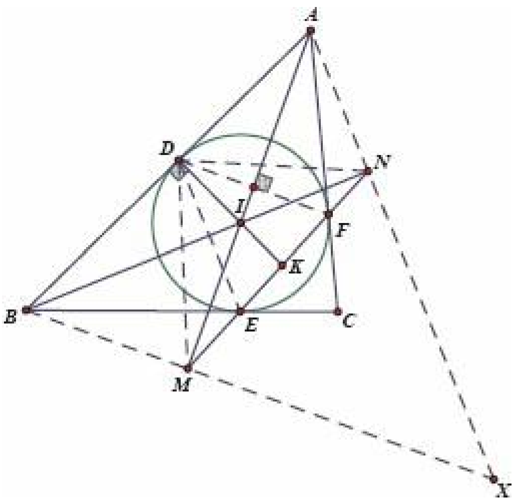


Now we have $\frac{KE}{KF} = \frac{NK}{MK}$ and hence, it suffices to show $\frac{DM}{DN} = \frac{MK}{NK}$.

Notice that this is equivalent to *DK* bisecting $\angle MDN$ (Angle Bisector Theorem).

It seems from the diagram that B, D, I, E are concyclic. One may easily see this because $\angle BDI = \angle BEI = 90°$. Now B, D, I, E, M are concyclic (where BI is a diameter). Hence, $\angle BMI = 90°$ and $AM \perp BM$.
Can you see that I is the orthocenter of a larger triangle? How is it related to the angle bisector of $\angle MDN$?

Proof. Refer to the diagram on the below. We have



$$\angle DEF = \frac{1}{2}\angle DIF = \angle AID.$$

Hence, $\angle DIM = \angle DEM$ and we must have D, I, E, M concyclic. It is easy to see that B, D, I, E are also concyclic.
We conclude that B, D, I, E, M are concyclic. Similarly, A, D, I, F, N are concyclic.

Now $KE \cdot MK = KI \cdot DK = KF \cdot NK$, which implies $\frac{KE}{KF} = \frac{NK}{MK}$. (1)

Notice that $\angle BMA = \angle BEI = 90°$, i.e., $AM \perp BM$. Similarly, we have $AN \perp BN$. Let the lines AM, BM intersect at X.
Now I is the orthocenter of ΔABX and hence, the incenter of ΔDMN (Example 3.1.6).

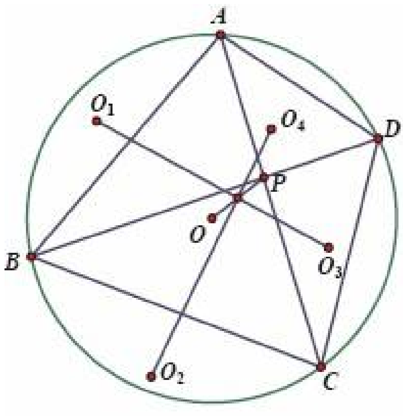
By the Angle Bisector Theorem, $\frac{MK}{NK} = \frac{DM}{DN}$. (2)

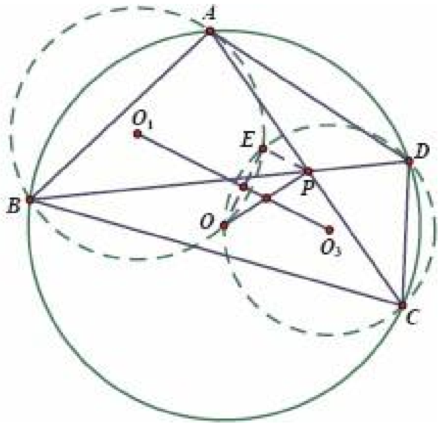
(1) and (2) give $\frac{DM}{DN} = \frac{KF}{KE}$, or equivalently, $DM \cdot KE = DN \cdot KF$.

□

Example 6.4.7 (CHN 06) Let $ABCD$ be a cyclic quadrilateral inscribed in $\odot O$, where O does not lie on any side of the quadrilateral. The diagonals AC, BD intersect at P. Let O_1, O_2, O_3, O_4 denote the circumcenters of ΔOAB, ΔOBC, ΔOCD, ΔODA respectively. Show that the lines O_1O_3, O_2O_4 and OP are concurrent.

Insight. Refer to the left diagram below. We draw the circumcenters only, but hide other related details like the perpendicular bisectors and the circumcircles. It seems not clear how the lines O_1O_3, O_2O_4 and OP are related. (Notice that applying Ceva's Theorem is not feasible.)

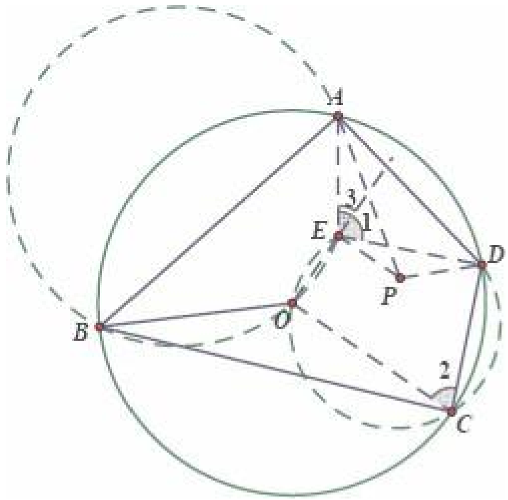




However, it *seems* that both O_1O_3 and O_2O_4 pass through the midpoint of OP. Is it true?

We focus on the line O_1O_3. Let $\odot O_1$ and $\odot O_2$ denote the circumcircles of ΔOAB and ΔOCD respectively and the circles intersect at O and E. Refer to the right diagram above. We know that O_1O_3 is the perpendicular bisector of OE. If O_1O_3 indeed passes through the midpoint of OP, we **should** have $PE \perp OE$ (Midpoint Theorem).

Can we show $PE \perp OE$? One may consider calculating the angles, as there are many circles (and circumcenters) in the diagram. Refer to the diagram on the below. It suffices to show that $\angle 1 + \angle DEP = 90°$.



We do not know much about $\angle DEP$, but we know

$$\angle 1 = \angle 2 = 90^\circ - \frac{1}{2}\angle COD \text{ (because } OC = OD\text{)}.$$

Similarly, $\angle 3 = \angle ABO = 90^\circ - \frac{1}{2}\angle AOB$.

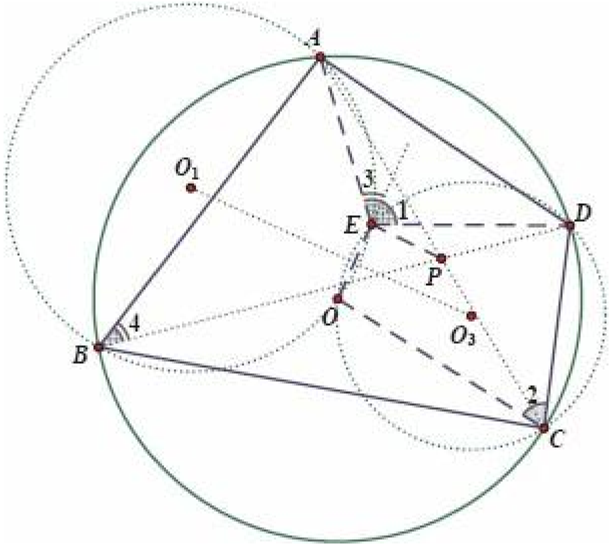
Now, $\angle 1 + \angle 3 = 180^\circ - \frac{1}{2}(\angle AOB + \angle COD) = 180^\circ - (\angle ADB + \angle CAD)$

= $\angle ADB$, which implies *A, D, P, E* are concyclic.

Notice that we have used the properties of the circumcenters extensively. Indeed, we are not given many conditions other than the circumcenters.

We have obtained one more circle. One should be able to show the conclusion easily using the properties of angles.

Proof.



Let the circumcircles of ΔABO and ΔCDO intersect at O and E. Refer to the diagram above. Consider the isosceles triangle ΔOCD.

We have $\angle 1 = \angle 2 = 90° - \frac{1}{2}\angle COD = 90° - \angle CAD.$ (1)

Similarly, $\angle 3 = \angle 4 = 90° - \angle ADB$.

Now $\angle AED = \angle 1 + \angle 3 = 180° - \angle ADB - \angle CAD = \angle APD$. We conclude that *A, D, P, E* are concyclic.

Hence, $\angle DEP = \angle DAP = 90° - \angle 1$ by (1), which implies $PE \perp OE$.

Since O_1O_3 is the perpendicular bisector of *OE*, we must have $PE \,//\, O_1O_3$ and hence, O_1O_3 passes through the midpoint of *OP*.

Similarly, O_2O_4 also passes through the midpoint of *OP*. It follows that O_1O_3, O_2O_4 and *OP* are concurrent (at the midpoint of *OP*).

□

6.5 Exercises

(CZE-SVK 89) Let *O* be the circumcenter of ΔABC. *D, E* are points on *AB, AC* respectively. Show that *B, C, E, D* are concyclic if and only if $DE \perp OA$.

2. (IWYMIC 14) In ΔABC, $\angle A = \angle C = 45°$. *M* is the midpoint of *BC*. *P* is a point on *AC* such that $BP \perp AM$. If $PC = \sqrt{2}$, find *AB*.

3. (JPN 14) Let *ABCDEF* be a cyclic hexagon where the diagonals *AD, BE, CF* are concurrent. If $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$ and $EF = 5$, find AF.

4. (IND 11) ΔABC is an acute angled triangle where D is the midpoint of BC. BE bisects $\angle B$, intersecting AC at E. $CF \perp AB$ at F. Show that if ΔDEF is an equilateral triangle, then ΔABC is also an equilateral triangle.

5. (USA 90) ΔABC is an acute angled triangle where AD, BE are heights. Let the circle with diameter BC intersect AD and its extension at M, N respectively. Let the circle with diameter AC intersect BE and its extension at P, Q respectively. Show that M, P, N, Q are concyclic.

6. (CAN 11) $ABCD$ is a cyclic quadrilateral. BA extended and CD extended intersect at X. AD extended and BC extended intersect at Y. If the angle bisector of $\angle X$ intersects AD, BC at E, F respectively, and the angle bisector of $\angle Y$ intersects AB, CD at G, H respectively, show that $EGFH$ is a parallelogram.

7. (ROU 08) Given ΔABC, D, E, F are on BC, AC, AB respectively such that $\frac{BD}{CD} = \frac{CE}{AE} = \frac{AF}{BF}$. Show that if the circumcenters of ΔABC and ΔDEF coincide, then ΔABC is an equilateral triangle.

8. (IMO 04) Given a non-isosceles acute angled triangle ΔABC where O is the midpoint of BC, draw $\odot O$ with diameter BC, intersecting AB, AC at D, E respectively. Let the angle bisectors of $\angle A$ and $\angle DOE$ intersect at P. If the circumcircles of ΔBPD and ΔCPE intersect at P and Q, show that Q lies on BC.

9. (CHN 04) Given ΔABC, D is a point on BC and P is on AD. A line ℓ passing through D intersects AB, PB at M, E respectively, and intersects AC extended and PC extended at F, N respectively. Show that if $DE = DF$, then $DM = DN$.

10. (IMO 08) Given an acute angled triangle ΔABC where O_1, O_2, O_3 are the midpoints of BC, AC, AB respectively, H is the orthocenter of ΔABC. Draw $\odot O_1$, $\odot O_2$, $\odot O_3$ whose radii are O_1H, O_2H, O_3H respectively. If $\odot O_1$ intersects BC at A_1, A_2, $\odot O_2$ intersects AC at B_1, B_2 and $\odot O_3$ intersects AB at C_1, C_2, show that A_1, A_2, B_1, B_2, C_1, C_2 are concyclic.

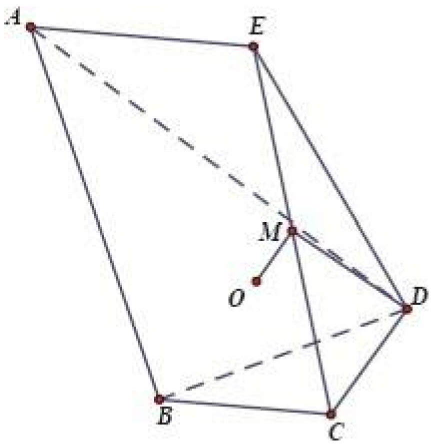
11. (IMO 14) Given an acute angled triangle ΔABC, P, Q are on BC such that $\angle PAB = \angle C$ and $\angle CAQ = \angle B$. M, N are on the lines AP, AQ respectively such that $AP = PM$ and $AQ = QN$. Show that the intersection of the lines BM and CN lies on the circumcircle of ΔABC.

12. (CHN 13) Given ΔABC where $AB < AC$, M is the midpoint of BC. $\odot O$ passes through A and is tangent to BC at B, intersecting the lines AM, AC at

D, E respectively. Draw $CF \parallel BE$, intersecting BD extended at F. Let the lines BC and EF intersect at G. Show that $AG = DG$.

13. (RUS 13) Let $\odot I$ denote the incircle of ΔABC, which touches BC, AC, AB at D, E, F respectively. Let J_1, J_2, J_3 be the ex-centers opposite A, B, C respectively. If J_2F and J_3E intersect at P, J_3D and J_1F intersect at Q, J_1E and J_2D intersect at R, show that I is the circumcenter of ΔPQR.

14. (IMO 10) Refer to the diagram below. $ABCDE$ is a pentagon such that $BC \parallel AE$, $AB = BC + AE$ and $\angle B = \angle D$. Let M be the midpoint of CE and O be the circumcenter of ΔBCD. Show that if $OM \perp DM$, then $\angle CDE = 2\angle ADB$.



Insights into Exercises

Chapter 1

1.1 Notice that $\angle B + \angle C = \angle A$. If $\angle PAB = \angle C$, what can you say about $\angle PAC$?

1.2 We are to show $AC = AB + BD$. If we choose E on AC such that $AB = AE$, it suffices to show $CE = BD$. Since AD bisects $\angle A$, can you see that E is the reflection of B about AD, i.e., $\Delta ABD \cong \Delta AED$?
How can we use the condition $\angle B = 2\angle C$? Can you see $\angle B = \angle AED$?

1.3 Can you see congruent triangles? It is similar to Example 1.2.6.

1.4 Notice that the ex-center is still about properties of angle bisectors. How did we show the existence of the incenter?

1.5 Notice that AI, AJ_1 are angle bisectors of neighboring supplementary angles. Can you see $AI \perp AJ_1$? Refer to Example 1.1.9.

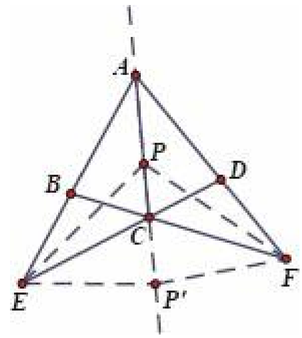
1.6 We have $\angle EAF = \frac{1}{2}\angle BAD$. However, the remaining portions of $\angle BAD$ are far apart. How can we put them together? Moreover, BE and DF are far apart as well. Cut and paste! It is similar to Example 1.2.9.

1.7 Can you see congruent triangles? Given that $BP = AC$ and $CQ = AB$, which two triangles are probably congruent?

1.8 We are given the angle bisector of $\angle CBE$ and $BE = AB$. Notice that ΔABC is an equilateral triangle. Can you see congruent triangles (say by the reflection about the angle bisector BD)? Can you see D is on the perpendicular bisector of AB?

1.9 Since I is the incenter of ΔABC, can you express both $\angle BID$ and $\angle CIH$ in terms of $\angle A$, $\angle B$ and $\angle C$? Alternatively, you may apply Theorem 1.3.3.

1.10 One may immediately see that $\Delta ABC \cong \Delta ADC$. Even though this is not related to PE and PF, we have more equal angles and line segments now. Can you find more congruent triangles which lead to $PE = PF$?



Note:

(1) P could be on the line segment AC or its extension.

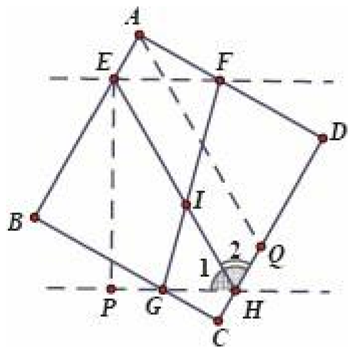
(2) One may see many pairs of congruent triangles in the diagram, but careful justification is needed for each pair and the argument must not depend on the diagram.

1.11 By definition, O lies on the perpendicular bisector of BC and I lies on the angle bisector of $\angle A$. What can you conclude if $AB = AC$?

1.12 Can you see D is an ex-center of ΔABP, i.e., AD bisects the exterior angle of $\angle BAC$? Now can you express both $\angle PAD$ and $\angle BDP$ in terms of $\angle ABP$ and $\angle APC$?

1.13 If $ABCD$ is a parallelogram, one sees that $BC - AB = AD - CD$ holds. Since $AD \,//\, BC$, we may draw a parallelogram $ABCD'$ such that D' lies on the line AD. Now $AD - CD = BC - AB = AD' - CD'$. This is only possible when D and D' coincide. (You may show it using triangle inequality. Notice that you need to discuss both cases when $AD > AD'$ and $AD < AD'$.)

1.14 Notice that the condition AB is equal to the distance between $\ell_1 \ell_2$ is important. If we move ℓ_2 downwards, $\angle GIH$ will be smaller, i.e., it is not a fixed value.



If we draw a perpendicular from E to ℓ_2, say $EP \perp \ell_2$ at P, we have $AB = EP$.

Does it help us to find congruent triangles?

Are there any other equal angles or sides? If $EH \perp FG$, then we have $EH = FG$ (Example 1.4.12). However, it seems from the diagram that EH and FG are **not** perpendicular. Moreover, EH and FG are apparently not equal. What should we do? It is difficult to calculate EH and FG because we do not know the positions of E, F, G, H on the sides of the square. Perhaps we can use the same technique as in Example 1.4.12, say to push EH upwards.

If we draw $AQ // EH$, intersecting CD at Q, it is easy to see that $AQ = EH$. Now we have $\Delta EPH \cong \Delta ADQ$ (H.L.) and hence, $\angle 1 = \angle 2$. This implies that EH bisects the exterior angle of $\angle CHG$. A similar argument applies for FG as well. Can you see I is an ex-center of ΔCGH (Exercise 1.4)? Now we can calculate $\angle GIH$ using the properties of angle bisectors.

Chapter 2

2.1 Can you express $[BCXD]$, $[ACEY]$ and $[ABZF]$ in terms of $[\Delta ABC]$?

2.2 It is easy to show $BG = CE$ (Example 1.2.6). How are BG, CE related to (the midpoints) O_1, O_2, M, N ?

2.3 Can you see right angled isosceles triangles in the diagram (for example, $CD = CF + AF$)? Since we are to show $AE < \frac{1}{2}CD$, what do we know about $CD - 2AE$?

2.4 M, N are midpoints, but we cannot apply the Midpoint Theorem directly on MN. What if we consider more midpoints (Example 2.2.8)?

2.5 Can you see $EFGH$ is a parallelogram? Now we can focus on the parallelogram $EFGH$, which is a simpler problem. Can we use the techniques of congruent triangles to solve it?

2.6 Notice that every point in the diagram is uniquely determined once the square is drawn. Let $AB = a$. We can calculate AP, for example, by drawing $PQ \perp AD$ at Q and applying Pythagoras' Theorem. Can you find AQ and PQ ? Can you find ? $\frac{PF}{PD}$?

2.7 Given $BG \perp CG$, can you see AB, BC, AC can all be expressed in terms of the medians BD, CE (by the Midpoint Theorem and Pythagoras' Theorem)?

2.8 Given ΔABC, we can calculate $[DEF]$ by subtracting $[\Delta ADF]$, $[\Delta BDE]$ and $[\Delta CEF]$ from $[\Delta ABC]$, while the areas of the small triangles are determined once the positions of D, E, F are known.

[$\Delta D'E'F'$] can be calculated in a similar manner, while the positions of D', E', F' are determined by D, E, F.

Since D, E, F are arbitrarily chosen, the conclusion **should** hold if we let $\frac{AD}{AB}=a,\ \frac{BE}{BC}=b,\ \frac{CF}{CA}=c$ and express both areas in terms of a, b, c and [ΔABC].

2.9 We are to show $BD \cdot CD = BE \cdot CF$ or equivalently, $\frac{BD}{BE}=\frac{CF}{CD}$. Since $\angle B = \angle C = 60°$, we **should** have $\Delta BDE \sim \Delta CFD$. Can we prove it, say by equal angles? Notice that A and D are symmetric about MN, i.e., $\angle EDF = \angle A = 60°$.

2.10 Example 1.2.7 is a special case of this problem, where $\angle A = 45°$ and $AH = BC$ We solved Example 1.2.7 using congruent triangles. Can you see a pair of similar triangles in this problem?

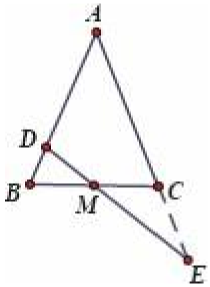
2.11 We know how to calculate a median, but what about trisection points? Can you see AD is a median of ΔABE? Similarly, AE is a median of ΔACD.

2.12 Notice that the parallel line is almost the only condition. If we apply Ceva's Theorem, the conclusion would be concurrency instead of collinearity. Nevertheless, we can show GM passes through D, which is equivalent to the conclusion.
Applying Menelaus' Theorem directly to D, G, M will probably not show the collinearity because it is not related to the condition $AB \parallel CE$. How about applying Menelaus' Theorem more than once?

2.13 This is similar to Example 2.5.3.

2.14 It seems natural to apply Menelaus' Theorem. Even though the line where D, E, F should lie does **not** intersect any triangle, Menelaus' Theorem still holds when the points of division are on the extension of the sides of the triangle.
One may also consider applying the Angle Bisector Theorem to the exterior angle bisectors.

2.15 Refer to the diagram below. If we apply Menelaus' Theorem when the line DE intersects ΔABC, we have $\frac{AD}{BD}\cdot\frac{BM}{CM}\cdot\frac{CE}{AE}=1$. Alternatively, if we consider the line BC intersecting ΔADE, we have



$\frac{AB}{DB}\cdot\frac{DM}{EM}\cdot\frac{EC}{AC}=1$. However, neither gives us a clue for $\frac{1}{AD}+\frac{1}{AE}$ or $\frac{2}{AB}$. Perhaps we shall apply Menelaus' Theorem to another triangle, but which triangle (and the line intersecting it) should we choose?

We are to show $\frac{AB}{AD}+\frac{AB}{AE}=2$, where $AB = AC$. How could we obtain say $\frac{AB}{AD}$, If we apply Menelaus' Theorem, *BD* should be a side of the triangle and the line should pass through *A*. It seems we should choose the line *AE* intersecting Δ*BDM*. Even though *AE* intersects *BD*, *DM* and *BM* only at the extension, we could still apply Menelaus' Theorem.

Can you give a similar argument for $\frac{AC}{AE}$?

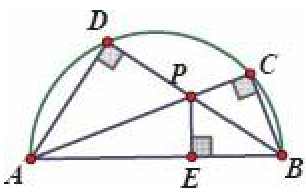
Chapter 3

3.1 Apply Corollary 3.1.4.

3.2 This is similar to Example 3.1.7. Connect *EF* and one could see concyclicity.

3.3 Can you see $\angle AIJ=\frac{1}{2}(\angle A+\angle B)$? How does this relate to the exterior angle of $\angle C$?

3.4 Since *AB* is the diameter, $AC\perp BC$ and $AD\perp BD$. Can you construct a triangle whose orthocenter is *P*? Example 3.1.6 relates the orthocenter of a triangle to the incenter of another triangle.



3.5 Let M be the midpoint of $\overset{\frown}{AB}$. Clearly the angle bisector of $\angle PAB$ passes through M. Can you find another angle bisector which passes through M? You may apply Theorem 3.2.10 for angles related to tangent lines.

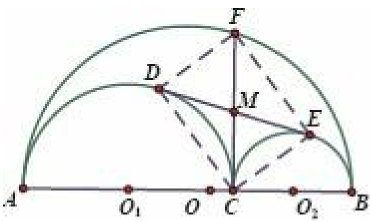
3.6 Notice that $\angle BHC = 180° - \angle A$ because H is the orthocenter.

3.7 It is easy to see that OD is the perpendicular bisector of BC. How can we show $OM \perp PM$? Draw a diagram and one may see many equal angles and right angles. It should not be difficult to find concyclicity.

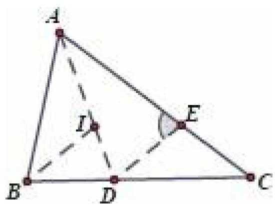
3.8 This is similar to Example 3.1.17. Besides, one may also recall the property of ΔACD, i.e., an isosceles triangle with 120° at the vertex (Example 2.3.4).

3.9 There are many right angles in this diagram due to the orthocenter and diameters. (Draw a diameter of $\odot O$.)

3.10 Since we are to show $CDEF$ is a rectangle, it suffices to show CF and DE bisect each other and are equal. We know $CM = DM = EM$ Hence, it suffices to show $CF = DE$.
Notice that both CF and DE are uniquely determined by AC and BC. In particular, CF and DE can be calculated by Pythagoras' Theorem.



3.11 How can we apply the condition $\angle B = 2\angle C$? Since AD is the angle bisector, it is natural to reflect ΔABD about AD, i.e., choose E on AC such that $AB = AE$.
Now $\angle AED = 2\angle C$, which implies $\angle C = \angle CDE$, i.e., $DE = CD$.
It *seems* that $BDEI$ is a rhombus. Can you show it? (Notice that if $BDEI$ is indeed a rhombus, then E is the circumcenter of ΔCDI.)



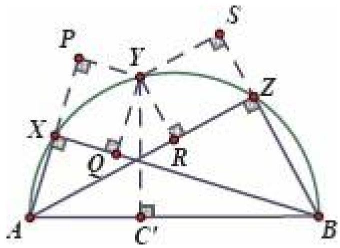
3.12 Let the circumcircle of ΔABP intersect AQ at M'. What do you know about M' ? Can you see M' is the midpoint of AQ?

3.13 Can you see $AA'CB$ and $ABB'C$ are isosceles trapeziums? Notice that there are many equal angles in the diagram due to concyclicity, heights, parallel lines and equal arcs.

3.14 This follows immediately from Example 3.4.2.

3.15 Can you see I is the orthocenter of $\Delta J_1J_2J_3$?

3.16 One may see many right angles from the diagram. (Notice that the diameter also gives right angles.) Moreover, P, Q, R, S are the feet of the perpendiculars from Y, a point on the circumference. Is it reminiscent of Simson's Line? What if you draw $YC' \perp AB$ at C'?



We are to show $\angle PCS = \frac{1}{2}\angle XOZ$. Can we replace $\frac{1}{2}\angle XOZ$ by an angle on the circumference? Those right angles should give plenty of concyclicity. It seems we are not far from the conclusion.

Alternatively, one may also notice that $PXQY$ and $SYRZ$ are rectangles. What can we say about these rectangles?

3.17 Since AD // BC and we are to show ℓ_1 // ℓ_2, we **should** have a parallelogram enclosed by AD, BC, ℓ_1 and ℓ_2. Can we show it?

By extending the sides of $ABCD$ and ℓ_1, ℓ_2, we will have many equal tangent segments. Hence, we may be able to find an equation of various line segments. (Refer to Example 3.2.7. You may need to draw a large diagram.)

Now we may identify the parallelogram by applying Exercise 1.13. Even though this is not a commonly used result, it is most closely related to the parallelogram given the sum or difference of neighboring sides. (If you are not familiar with this result, you may prove it first as a lemma.)

Chapter 4

4.1 Draw a common tangent at C. Can you see AB // DE ? What other equal angles can you obtain if *A*, *C*, *D*, *E* are concyclic?

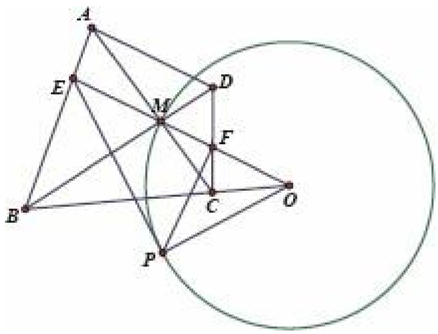
4.2 Draw $DE \perp AP$ at *E*. By definition, $\sin\angle PAD = \dfrac{DE}{AD}$.

Can you see *C* is the midpoint of *AP*? Can you see a number of right angled isosceles triangles?

4.3 It seems not easy to see the geometrical sense of AB^3 and AD^3. However, there are many right angles and we know AB^2 and AD^2 (by Example 2.3.1). In particular, if *G*, *H* are the feet of the perpendiculars from *D*, *B* to *AC* respectively, one can show that $\dfrac{AB^2}{AD^2} = \dfrac{AG}{AH}$.

Now it suffices to show $\dfrac{AB}{AD} \cdot \dfrac{AG}{AH} = \dfrac{AF}{AE}$. Since $DE \ // \ BF$, applying the Intercept Theorem will probably solve the problem. Are you fluent and skillful in manipulating ratios?

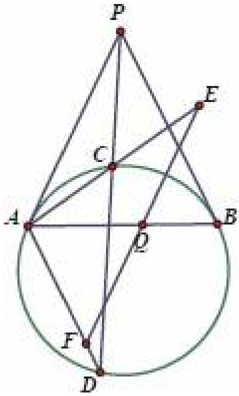
4.4 Since *P* is an arbitrary point and $\angle OPF = \angle OEP$ **should** always hold, can we replace *P* by a special point on the circumference? Unfortunately, we cannot use *M* because *M* lies on the line *OE*.



What can we say about *P*? Notice that we **should** have $\Delta OPE \sim \Delta OFP$, or equivalently, $OP^2 = OE \cdot OF$. Since $OP = OM$, can we show that $OM^2 = OE \cdot OF$? Notice that we do not need the circle anymore! Since $EF \ // \ AD$, we may probably show $\dfrac{OE}{OM} = \dfrac{OM}{OF}$ using the Intercept Theorem. (Are you skillful in applying the Intercept Theorem? Refer to the remarks after Corollary 2.2.2.)

4.5 The only equal lengths we have are $PA = PB$. Apparently, it is not easy to place *QE*, *QF* in congruent triangles. Notice that there are many equal angles in the diagram due to the circle, tangents and parallel lines. Can you identify similar triangles involving *QE* and *QF*? For example, can you see

$\Delta AEQ \sim \Delta ABC$? If we express *QE*, *QF* as ratios of line segments, perhaps we can show that the ratios are the same.



Note that it is not easy to solve the problem by applying the Intercept Theorem even though we have *AP* // *EF* : we do not know $\frac{QE}{AP}$ or $\frac{QF}{AP}$.

4.6 Apply the Tangent Secant Theorem. (You may need Example 2.3.1.)

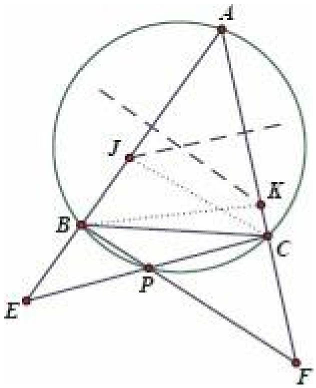
4.7 Can you see *AD* is both an angle bisector and a height? Can you construct the isosceles triangle? Can you find *BC* using similar triangles or the Tangent Secant Theorem? (You are given *CE* and *BD*. How are they related to *BC*?)

4.8 We are to show *D*, *E*, *F* are collinear where *D*, *E*, *F* are closely related to ΔABC : shall we apply Menelaus' Theorem? Can you show that $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$? What do we know about $\frac{AF}{BF}$, $\frac{BD}{CD}$ and $\frac{CE}{AE}$? We know $AF \cdot BF = CF^2$ by the Tangent Secant Theorem, i.e., $\frac{AF}{CF}=\frac{CF}{BF}$. Can you see that $\frac{AF}{BF}=\frac{AF}{CF}\cdot\frac{CF}{BF}$?

Notice that the circumcircle of ΔABC and the tangent lines give similar triangles. For example, can you see that $\Delta BCF \sim \Delta CAF$?

Now $\frac{AF}{CF}=\frac{CF}{BF}=\frac{AC}{BC}$ and hence, $\frac{AF}{BF}=\frac{AF}{CF}\cdot\frac{CF}{BF}=\left(\frac{AC}{BC}\right)^2$. This implies $\frac{AF}{BF}$ is uniquely determined by ΔABC. Can you express $\frac{BD}{CD}$ and $\frac{CE}{AE}$ similarly?

4.9 We see that *AJ*, *AK* are not related to the choice of *P*. How are *CE*, *BF* related to ΔABC? One easily sees that $CJ = AJ$ and $BK = AK$ (because of the perpendicular bisectors). Now CE, BF are in ΔCEJ and ΔBFK respectively.

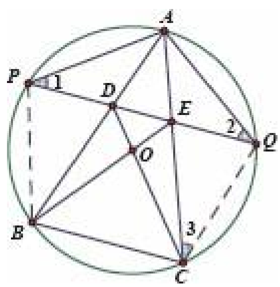


If we have $\Delta CEJ \sim \Delta FBK$, then $\frac{CE}{BF}=\frac{JE}{BK}=\frac{CJ}{KF}$.

Hence, $\frac{CE^2}{BF^2}=\frac{JE}{BK}\cdot\frac{CJ}{KF}$. The conclusion follows because $\frac{CJ}{BK}=\frac{AJ}{AK}$.

Can we show $\Delta CEJ \sim \Delta BFK$? There are many equal angles in the diagram due to the circle and the perpendicular bisectors.

4.10 Since $AP = AQ$, one immediately sees that $\angle 2 = \angle 1 = \angle 3$ (angles in the same arc). Can you see similar triangles?



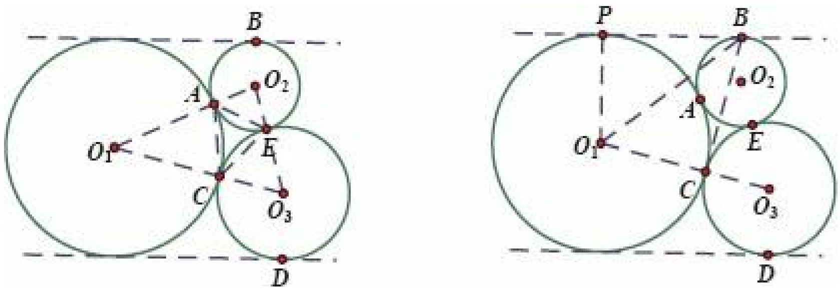
Since we are to show $DE \,//\, BC$, $BCDE$ **should** be a trapezium. Can you see that $BCDE$ **should** be an isosceles trapezium? How is O related to $BCDE$? (**Hint**: $OB = OC$) Now it suffices to show that B, C, D, E are concyclic. What can you conclude from $AP = AQ$ and the similar triangles?

4.11 Naturally, we suppose two common tangents intersect at P and show that P lies on the third common tangent. One may see this as a special case of Theorem 4.3.6, while the radical axes are the common tangent. We still apply the Tangent Secant Theorem and construct a proof by contradiction.

4.12 Since three circles intersect (or touch) each other, one may consider applying Theorem 4.3.6. Can you see which lines are the radical axes? What can you obtain by applying the Tangent Secant Theorem?

4.13 Since $\angle B = 2\angle C$, drawing the angle bisector of $\angle B$ gives an isosceles triangle. One may attempt a few techniques with the angle bisector, but notice that applying the Angle Bisector Theorem or reflecting the diagram about the angle bisector would not give AC^2. Since we have an isosceles triangle, how about reflecting the diagram about the perpendicular bisector of BC?

4.14 Refer to the left diagram below. What property do we know about the circumcenter of ΔACE? By Example 4.3.3, the circumcircle of ΔACE, say I, is the incircle of $\Delta O_1O_2O_3$ and moreover, A, C, E are the feet of the perpendiculars from I to O_1O_2, O_2O_3, O_1O_3 respectively.

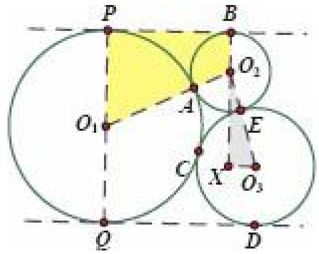


We are to show B, I, C are collinear. It suffices to show $BC \perp O_1O_2$. Let $O_1P \perp \ell_1$ at P. Refer to the right diagram above. Since $O_1C = O_1P$, we **should** have $\Delta BPO_1 \cong \Delta BCO_1$. However, it may not be easy to find equal angles since we do not know how the line segments, say O_1B or BC, intersect the circles given. Can we show $BC = BP$?

Would it be easier to show $O_1B^2 - O_1C^2 = O_3B^2 - O_3C^2$? Notice that all these line segments are uniquely determined by the radii of the three circles (by Pythagoras' Theorem).

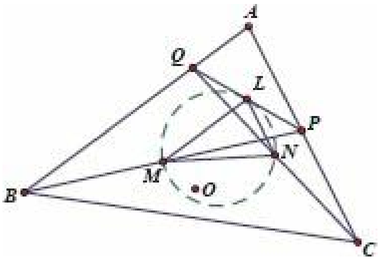
Observe that those radii are not independent. Let the radii of $\odot O_1$, $\odot O_2$ and $\odot O_3$ be r_1, r_2, r_3 respectively. For example, if we draw $O_3X \perp O_2B$ at X, we have $O_2X^2 + O_3X^2 = O_2O_3^{\,2}$, where $O_2X = 2r_1 - r_2 - r_3$, $O_2O_3 = r_2 + r_3$ and $O_3X = DQ - BP$.

Refer to the diagram below. One may find BP via the right angled trapezium BPO_1O_2 and similarly DQ as well. Applying Pythagoras' Theorem repeatedly should lead to the conclusion.



4.15 How can we use the condition that PQ is tangent to the circumcircle of ΔMNL ? Notice that PQ only touches the circumcircle of ΔMNL once, i.e., at L. We are to show $OP = OQ$. Hence, it suffices to show $OL \perp PQ$. Regrettably, this seems not clear because O is **not** the circumcenter of ΔMNL.

Refer to the diagram below. Once we draw $AMNL$, it is easy to see $AB // ML$ and $AC // NL$ because M, N, L are midpoints. Are there any similar triangles?



Clearly, $\angle BAC = \angle MLN$. We also have $\angle LMN = \angle PLN = \angle APQ$ because of Theorem 3.2.10 and $AC // NL$. Similarly, $\angle LNM = \angle AQP$. We must have $\Delta LMN \sim \Delta APQ$.

Notice that L is the midpoint, i.e., $\frac{LM}{BQ} = \frac{PL}{PQ} = \frac{QL}{PQ} = \frac{LN}{CP}$. Can you see this implies $AP \cdot CP = AQ \cdot BQ$. How does this remind you of OP and OQ? Consider the power of points P, Q with respect to $\odot O$, the circumcircle of ΔABC !

Chapter 5

5.1 Recall Example 3.4.1.

5.2 There are many right angles in the diagram. One immediately sees that $PH^2 = MH \cdot BH$. Hence, it suffices to show $MH \cdot BH = AH \cdot OH$, or $\frac{MH}{AH} = \frac{OH}{BH}$. Can we show it by similar triangles?

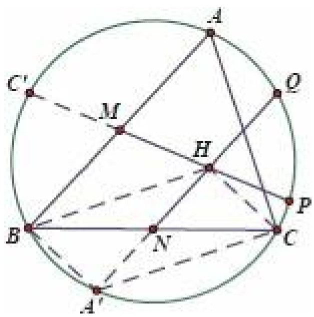
Notice that M and O are midpoints. If we cannot find many angle properties related to them, perhaps we can calculate more lengths.

On a side note, all the points are uniquely determined in the circle because ΔPAB is a right angled isosceles triangle. One may calculate *PH*, *AH*, *OH* explicitly, say by Pythagoras' Theorem and Cosine Rule. Of course, this would not lead to an elegant solution, but is still a valid proof.

5.3 One may solve it by either similar triangles or angle properties in a circle. Can you see any pair of angles which **should** be equal? Can you see that *A*, *I*, *E*, *P* **should** be concyclic?

5.4 Can you see $DR = DQ$? Can you see that *DX* is the perpendicular bisector of *QR*? What can you say about *EY* and *CZ*?
Hint: This is an easy question if you construct the diagram wisely. Do **not** draw all the points explicitly as it only complicates the diagram unnecessarily and distracts you from seeking the clues.

5.5 Given the orthocenter *H* and the midpoint *M*, one immediately sees that *A'BHC* is a parallelogram, where *AA'* is a diameter of ⊙*O* (Example 3.4.4).
In particular, *A'*, *H*, *Q* are collinear and *N* is the midpoint of *A'H*.
We are to show *M*, *N*, *P*, *Q* are concyclic.



It seems we may consider the Intersecting Chords Theorem. Refer to the diagram above. Can you see that $A'H \cdot QH = C'H \cdot PH$, where *C'* is obtained by *PH* extended intersecting ⊙*O*? (*CC'* is also a diameter!)

5.6 Recall that $r = \frac{1}{2}(AB + AC - BC)$ since $\angle A = 90°$. Notice that a similar argument applies for r_1, r_2 as well.

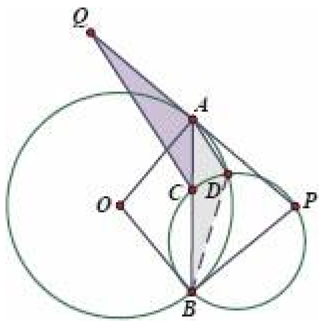
5.7 Consider the reflection of *C* about *BD*, called *C'*. Can you see that $CP + PQ = C'P + PQ \geq C'Q$? What is the smallest possible value of *C'Q* ? (Notice that *C'Q* does not depend on the choice of *P* and *Q*.)

5.8 We **should** have $PM = QM$ However, it is not easy to show because *BQ*, *CP* are **not** the altitudes. How are ΔAPQ and ΔBCH related? Notice that

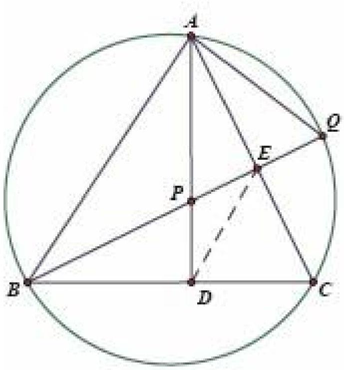
$\angle BHC = 180° - \angle A$. Does it remind you of any technique? Double the median HM!

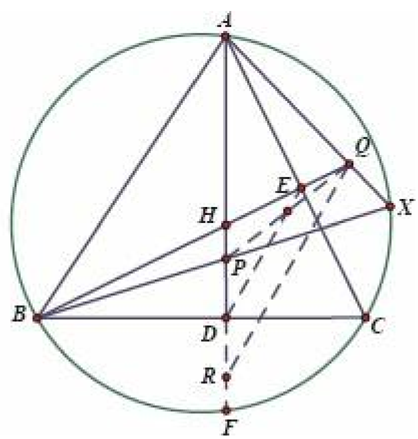
5.9 It suffices to show $\angle ACQ = \angle BAD$. Notice that $\angle CAQ = \angle ADB$ (because AP is tangent to $\odot O$). Hence, we **should** have $\Delta ACQ \sim \Delta DAB$. Can we show $\frac{AQ}{AC} = \frac{BD}{AD}$?

Notice that $AQ = AP = BP$. Can we show $\frac{BP}{AC} = \frac{BD}{AD}$? Is there another pair of similar triangles which imply this? We have two circles and hence, plenty of equal angles.



5.10 If X lies on BE extended, then P, H coincide and Q, X coincide, where H is the orthocenter of ΔABC. It is easy to see that E is the midpoint of PQ (Example 3.4.3). Refer to the left diagram below.



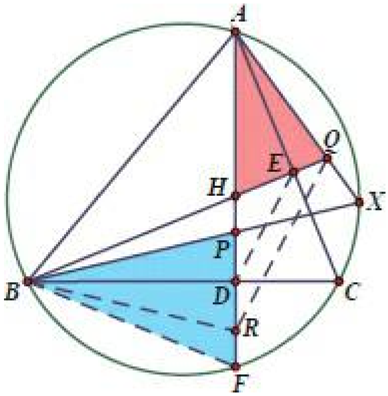


Let X be an arbitrary point. Now it is not easy to show DE passes through the midpoint of PQ since P, H do not coincide and Q does not lie on the circumcircle of ΔABC. Nevertheless, since P still lies on the line AD, perhaps we can draw $QR \parallel DE$, intersecting the line AD at R. Refer to the right diagram above.

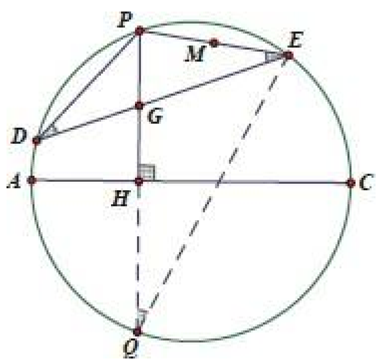
We **should** have $PD = DR$. Since ΔBFH is an isosceles triangle (Example 3.4.3),

P and *R* **should** be symmetric about the line *BC*.
On the other hand, what properties do we know about *Q*? It is easy to see that $\angle BFH = \angle BHF = \angle AHQ$ and $\angle QAH = \angle PBF$.

Hence, we have $\Delta AHQ \sim \Delta BFP$, where ΔBFP **should** be the reflection of ΔBHR. Refer to the diagram below. Note that ΔBHR and ΔAHQ are related by the parallel lines *DE* and *QR*.
If we equate the ratios of the line segments via the similar triangles and the parallel lines, we will probably see the conclusion.



5.11 One immediately notices that the point *A* could be neglected. Let *DE* intersect *PH* at *G*. We are to show that *G* is the midpoint of *PH*. In fact, we have a midpoint *H* if we extend *PH*, intersecting Γ at *Q*. How can we apply the condition *PD* = *PE*? Can you see that $\Delta PEG \sim \Delta PQE$?

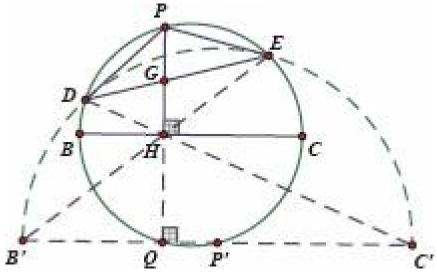


If we choose *M* as the midpoint of *PE, M* and *H* are corresponding points in the similar triangles. Is it reminiscent of Example 5.2.8?

Alternatively, one recognizes that *P* is the circumcenter of ΔDEH. Notice that the circumcircle of ΔDEH intersects Γ exactly at *D* and *E*. If *DE* intersects *PH* at *G*, one may probably show *PG* = *HG* by considering the power of point *G* (or by the Intersecting Chords Theorem).

Note: One may refer to Example 3.5.1, the diagram of which apparently shows a similar structure.

In fact, if PP' is a diameter of Γ, one may draw ⊙P' with radius $P'D$ (where PP' is the perpendicular bisector of DE). Notice that $PD \perp P'D$ and $PE \perp P'E$, i.e., PD, PE are tangent to ⊙ P'. Refer to the diagram below.



Let $B'C'$ be the diameter passing through Q. It is easy to see that $B'C' \parallel BC$. B y Example 3.5.1, $B'E$, $C'D$ and AQ are concurrent at H. Unfortunately, knowing this fact is not helpful when showing $PG = GH$.

5.12 Can you see $\angle CPD = 90° - \angle CAD$ and $\angle CQD = 180° - 2\angle CAD$? How are P and Q related?

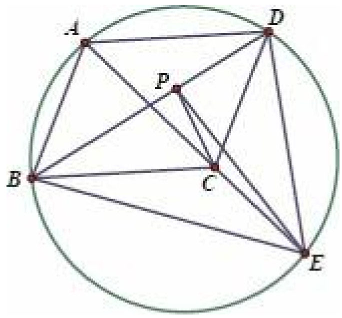
Since B, C, E **should** be related, can you see that B should be the orthocenter of ΔAPE? Can you show that $AB \perp PE$?

5.13 Can you see that DE is the perpendicular bisector of CI? Can you show F lies on the perpendicular bisector of CI? It may not be easy because we do not know much about the line segments CF and FI. We are given a parallel line ℓ_1 and a tangent line ℓ_2. If ℓ_1 and the line DE intersect at F', can we show that $F'C$ is tangent to ⊙O (i.e., F and F' coincide) by angle properties?

5.14 We are to show $AB - AC = BP - CP$, where $\angle A = 90°$ and angle bisectors are given. It is natural to consider reflecting A about the angle bisectors. In particular, if we draw $DF \perp BC$ at F and $EG \perp BC$ at G, it is easy to see that $AB - AC = BG - CF$.

Hence, P **should** be the midpoint of FG. Can we show it? (Notice that there are many right angles in the diagram.)

5.15 One may notice that the condition and the conclusion are probably related to similar triangles sharing a common vertex. In particular, we are to show $\angle AED = \angle BEP$ and we know that $\angle DAE = \angle DBE$. Hence, we **should** have $\Delta ADE \sim \Delta BPE$.



However, showing $\Delta ADE \sim \Delta BPE$ may not be easy because we know neither *BP* nor $\angle BPE$. Can we show $\Delta ABE \sim \Delta DPE$ instead? It seems the difficulties remain: what do we know about *P*?

Perhaps we should seek more clues from the condition. We are given a circle and a parallelogram, the properties of which should give us many pairs of equal angles. For example, $\angle BDC = \angle ABD = \angle AED$. Notice that we also have $\angle PCD = \angle ACB = \angle CAD$ from the given condition.

It follows that $\Delta PCD \sim \Delta DAE$, which gives us $\frac{PD}{CD} = \frac{DE}{AE}$. Now we know more properties of *P*. Can you see $\Delta ABE \sim \Delta DPE$?

5.16 Upon constructing the diagram, one may notice that this problem is very similar to Example 5.2.8. Can we still apply the technique by introducing a perpendicular from *O* to the chord *AB*?

We are to show $AC \perp CE$, i.e., if *CE* intersects $\odot O$ at *A*', then *AA*' must be a diameter of $\odot O$.

5.17 Given *A*, *B*, *C*, *D* are concyclic and *A*, *B*, *F*, *E* are concyclic, can you see that $\angle DAE = \angle CBF$? Is this useful? (Notice that *AD* and *BF* should **not** be parallel because *F* could be arbitrarily chosen on *CD*).

Given the circumcenters *G* and *H*, can you see that $\angle DGE = \angle 2DAE$? What can you conclude about the (isosceles) triangles ΔDEG and ΔCFH ? Can you see that $DG \parallel FH$?

Since *P*, *G*, *H* **should** be collinear, can you see similar triangles from $DG \parallel FH$? How are ΔAPE and ΔBPC related? Clearly they are not similar, but how are $\frac{PD}{DE}$ and $\frac{PF}{CF}$ related?

Chapter 6

6.1 Recall Example 3.4.1.

6.2 Let AM and BP intersect at D. It is easy to find $\frac{AD}{DM}$ in the right angled triangle ΔABM. Can you find $\frac{AP}{CP}$ by Menelaus' Theorem?
Alternatively, one may draw $PE \perp BC$ at E. Can you see that ΔPEC is also a right angled isosceles triangle?

Indeed, there are many ways to calculate $\frac{AP}{CP}$. One may also draw the square $ABCX$. Can you see that BP extended pass through the midpoint of CX, called F? Can you see $\Delta ABM \cong \Delta BCF$? Can you see $\frac{AP}{CP} = \frac{AB}{CF}$?

6.3 Since AD, BE, CF are concurrent, can you see many pairs of similar triangles?

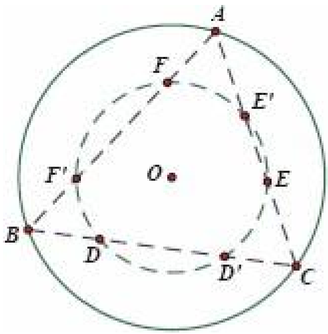
6.4 We are given a median, an angle bisector and an altitude. Can you show that BE is an altitude as well (by considering the median on the hypotenuse BC)? Can you see $EF = \frac{1}{2}AC$?

6.5 How will the circles drawn (with diameters BC and AC) intersect ΔABC ? If you draw a circle with a diameter AC, can you see that it must intersect BC, AC at D, E respectively?
Can you see that MN, PQ intersect at the orthocenter of ΔABC, called H? Can you show that $MH \cdot NH = PH \cdot QH$ by the Intersecting Chords Theorem?

Alternatively, one easily sees that $CM = CN$ and $CP = CQ$. Since M, N, P, Q **should** be cyclic, this circle **should** be centered at C. Can you show $CM = CP$? (Notice that they are in right angled triangles!)

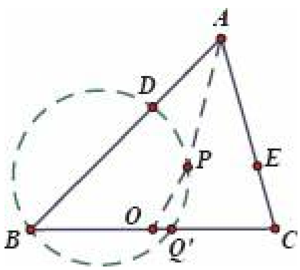
6.6 What can you say about $\frac{AX}{DX}$, by the Angle Bisector Theorem or similar triangles? How are ΔADX and ΔCDY (**not** similar) related?

6.7 Since ΔABC **should** be an equilateral triangle, one should draw an *almost* equilateral triangle. Suppose the circumcircle of ΔDEF intersects BC at D, D'. Can you see that D and D' are symmetric about the midpoint of BC? (Notice that the perpendicular bisector from O to BC is also the perpendicular bisector of DD'.)
How are BD and BD' related? How are BD and BF' related?



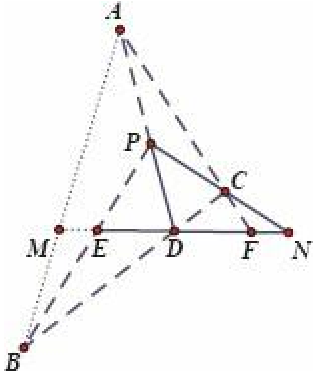
6.8 How to show *Q* lies on *BC*? One strategy is to show that if the circumcircle of Δ*BPD* intersects *BC* at *Q*', then *C*, *E*, *P*, *Q*' are concyclic (say by angle properties). However, this may not be easy because we do not know how Δ*BDP* and Δ*CEP* are related.

Where should Δ*BDP* intersect *BC*? Refer to the diagram below. It *seems* that *A*, *P*, *Q*' are collinear. If this is true, we **should** have ∠*B* = ∠*APD*.



Notice that ∠*B* = ∠*AED* (because *BE*, *CD* are heights). Hence, we **should** have ∠*AED* = ∠*APD*, i.e., *A*, *D*, *P*, *E* **should** be concyclic.
It is easy to see that *PD* = *PE* because *OP* is the perpendicular bisector of *DE*. Can you see why *A*, *D*, *P*, *E* are concyclic?

6.9 How can we construct such a diagram? If we choose *D* and *P* casually, it is difficult to introduce ℓ which gives *DE* = *DF*.
Let us construct the diagram in the reverse manner. Refer to the diagram below. First we draw a line segment *EF* with its midpoint *D*, and *N* is on *EF* extended. Now if *P* and *C* are chosen, *A* and *B* will be uniquely determined (illustrated by the broken lines).

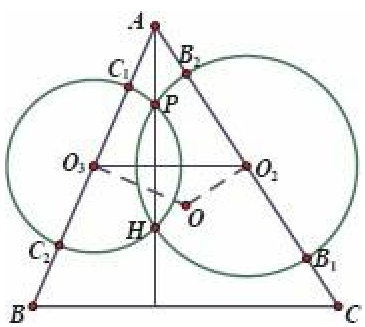


Hence, M is uniquely determined (by the dotted lines), where we **should** have $DM = DN$.
It seems that DM can be calculated via other line segments. Is it reminiscent of Menelaus' Theorem?
Which triangle should we apply Menelaus' Theorem to? We should have line segments DM, DN (or equivalently, EM, FN) in the equation, and probably DE, DF as well. Apparently, more than one triangle will be involved.

6.10 Notice that drawing all the circles given, $\odot O_1$, $\odot O_2$ and $\odot O_3$, only makes the diagram unnecessarily complicated. Instead, we may study the properties of two circles, say $\odot O_2$ and $\odot O_3$. Similar properties should apply to $\odot O_1$ as well.

Let $\odot O_2$ and $\odot O_3$ intersect at P and H. One immediately sees that $PH \perp O_2O_3$.

Since O_2, O_3 are the midpoints of AC, AB respectively, we have $O_2O_3 \,//\, BC$ and hence, $PH \perp BC$. This implies A, P, H are collinear.

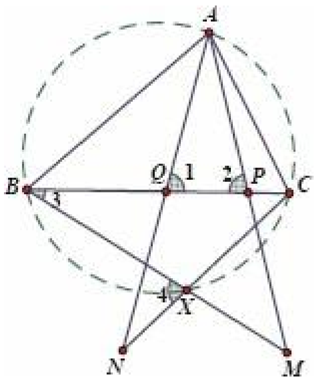


Now a simple application of the Tangent Secant Theorem shows that B_1, B_2, C_1, C_2 are concyclic. Similarly, we should have A_1, A_2, B_1, B_2 concyclic as well. How can we show that A_1, A_2, B_1, B_2, C_1, C_2 all lie on the same circle?

Which circle does B_1, B_2, C_1, C_2 lie on? Do you know the center and the radius of that circle? (You may identify the center by drawing the perpendicular bisectors of B_1B_2 and C_1C_2.) How about the circle which A_1, A_2, B_1, B_2 lie on?

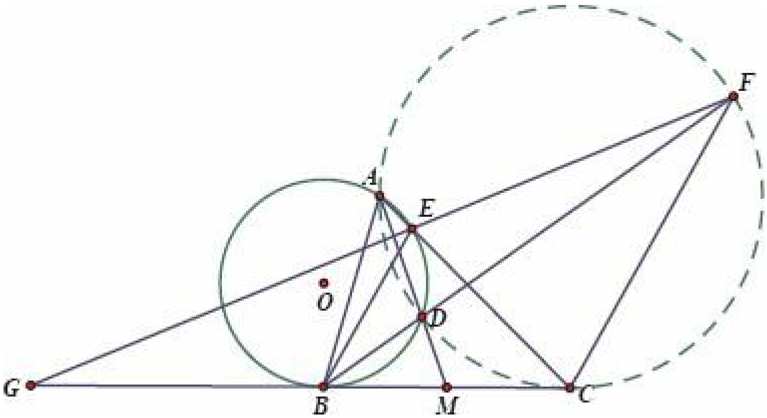
6.11 Suppose BM and CN intersect at X. Since we are to show X lies on the circumcircle of ΔABC, the most straightforward method might be showing that $\angle BXC = 180° - \Delta BAC$.

One notices that $\angle PAB = \angle C$ is a useful condition, with which one easily sees that $\Delta ABC \sim \Delta PBA$.
Similarly, $\Delta ABC \sim \Delta QAC$. (*)
Refer to the diagram below. Can you see that $\angle 1 = \angle 2 = \angle BAC$?
Hence, we **should** have $\angle 4 = \angle BAC = \angle 1 = \angle BQN$, which implies B, N, X, Q are concyclic.



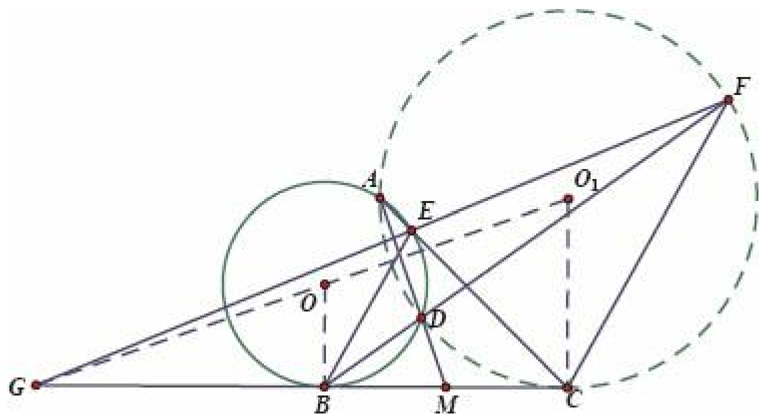
Now we **should** have $\angle 3 = \angle N$ and similarly, $\angle BCN = \angle M$. This implies that $\Delta BPM \sim \Delta NQC$. Can we show it? Since $\angle 1 = \angle 2$, it suffices to show $\frac{BP}{PM} = \frac{NQ}{CQ}$. Notice that we have **not** used the condition that P, Q are midpoints of AM, AN respectively. Now it suffices to show $\frac{BP}{AP} = \frac{AQ}{CQ}$. Can you see it from (*)?

6.12 We are given a circle and a triangle, but the condition $CF \parallel BE$ seems not closely related to circle geometry. Perhaps we can find equal angles through the parallel lines and the property of $\odot O$.



Refer to the diagram above. Since $BE // CF$, we have $\angle BFC = \angle EBF = \angle CAD$, which implies A, D, C, F are concyclic. Alternatively, one may obtain this result by $\angle ACF = \angle BEC = \angle ADF$. Suppose A, D, C, F lie on $\odot O_1$.

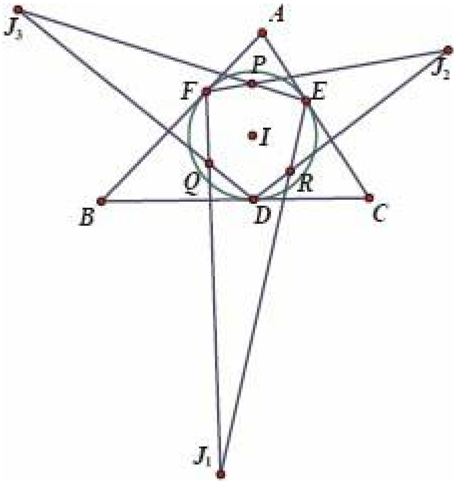
We are to show $AG = DG$, which implies G **should** lie on the perpendicular bisector of AD. Since AD is the common chord of $\odot O$ and $\odot O_1$, its perpendicular bisector is the line OO_1. Can we show that G, O, O_1 are collinear? Refer to the diagram below. Can we show $\angle BGO = \angle BGO_1$? Notice that $BE // CF$ gives us similar triangles $\Delta BEG \sim \Delta CFG$. Hence, it suffices to show O and O_1 are corresponding points in ΔBEG and ΔCFG.



O is obtained by intersecting the perpendicular bisector of BE and the line passing through B perpendicular to BC. Hence, it suffices to show BC is tangent to $\odot O_1$. Notice that we have not used the condition $BM = CM$ Observe the position of M and the two circles. Does it remind you of the Tangent Secant Theorem?

6.13 Refer to the diagram below. We are to show I is the circumcenter of ΔPQR, which is equivalent to $PI = QI = RI$ How is I related to P, Q, R? We know $EI = FI$ and indeed, AI is the perpendicular bisector of EF. Notice that I

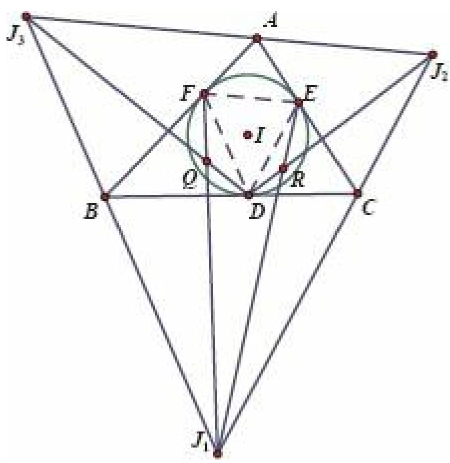
should lie on the perpendicular bisector of QR.



It *seems* from the diagram that EF // QR. Is it true?

If we can show $\frac{J_1Q}{FQ}=\frac{J_1R}{ER}$, then QR // EF, which implies $QI = RI$(because A, I, J_1 are collinear and $J_1E = J_1F$). Similarly, $PI = QI$ and the conclusion follows. This is probably the critical step we need!

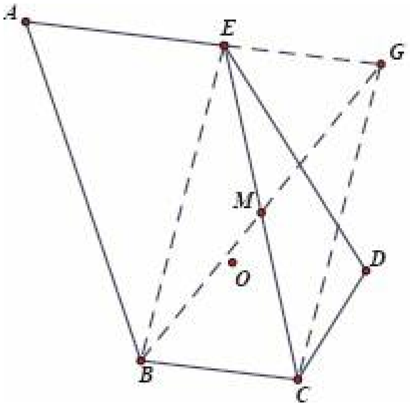
On the other hand, we know EF // J_2J_3 because $AI \perp J_2J_3$ (Exercise 1.5). Similarly, we have DF // J_1J_3 and DE // J_1J_2. Refer to the diagram below.

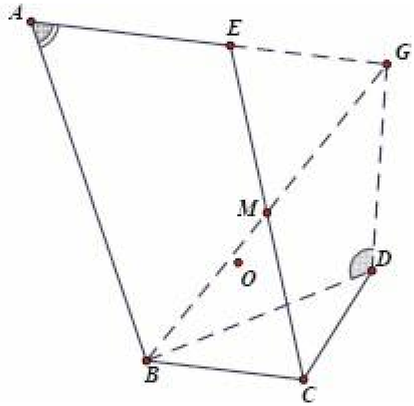


Notice that the parallel lines give $\Delta DEF \sim \Delta J_1J_2J_3$. Now can you see $\frac{J_1Q}{FQ}=\frac{J_1R}{ER}$?

6.14 We are given many conditions. It is easy to seek clues from some of the conditions. Refer to the left diagram below. Since $AB = BC + AE$ and AE //

BC, it is natural to move *BC* up (i.e., extend *AE* to *G* such that *BC* = *EG*). (*)





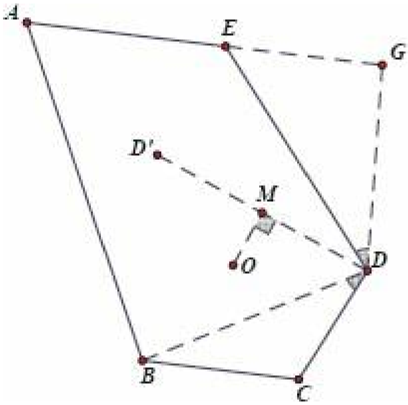
We obtain a parallelogram *BCGE* where *M* is the center, as well as an isosceles triangle ΔABG. Given $AE // BC$, can you see that *BG* bisects $\angle ABC$? Now it suffices to show that $\angle ADB = \frac{1}{2}\angle CDE = \frac{1}{2}\angle ABC = \angle AGB$, i.e., we **should** have *A*, *B*, *D*, *G* concyclic.

Apparently, we do not know much about the line segments, but only about the angles. Refer to the right diagram above. Can we show that $\angle BDG = 180° - \angle A$? Notice that $180° - \angle A = \angle ABC = \angle CDE$. Hence, we **should** have $\angle BDG = \angle CDE$, or equivalently, $\angle BDC = \angle EDG$.

Notice that we have not used the following conditions:

- *M* is the midpoint of *CE* (and hence the center of the parallelogram *BCG*
- *O* is the circumcenter of ΔBCD.
- $OM \perp DM$

It seems that these properties are related to *symmetry*. Refer to the diagram below. Let *D*' be the reflection of *D* about *OM*.
What can you say about *D*' ? Can you see congruent triangles related to *D*' ? How is the parallelogram *BCGE* related to *D*' ? How is *D*' related to *O*, the circumcenter of ΔBCD ?

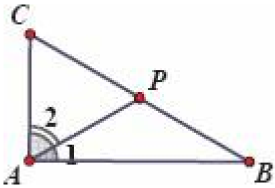


Note: If one extends BC instead of AE at (*) to G such that $CG = AE$, an isosceles triangle ΔABG will be obtained where AG bisects $\angle A$. Unfortunately, this is not useful because we need angles related to half of $\angle ABC$ or $\angle CDE$.

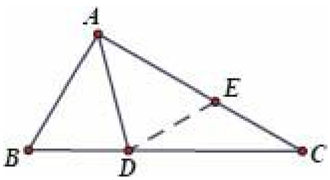
Solutions to Exercises

Chapter 1

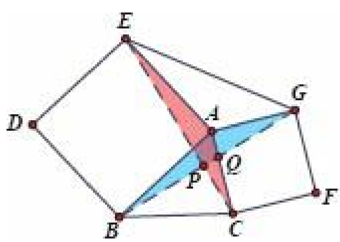
1.1 Since $AP = BP$, we have $\angle 1 = \angle B$. Now $\angle 2 = 90° - \angle 1 = 90° - \angle B = \angle C$, which implies $AP = CP$. The conclusion follows.



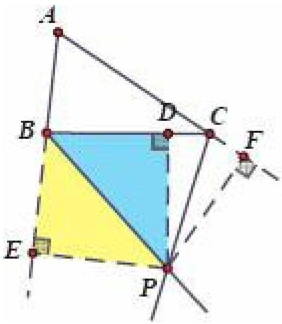
1.2 Choose E on AC such that $AB = AE$ Since AD bisects $\angle BAC$, one sees that $\Delta ABD \cong \Delta AED$ (S.A.S.). Hence, $BD = DE$ and $\angle AED = \angle ABD = 2\angle C$. Since $\angle AED = \angle C + \angle CDE$, we conclude that $\angle C = \angle CDE$, i.e., $CE = DE$. Now $CE = DE = BD$. We have $AC = AE + CE = AB + BD$.

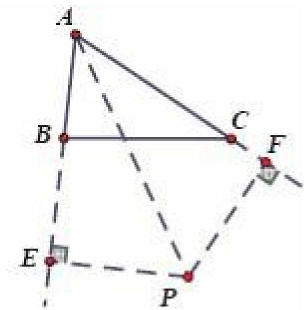


1.3 It is easy to see that $\Delta ACE \cong \Delta AGB$ (S.A.S.). Hence, we have $BG = CE$ and $\angle ACE = \angle AGB$. Let BG and CE intersect at P. Notice that $\angle CPG = \angle CAG = 90°$ (Example 1.1.6) and hence, $BG \perp CE$.



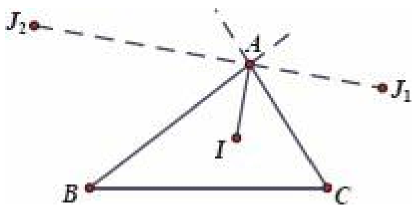
1.4 Refer to the left diagram below. Let BP, CP bisect the exterior angles of $\angle B$, $\angle C$ respectively. We are to show AP bisects $\angle A$. Draw $PD \perp BC$ at D, $PE \perp AB$ at E and $PF \perp AC$ at F. It is easy to see that $\Delta BPE \cong \Delta BPD$ (A.A.S.) and hence, $PD = PE$. Similarly, $PD = PF$.





Now we have $PE = PF$. Refer to the right diagram above. One sees that $\Delta APE \cong \Delta APF$ (H.L.) and hence, AP bisects $\angle A$.

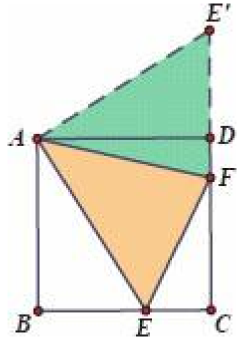
1.5 Connect AJ_1. Since AI and AJ_1 are the angle bisectors of neighboring supplementary angles, we have $AI \perp AJ_1$ (Example 1.1.9, or one may simply see that



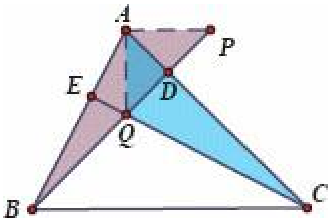
$$\angle IAJ_1 = \angle CAI + \angle CAJ_1 = \frac{1}{2}\angle BAC + \frac{1}{2}(180° - \angle BAC) = 90°\ .)$$

Similarly, $AI \perp AJ_2$. Now J_1AJ_2 = 90° + 90° = 180° which implies A, J_1, J_2 are collinear and hence, $AI \perp J_1J_2$.

1.6 Choose E' on CD extended such that $DE' = BE$. Connect AE' It is easy to see that $\Delta ABE \cong \Delta ADE'$ (S.A.S.). Hence, $AE = AE'$ and $\angle BAE = \angle DAE$.' Now we see that $\angle EAF = \angle E'AF = 45°$ and $\Delta AEF \cong \Delta AE'F$ (S.A.S.). Hence, $EF = E'F = DF + BE$.

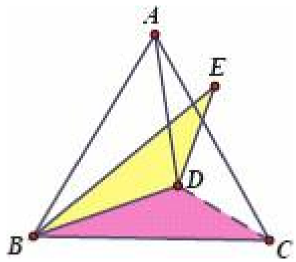


1.7 We have $\angle ABD = \angle ACE = 90° - \angle BAC$. Hence, $\Delta ABP \cong \Delta QCA$ (S.A.S.). It follows that $AQ = AP$ and $\angle QAD = \angle APD = 90° - \angle PAC$, i.e., $\angle QAD + \angle PAC = \angle PAQ = 90°$. Thus, $\angle AQP = 45°$.



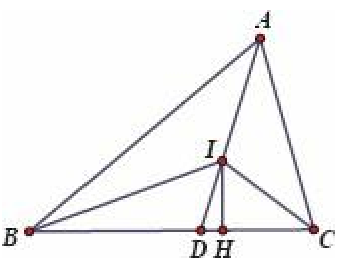
1.8 Connect CD. Since $BE = AB = BC$ and BD bisects $\angle CBE$, we have $\Delta BCD \cong \Delta BED$ (S.A.S.). Hence, $\angle BED = \angle BCD$.

Since $AD = BD$, D (and similarly C) lie on the perpendicular bisector of AB, which is indeed the line CD. It follows that CD bisects $\angle ACB$.

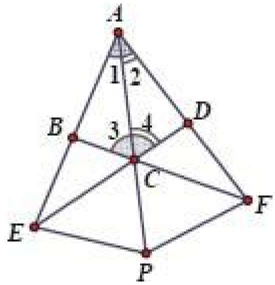


Now $\angle BED = \angle BCD = \frac{1}{2}\angle ACB = 30°$.

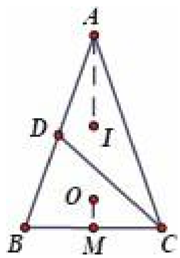
1.9 Since I is the incenter, CI bisects $\angle C$. Theorem 1.3.3 gives $\angle AIB = 90° + \frac{1}{2}\angle C$. Hence, $\angle BID$ = 180°’ $\angle AIB = 90° + \frac{1}{2}\angle C$ = 90° − $\angle BCI = \angle CIH$.



1.10 Since $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, we have $\Delta ABC \cong \Delta ADC$ (A.A.S.). Hence, $AB = AD$ and $\angle ABF = \angle ADE$. Now $\Delta ABF \cong \Delta ADE$ (A.A.S.), which implies $AE = AF$. It follows that $\Delta AEP \cong \Delta AFP$ (S.A.S.) and $PE = PF$. Note that the proof holds regardless of the position of P.

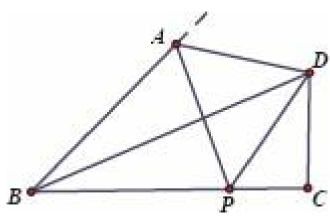


1.11 Let M be the midpoint of BC. Since O is the circumcenter of ΔBCD, OM is the perpendicular bisector of BC. On the other hand, since I is the incenter of ΔACD, AI is the angle bisector $\angle A$, which passes through M since $AB = AC$. Thus, A, I, O lie on the perpendicular bisector of BC. The conclusion follows.



1.12 Let $\angle ABC = 2\alpha$ and $\angle APC = 2\beta$. We have $\angle BAP = \angle APC - \angle ABC = 2(\alpha - \beta)$. Since BD, PD are angle bisectors, we have $\angle CBD = \alpha$ and $\angle CPD = \beta$. It follows that $\angle BDP = \angle CPD - \angle CBD = \alpha - \beta$

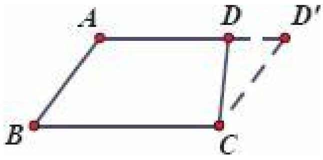
Notice that D is the ex-center of ΔABP opposite B (Exercise 1.4), which implies that AD bisects the exterior angle of $\angle BAP$.



Now $\angle PAD = \frac{1}{2}(180° - \angle BAP) = 90° - \frac{1}{2}\cdot 2(\alpha - \beta) = 90° - \angle BDP.$

This completes the proof.

1.13 Suppose otherwise. Draw $CD' \parallel AB$, intersecting the line AD at D' Now $ABCD'$ is a parallelogram and $AB = CD'$ $BC = AD'$ We have $AD' - CD' = BC - AB = AD - CD$.



Case I: $AD < AD'$

Refer to the diagram below.

We have $DD' = AD' - AD = CD' - CD$, i.e., $DD' + CD = CD'$. This contradicts triangle inequality.

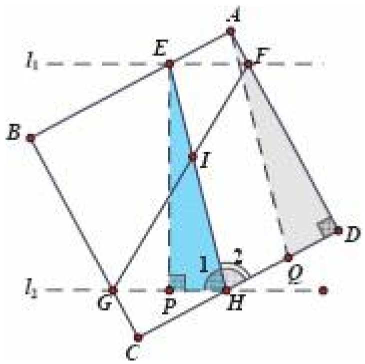
Case II: $AD > AD'$

Similarly, we have $DD' = AD - AD' = CD - CD'$, i.e., $DD' + CD' = CD$. This contradicts triangle inequality.

It follows that D and D' coincide, i.e., $ABCD$ is a parallelogram.

1.14 Draw $EP \perp \ell_2$ at P and $AQ \parallel EH$, intersecting CD at Q. It is easy to see that $AEHQ$ is a parallelogram and hence, $EH = AQ$. Given that $EP = AD$, we must have $\Delta EPH \cong \Delta ADQ$ (H.L.). It follows that $\angle 1 = \angle AQD = \angle 2$.

Similarly, we have $\angle BGF = \angle HGF$.



Now $\angle GIH = 180° - \angle HGF - \angle 1$, where $\angle 1 = \frac{1}{2}(180° - \angle CHG)$

$= 90° - \frac{1}{2}\angle CHG$ and similarly, $\angle HGF = 90° - \frac{1}{2}\angle CGH$.

Hence, $\angle GIH = 180° - \left(90° - \frac{1}{2}\angle CHG\right) - \left(90° - \frac{1}{2}\angle CGH\right)$

$= \frac{1}{2}(\angle CGH + \angle CHG) = 45°$, because ΔCGH is a right angled triangle where $\angle C = 90°$.

Note: One may observe that I is the ex-center of ΔCGH opposite C (Exercise

1.4). Indeed, one may show, following a similar argument as above, that if *J* is the ex-center of ΔABC opposite *A*, then we always have $\angle BJC = 90° - \frac{1}{2}\angle A$. (You may compare this result with Theorem 1.3.3.)

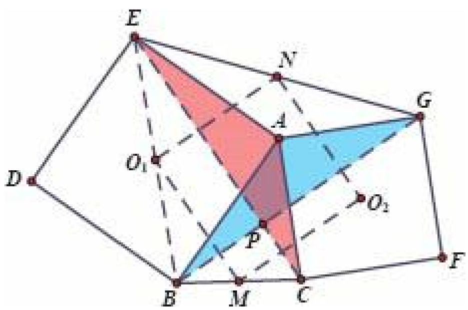
Chapter 2

2.1 Since $AB = BD$, $[\Delta ABC] = [\Delta BCD] = \frac{1}{2}[BCXD]$, i.e., we have $[BCXD] = 2[\Delta ABC]$.

Similarly, $[ACEY] = [\Delta ABC]$ and $[ABZF] = 4[\Delta ABC]$.

Now the total area of parallelograms is $175 = 7[\Delta ABC]$. It follows that $[\Delta ABC] = 25\text{cm}^2$.

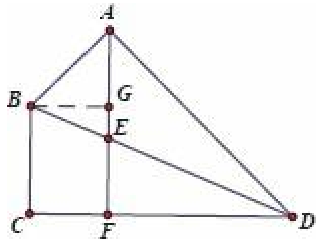
2.2 It is easy to see that $\Delta ACE \cong \Delta AGB$, which implies $CE = BG$ and $BG \perp CE$ (Exercise 1.3). Since O_1M is a midline of ΔBEC, we have $O_2N = \frac{1}{2}CE$ and $O_1M \parallel CE$.



Similarly, $O_2N = \frac{1}{2}CE$ and $O_2N \parallel CE$. Now $O_1M = O_2N$ and $O_1M \parallel O_2N$ imply MO_1NO_2 is a parallelogram.

A similar argument gives $O_1N = O_2M = \frac{1}{2}BG$ and $O_1N \parallel O_2M \parallel BG$. Now $BG = CE$ implies $O_1M = O_1N$ while $BG \perp CE$ implies $O_1M. \perp O_2N$ It follows that MO_1NO_2 is a square.

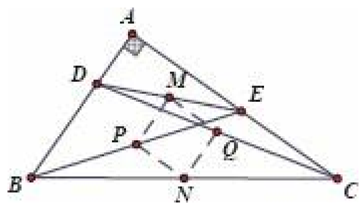
2.3 Draw $BG \perp AF$ at *G*. It is easy to see that ΔABG and ΔADF are right angled isosceles triangles and *BCFG* is a rectangle.
Hence, $CF = BG = AG$ and $AF = DF$. Now $CD - 2AE = CF + DF - 2AE = AG + AF - 2AE = (AG - AE) + (AF - AE) = EF - EG$.



Since $DF = AF > AG = CF$, $\frac{EF}{BC} = \frac{DF}{DF+CF} > \frac{1}{2}$, i.e., $EF > EG$.

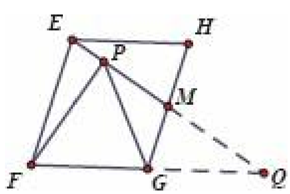
In conclusion, $CD - 2AE > 0$ and $AE < \frac{1}{2}CD$.

2.4 Notice that *PM* is a midline of Δ*BDE*. Hence, $PM = \frac{1}{2}BD$ and *PM* // *BD*. Similarly, $QN = \frac{1}{2}BD = PM$ and *QN* // *BD* // *PM*.



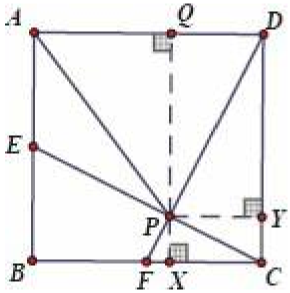
We also have $QM = PN = \frac{1}{2}CE$ and *QM* // *PN* // *CE*. It follows that *MPNQ* is a parallelogram. Since *PM* // *AB*, *QM* // *AC* and *AB* ⊥ *AC*, we must have *PM* ⊥ *QM*. Hence, *MPNQ* is a rectangle and *MN* = *PQ*.

2.5 It is easy to see that *EFGH* is a parallelogram (Example 2.2.6). We focus on *EFGH*. Refer to the diagram below. Let *EM* extended and *FG* extended intersect at *Q*.



Since *EH* // *FQ* and *GM* = *HM*, Δ*EHM* ≅ Δ*QGM* (A.A.S.). Hence, *QG* = *EH* = *FG*. It is given that *FG* = *PG*. We have $PG = FG = QG = \frac{1}{2}FQ$. It follows that *FP* ⊥ *PQ* (Example 1.1.8).

2.6 Let *AB* = *a*. Since *ABCD* is a square, it is easy to see that *BE* = *CF* and Δ*BCE* ≅ Δ*CDF*. Now ∠*BCE* = ∠ *CDF* = 90° – ∠*CFD*, which implies *CE* ⊥ *DF*.

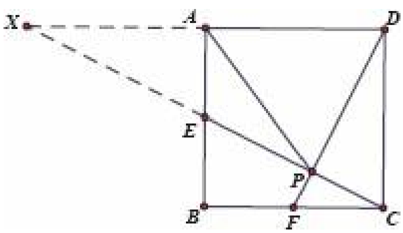


Notice that $\frac{PF}{PD}=\left(\frac{CF}{CD}\right)^2=\frac{1}{4}$ (Example 2.3.1).

Draw $PX \perp BC$ at X, $PY \perp CD$ at Y and $PQ \perp AD$ at Q. We have $\frac{CY}{DY}=\frac{PF}{PD}=\frac{1}{4}$ and $\frac{FX}{CX}=\frac{PF}{PD}=\frac{1}{4}$. Hence, $DY=\frac{4}{5}a=PQ$ and $CX=\frac{4}{5}CF=\frac{2}{5}BC$, which implies $AQ=BX=\frac{3}{5}a$.

By Pythagoras' Theorem, $AP=\sqrt{AQ^2+PQ^2}=a\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=a$, i.e., $AP = AB$.

Note: There is an alternative solution based on the median *CE* doubled. Refer to the diagram below. Extend *CE* to *X* such that *CE* = *EX* It is easy to see that $\Delta BCE \cong \Delta AXE$.



Hence, *X* lies on the line *AD* and *AD* = *AX* Notice that $CE \perp DF$ as shown in the proof above. It follows that *AP* is the median on the hypotenuse *DX* of the right angled triangle Δ*PXD*. Hence, $AP=\frac{1}{2}DX$ = *AD* = *AB* (Theorem 1.4.6).

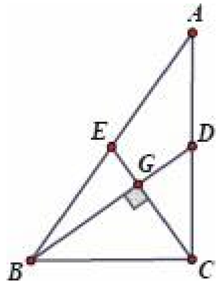
This is an elegant solution, even though the previous solution using Pythagoras' Theorem is more straightforward.

2.7 Let *BD,CE* be the medians. By the Midpoint Theorem, *BG* = 2*DG* and *CG* = 2*EG*.

Let $DG = a$ and $EG = b$. Since $BG \perp CG$, we have $BE^2 = (2a)^2 + b^2 = 4a^2 + b^2$.

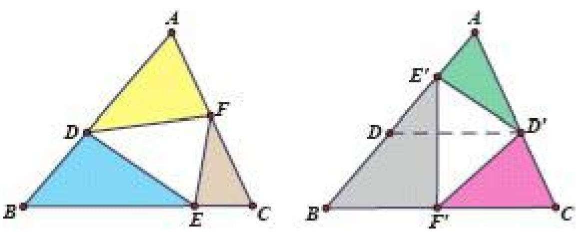
Hence, $AB^2 = (2BE)^2 = 4 \cdot (4a^2 + b^2) = 16a^2 + 4b^2$.

Similarly, $AC^2 = 4a^2 + 16b^2$. It follows that $AB^2 + AC^2 = 20(a^2 + b^2)$, while $BC^2 = (2a)^2 + (2b)^2 = 4(a^2 + b^2)$. The conclusion follows.



2.8 Refer to the following diagrams. Since $DD' \,//\, BC$, by the Intercept Theorem, we have $\frac{AD}{BD} = \frac{AD'}{CD'}$. Let $\frac{AD}{AB} = \frac{AD'}{AC} = a$.

Similarly, let $\frac{BE}{BC} = \frac{BE'}{AB} = b$ and $\frac{CF}{AC} = \frac{CF'}{BC} = c$.



Let S = Δ[ABC]. Note that $[\Delta DEF] = S - ([\Delta ADF] + [\Delta BDE] + [\Delta CEF])$ and $[\Delta D'E'F'] = S - ([\Delta AD'E'] + [\Delta BE'F'] + [\Delta CD'F'])$.

One sees that $\frac{[\Delta ADF]}{[\Delta ABC]} = \frac{AD}{AB} \cdot \frac{AF}{AC} = a(1-c)$, i.e., $[\Delta ADF] = a(1-c)S$.
Similarly, $[\Delta BDE] = b(1-a)S$ and $[\Delta CEF] = c(1-b)S$.

We also have $[\Delta AD'E'] = \frac{AD'}{AC} \cdot \frac{AE'}{AB} = a(1-b)S$, $[\Delta BE'F'] = b(1-c)S$ and $[\Delta CD'F'] = c(1-a)S$.

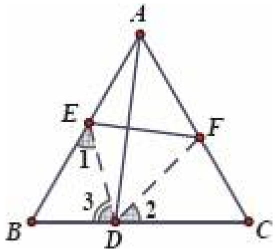
Hence, $[\Delta ADF] + [\Delta BDE] + [\Delta CEF] = [a(1-c) + b(1-a) + c(1-b)] \cdot S$

$= [a + b + c - ac - ab - bc] \cdot S$

$$= [a(1-b)+b(1-c)+c(1-a)]\cdot S = [\Delta AD'E'] + [\Delta BE'F'] + [\Delta CD'F'].$$

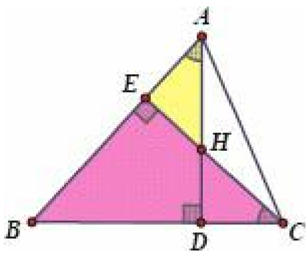
The conclusion follows.

2.9 Connect *DE, DF*. We claim that Δ*BDE* ~ Δ*CFD*. Notice that ∠*B* = ∠*C* = 60°. It suffices to show that ∠1 = ∠2. Since *EF* is the perpendicular bisector of *AD*, we must have *AE* = *DE* and *AF* = *DF*. Hence, Δ*AEF* ≅ Δ*DEF* (S.S.S). Now ∠*EDF* = ∠*EAF* = 60° and hence, ∠2 = 180° – ∠*EDF* – ∠3 = 180° – 60° – ∠3 = 180° – ∠B – ∠3 = ∠1.

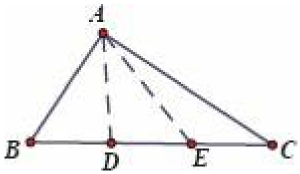


We conclude that Δ*BDE* ~ Δ*CFD*. It follows that $\frac{BD}{BE} = \frac{CF}{CD}$, or equivalently, $BD \cdot CD = BE \cdot CF$.

2.10 It is easy to see that ∠*EAH* = ∠*DCH*. Hence, Δ*BCE* ~ Δ*HAE* and we have $\frac{BC}{AH} = \frac{CE}{AE} = \tan\angle A$.



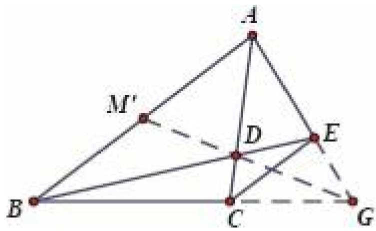
2.11 Let *BC* = *a*, *AC* = *b*, *AB* = *c*, *AD* = *x* and *AE* = *y*. Clearly, $BE = CD = \frac{2}{3}a$.



Since *AD* is a median of Δ*ABE*, we have $x^2 = \frac{1}{2}c^2 + \frac{1}{2}y^2 - \frac{1}{4}\left(\frac{2}{3}a\right)^2$ by Theorem 2.4.3. Similarly, $y^2 = \frac{1}{2}b^2 + \frac{1}{2}x^2 - \frac{1}{4}\left(\frac{2}{3}a\right)^2$ because *AE* is a median of ΔACD. Hence, we have $x^2+y^2=\frac{1}{2}b^2+\frac{1}{2}c^2+\frac{1}{2}x^2+\frac{1}{2}y^2-\frac{1}{4}\left(\frac{2}{3}a\right)^2-\frac{1}{4}\left(\frac{2}{3}a\right)^2$, which could be simplified to $x^2+y^2=\left(b^2+c^2\right)-\frac{4}{9}a^2$.

Pythagoras' Theorem gives $b^2+c^2=a^2$ and the conclusion follows.

2.12 Let *AE* and *BC* intersect at *G*. Suppose *GD* extended intersects *AB* at *M*'. By Ceva's Theorem, $\frac{AM'}{BM'}\cdot\frac{BC}{GC}\cdot\frac{GE}{AE}=1$. Since *CE* // *AB*, we have $\frac{BC}{GC}=\frac{AE}{GE}$.

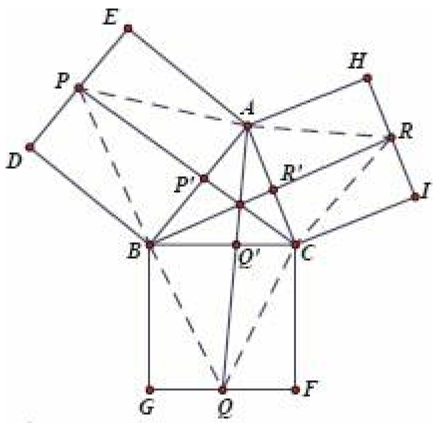


Hence, $\frac{AM'}{BM'}=1$, i.e., *M*' coincides with *M*. We conclude that the line *MD* passes through *G*, i.e., the lines *AE, BC, MD* are concurrent at *G*.

Note: One may solve this problem by Menelaus' Theorem as well.

Consider the line *BE* intersecting ΔACG: $\frac{AE}{GE}\cdot\frac{GB}{CB}\cdot\frac{CD}{AD}=1$. (*) Since *AB* // *CE*, $\frac{AE}{GE}=\frac{BC}{CG}$. We obtain $\frac{GB}{GC}\cdot\frac{CD}{AD}=1$ from (*). Now $\frac{AM}{BM}=1$ implies $\frac{AM}{BM}\cdot\frac{BG}{CG}\cdot\frac{CD}{AD}=1$. It follows from Menelaus' Theorem that *D, G, M* are collinear.

2.13 Refer to the diagram below. Let *AQ* intersect *BC* at *Q*', *BR* intersect *AC* at *R*' and *CP* intersect *AB* at *P*'. We claim that $\frac{BQ'}{CQ'}\cdot\frac{CR'}{AR'}\cdot\frac{AP'}{BP'}=1$.



Notice that $\frac{BQ'}{CQ'}=\frac{[\Delta ABQ]}{[\Delta ACQ]}$

$=\frac{\frac{1}{2}AB\cdot BQ\sin\angle ABQ}{\frac{1}{2}AC\cdot CQ\sin\angle ACQ}=\frac{AB\sin(\angle ABC+\alpha)}{AC\sin(\angle ACB+\alpha)}$ where $BQ = CQ$ and $\alpha = \angle BCQ = \angle CBQ$. It is easy to see that $\alpha = \angle ACR = \angle ABP$.

Similarly, $\frac{CR'}{AR'}=\frac{BC\sin(\angle ACB+\alpha)}{AB\sin(\angle BAC+\alpha)}$ and $\frac{AP'}{BP'}=\frac{AC\sin(\angle BAC+\alpha)}{BC\sin(\angle ABC+\alpha)}$. It follows that $\frac{BQ'}{CQ'}\cdot\frac{CR'}{AR'}\cdot\frac{AP'}{BP'}=1$ and by Ceva's Theorem, AQ, BR, CP are concurrent.

2.14 By Menelaus' Theorem, it suffices to show $\frac{AF}{BF}\cdot\frac{BD}{CD}\cdot\frac{CE}{AE}=1$. By the Angle Bisector Theorem, $\frac{AF}{BF}=\frac{AC}{BC}$, $\frac{BD}{CD}=\frac{AB}{AC}$ and $\frac{CE}{AE}=\frac{BC}{AB}$. The conclusion follows.

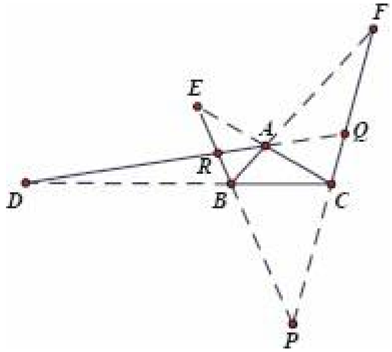
Note:

(1) One may find it easier to solve this problem by applying Menelaus' Theorem and the Angle Bisector Theorem *mechanically* instead of referring to the diagram.

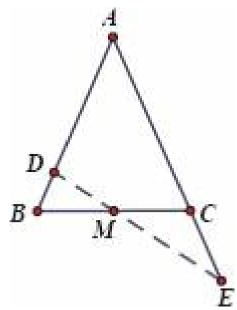
(2) One may also solve this problem using Desargues' Theorem. Refer to the diagram below, where P, Q, R are the ex-centers of ΔABC opposite A, B, C respectively. Apply Desargues' Theorem to ΔABC and ΔPQR. One sees that D, E, F are the intersections of the corresponding sides

extended: AB, PQ intersect at F, BC, QR intersect at D, AC, PR intersect at E.

Now D, E, F are collinear if the lines AP, BQ, CR are concurrent. This is clear because they all pass through the incenter of ΔABC.



2.15 Refer to the diagram below. Apply Menelaus' Theorem when the line *AE intersects* ΔBDM : $\frac{BA}{DA}\cdot\frac{DE}{ME}\cdot\frac{MC}{BC}=1$.



Since $\frac{MC}{BC}=\frac{1}{2}$, we have $\frac{AB}{AD}=2\frac{ME}{DE}$.

Apply Menelaus' Theorem when the line AB intersects ΔCEM :

$\frac{CA}{EA}\cdot\frac{ED}{MD}\cdot\frac{MB}{CB}=1$. Since $\frac{MB}{CB}=\frac{1}{2}$, we have $\frac{AC}{AE}=2\frac{MD}{DE}$.

Since $AB = AC$, we have $\frac{AB}{AD}+\frac{AB}{AE}=\frac{AB}{AD}+\frac{AC}{AE}=2\left(\frac{DM}{DE}+\frac{EM}{DE}\right)$
$=2\cdot\frac{DM+EM}{DE}=2$, i.e., $\frac{AB}{AD}+\frac{AB}{AE}=2$. This completes the proof.

Note: One may find an alternative solution using the area method. We are to show $\frac{1}{AD}+\frac{1}{AE}=\frac{2}{AB}$, i.e., $\frac{AD+AE}{AD\cdot AE}=\frac{2}{AB}$. We claim that $AD \cdot AE$
$+ AB \cdot 2AD \cdot AE$

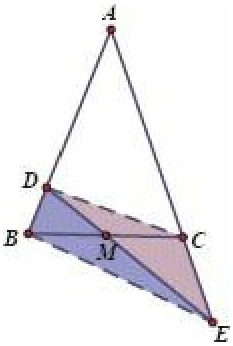
Notice that $[\Delta ACD] = \frac{1}{2} AD \cdot AB \sin \angle A$ (since $AB = AC$),

$[\Delta ABE] = \frac{1}{2} AB \cdot AE \sin \angle A$ and $[\Delta ADE] = \frac{1}{2} AD \cdot AE \sin \angle A$.

Hence, it suffices to show that $[\Delta ACD] + [\Delta ABE] = 2[\Delta ADE]$.

Refer to the diagram below. Since $BM = CM$, we have $[\Delta BDE] = [\Delta CDE]$. (Can you see it?)

Hence, $[\Delta ADE] - [\Delta ACD] = [\Delta ABE] - [\Delta ADE]$, which completes the proof.

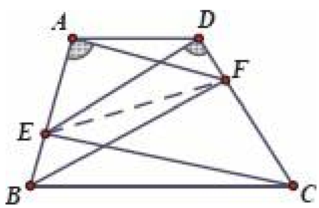


Chapter 3

3.1 (a) We always have $\angle A = \angle C$ in the parallelogram $ABCD$. Now $ABCD$ is cyclic if and only if $\angle A + \angle C = 180°$, which implies $\angle A = \angle C = 90°$. Hence, $ABCD$ is cyclic if and only if $ABCD$ is a rectangle.

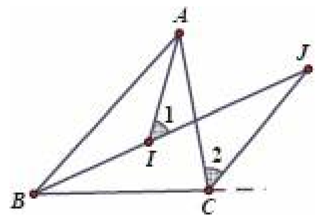
(b) In a trapezium $ABCD$, say $AD // BC$, we always have $\angle A + \angle B = 180°$. Now $ABCD$ is cyclic if and only if $\angle A + \angle C = 180°$, which implies $\angle B = \angle C$, i.e., $ABCD$ is cyclic if and only if it is an isosceles trapezium.

3.2 Since $\angle BAF = \angle CDE$, A, D, F, E are concyclic. Hence, $\angle BAD = \angle CFE$ (Corollary 3.1.5). Since $\angle BAD + \angle ABC = 180°$, we have $\angle ABC + \angle CFE = 180°$, i.e., B, C, F, E are concyclic.



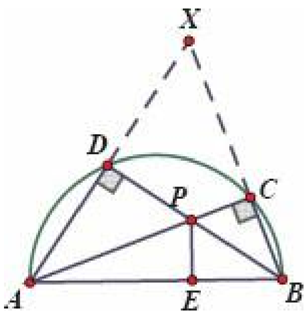
Since $\angle AFE = \angle ADE$ and $\angle BFE = \angle BCE$ (Corollary 3.1.3), we have $\angle AFB = \angle AFE + \angle BFE = \angle ADE + \angle BCE$. One can easily see that $\angle ADE + \angle BCE = \angle CED$ (Example 1.4.15). The conclusion follows.

3.3 Since $\angle 1 = \angle BAI + \angle ABI = \frac{1}{2}(\angle A + \angle B)$ and $\angle 2 = \frac{1}{2}(180° - \angle C) = \frac{1}{2}(\angle A + \angle B)$, we have $\angle 1 = \angle 2$. Hence, *A*, *I*, *C*, *J* are concyclic.

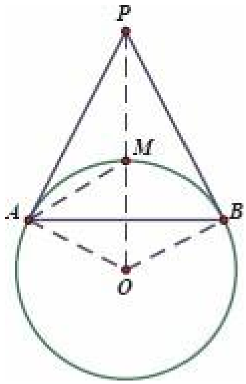


Note : One may also show that $\angle CAI = \angle CJI$.

3.4 Let the lines *AD*, *BC* intersect at *X*. Since *AB* is the diameter of the semicircle, we must have $AC \perp BC$, $AD \perp BD$ (Corollary 3.1.13). Hence, *P* is the orthocenter of Δ*ABX*. It follows that $XP \perp AB$, i.e., *X*, *P*, *E* are collinear. Example 3.1.6 states that *P* is the incenter of Δ*CDE*.



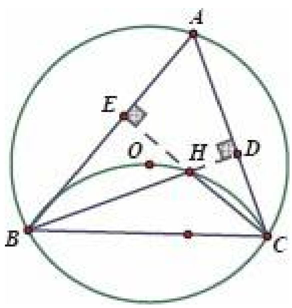
3.5 Let *OP* intersect ⊙*O* at *M*. It is easy to see Δ*PAO* ≅ Δ*PBO* (H.L.). Hence, $\angle AOM = \angle BOM$ and they must correspond to equal arcs. It follows that *M* is the midpoint of $\overset{\frown}{AB}$.



Since $\angle PAM = \frac{1}{2}\angle AOM$ (Theorem 3.2.10), we have $\angle BAM = \frac{1}{2}\angle BOM = \frac{1}{2}\angle AOM = \angle PAM$.

Now AM bisects $\angle PAB$ and clearly, PM bisects $\angle APB$. It follows that M is the incenter of ΔPAB.

3.6 (a) Since H is the orthocenter, $\angle BHC = 180° - \angle A$ (Example 2.5.5). Since B, C, O, H are cyclic, we have $\angle BOC = 2\angle A = \angle BHC$. It follows that $2\angle A = 180° - \angle A$, or $\angle A = 60°$.

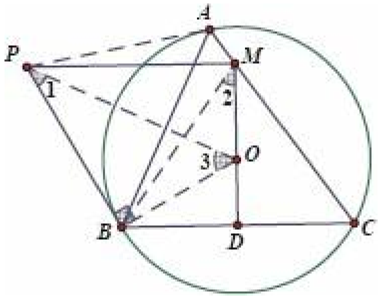


(b) Let the circumradii of ΔABC and ΔBHC be R_1, R_2 respectively.

By Sine Rule, $\frac{BC}{\sin\angle A} = 2R_1$ and $\frac{BC}{\sin\angle BHC} = 2R_2$.

Since $\angle BHC = 180° - \angle A$, we have $\sin\angle A = \sin(180° - \angle A)$. It follows that $R_1 = R_2$.

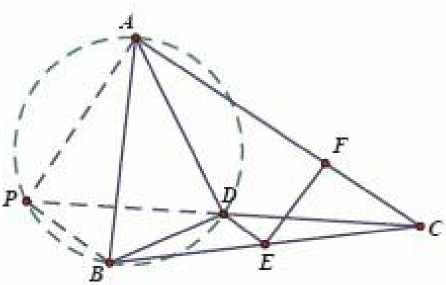
3.7 It is easy to see that OD is the perpendicular bisector of BC. Hence, $BM = CM$ and we have $\angle 2 = \angle CMD = 90° - \angle C$. On the other hand, consider the right angled triangles ΔAOP and ΔBOP.

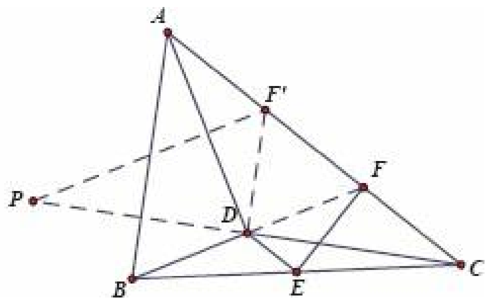


We have $\angle 1 = 90° - \angle 3$ and $\angle 3 = \frac{1}{2}\angle AOB = \angle C$ (Theorem 3.1.1).

It follows that $\angle 1 = 90° - \angle C = \angle 2$ and hence, *B*, *O*, *M*, *P* are concyclic. Now $\angle OMP = \angle OBP = 90°$. This completes the proof.

3.8 Extend *CD* to *P* such that *CD* = *PD*. We have *BP* = 2*DE* and Δ*ADP* is an equilateral triangle (because Δ*DAC* is an isosceles triangle and $\angle ADC$ = 120°). It follows that $\angle APD = 60° = \angle ABD$, i.e., *A*, *P*, *B*, *D* are concyclic. Refer to the left diagram below.

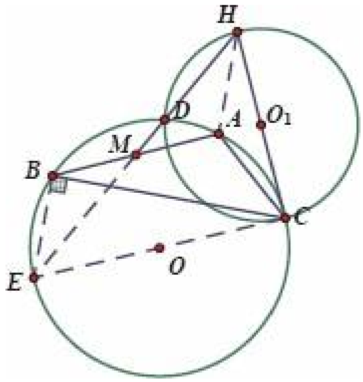




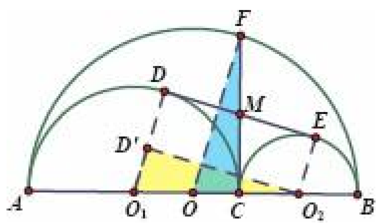
Let *F* and *F*' be the trisection points of *AC*. Notice that Δ*DFF*' is an equilateral triangle (Example 2.3.4). Clealy, *PF*' // *DF*. We must have $\angle AF'P = \angle AFD$ = 60° = $\angle ADP$. It follows that *A*, *F*' , *D*, *P* are concyclic. Refer to the right diagram above.

Now *A*, *P*, *B*, *D*, *F*' lie on the same circle where *PF*' is a diameter (since $\angle PAC$ = 90º). We have *DE* // *BP, EF* // *BF*' and *BP* ⊥ *BF*' (since *PF*' is the diameter). It follows that *DE* ⊥ *EF*.

3.9 Let *CE* be a diameter of ⊙*O*. Now *BE* ⊥ *BC* and *AH* ⊥ *BC*, which implies *BE* // *AH*. Similarly, *AE* // *BH* since both are perpendicular to *AC*. It follows that *AEBH* is a parallelogram. It suffices to show that *H*, *M*, *E* are collinear, in which case the diagonals of *AEBH* bisect each other.
Notice that $\angle CDH = 90° = \angle CDE$. Hence, *H*, *D*, *M*, *E* are collinear.



3.10 Let the midpoints of AB, AC, BC be O, O_1,O_2 respectively. We have $CM = DM = EM$ (equal tangent segments). Draw $O_2D' \perp O_1D$ at D'. Notice that DEO_2D' is a rectangle and hence, $DE = D'O_2$.



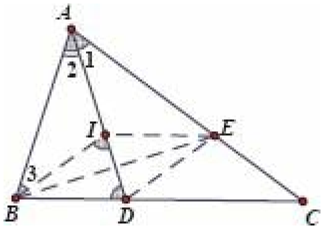
We denote $O_1C = r_1$ and $O_2C = r_2$. Notice that $AB = 2(r_1 + r_2)$ and hence, $OF = OB = r_1 + r_2$. It follows that $O_1D' = O_1D - O_2E = r_1 - r_2$ and $OC = OB - BC = (r_1 + r_2) - 2r_2 = r_1 - r_2$, i.e., $O_1D' = OC$. We also notice that $O_1O_2 = O_1C + O_2C = \frac{1}{2}AC + \frac{1}{2}BC = \frac{1}{2}AB = r_1 + r_2 = OF$.

Now $\Delta OCF \cong \Delta O_1D'O_2$ (H.L.), which implies $CF = D'O_2$. Hence, $CF = DE$. Since $CM = DM = EM$, we must have $FM = DM$.

Now $CDFE$ is a parallelogram since CF and DE bisect each other. Moreover, $CDFE$ is a rectangle since $CF = DE$.

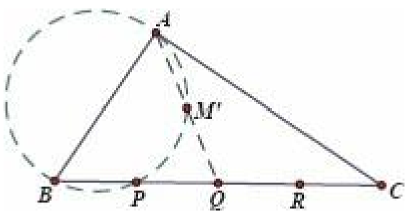
Note: One may also show $CF = DE$ using Pythagoras' Theorem, i.e., $CF = \sqrt{OF^2 - OC^2}$ and $DE = D'O_2 = \sqrt{O_1O_2{}^2 - O_1D'^2}$.

3.11 Choose E on AC such that $AB = AE$. It is easy to see that $\Delta ABD \cong \Delta AED$, $BD = DE$ and AD is the perpendicular bisector of BE.
Now $\angle AED = \angle ABD = 2\angle C$, which implies $\angle CDE = \angle AED - \angle C = \angle C$. Hence, $CE = DE = BD$. We claim that E is the circumcenter of ΔCDI and it suffices to show that $EI = DE$, or equivalently, $BI = BD$.



Notice that $\angle BDI = \angle 1 + \angle C = \frac{1}{2}\angle A + \angle C$ and $\angle BID = \angle 2 + \angle 3 = \frac{1}{2}\angle A + \angle 3$. Since $\angle 3 = \frac{1}{2}\angle ABC = \angle C$, we have $\angle BDI = \angle BID$, i.e., $BI = BD$. This completes the proof.

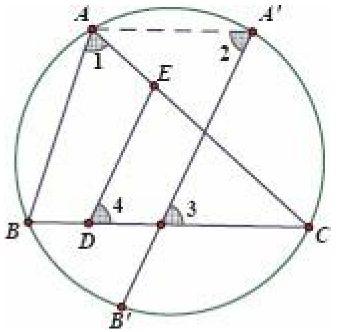
3.12 Let M' denote the midpoint of AQ. Since $\angle A = 90°$ and Q is the midpoint of BC, we have $AQ = BQ = CQ$.



Hence, $AM' = \frac{1}{2}AQ = \frac{1}{2}BQ = BP$.

Since $QA = QB$, we have $PM' \,//\, AB$ by the Intercept Theorem. It follows that $ABPM'$ is an isosceles trapezium. Hence, A, B, P, M' are concyclic (Exercise 3.1), i.e., M' lies on the circumcircle of ΔABP. Similarly, M' also lies on the circumcircle of ΔACQ. We conclude that M and M' coincide and hence, A, M, Q are collinear.

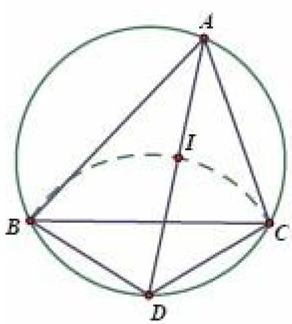
3.13 It is easy to see that $AA'CB$ and $ABB'C$ are isosceles trapeziums. Hence, $\overset{\frown}{BC} = \overset{\frown}{AB'}$, which extend equal angles on the circumference, i.e., $\angle 1 = \angle 2$.

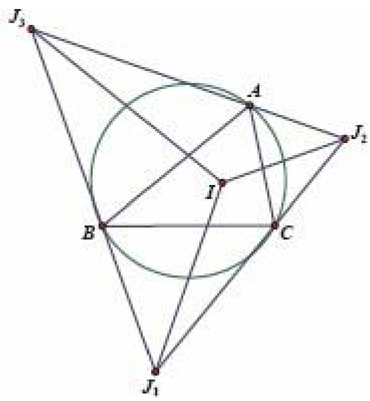


We also have ∠2 = ∠3 since *AA'* // *BC*.

Notice that *A*, *B*, *D*, *E* are concyclic (because *AD*, *BE* are heights) and hence, ∠1 = ∠4. It follows that ∠3 = ∠4 and hence, *A'B'* // *DE*.

3.14 Refer to the left diagram below. Let *AI* extended intersect the circumcircle of Δ*ABC* at *D*. Example 3.4.2 gives *BD* = *CD* = *DI*, which implies that *D* is the circumcenter of Δ*BIC*.



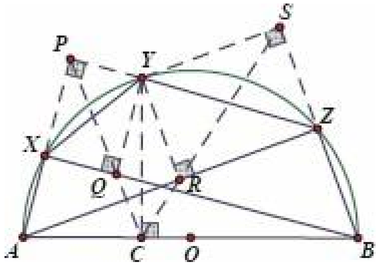


3.15 Refer to the right diagram above. By definition, *A*, *I*, J_1 are collinear. Since $AI \perp J_2J_3$ (Exercise 1.5), we have $AJ_1 \perp J_2J_3$. Similarly, $BJ_2 \perp J_1J_3$ and $CJ_3 \perp J_1J_2$.

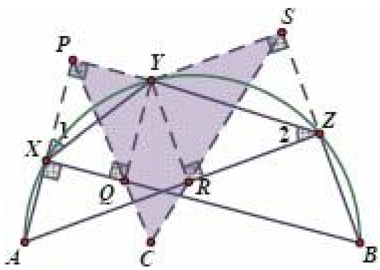
Now *A*, *B*, *C* are the feet of altitudes of $\Delta J_1J_2J_3$ whose orthocenter is *I*. It follows that the midpoints of IJ_1, IJ_2, IJ_3, JJ_1, JJ_2, JJ_3 lie on the nine-point circle of $\Delta J_1J_2J_3$.

3.16 Draw *YC'* ⊥ *AB* at *C'*. Since *P*, *Q*, *C'* are the feet of the perpendiculars from *Y* to the sides of Δ*ABX*, we must have *P*, *Q*, *C'* collinear (Simson's Line). Similarly, *S*, *R*, *C'* are also collinear. It follows that *PQ* and *SR* intersect at *C'*, i.e., *C* and *C'* coincide.

Since $\angle APY = \angle ARY = \angle ACY = 90°$, we have A, P, Y, R are concyclic and A, P, Y, C are concyclic. It follows that A, P, R, Y, C are concyclic. Now $\angle PCS = \angle PAR = \frac{1}{2}\angle XOZ$, which completes the proof.



Note: One may also find the following alternative solution, which does not requires the fact that C lies on AB. Refer to the diagram below. It is easy to see that

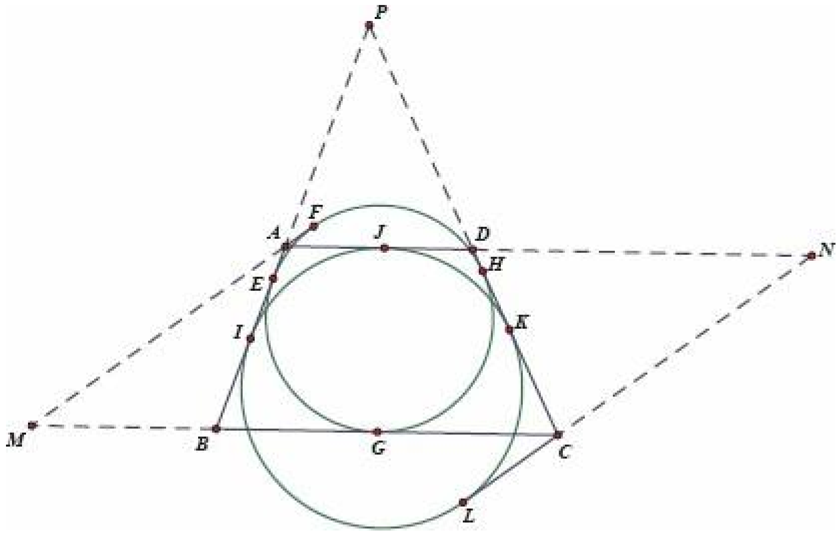


$$\frac{1}{2}\angle XOZ = \angle XAZ = 180° - \angle XYZ.$$

Hence, it suffices to show that $\angle PCS + \angle XYZ = 180°$.

Consider the shaded quadrilateral $PCSY$, where the sum of the interior angles is 360°, i.e.,
$\angle PCS + \angle XYZ + \angle PYX + \angle CPY + \angle SYZ + \angle CSY = 360°$. (*) Since AB is the diameter, we have $\angle AXB = 90° = \angle XPY = \angle XQY$. Hence, $PXQY$ must be a rectangle. Now $\angle PYX = \angle CPY = 90° - \angle 1$. Similarly, $SYRZ$ is also a rectangle and we have $\angle SYZ = \angle CSY = \angle 2$. Now (*) gives $\angle PCS + \angle XYZ + 2 \times (90° - \angle 1 + \angle 2) = 360°$.
This leads to the conclusion $\angle PCS + \angle XYZ = 180°$ as one observes that $\angle 1 = \angle 2$ (Corollary 3.1.5).

3.17 Refer to the diagram below.

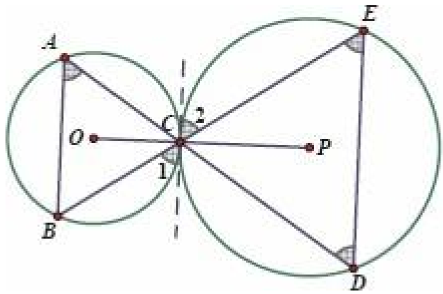


Let the lines *AB*, *CD* intersect at *P*. Let ℓ_1 touch Γ_2 at *F* and intersect the line *BC* at *M*. Let ℓ_2 touch Γ_1 at *L* and intersect the line *AD* at *N*. It suffices to show that *AMCN* is a parallelogram.
We claim that $CM - AM = AN - CN$. (*)
By applying equal tangent segments repeatedly, we have $CM - AM = (CG + MG) - (MF - AF) = CH + AE$, because $MG = MF$.
Similarly, $AN - CN = (AJ + NJ) - (NL - CL) = AI + CK$
Now $(CH + AE) - (AI + CK) = HK - EI = (PK - PH) - (PI - PE) = 0$ since $PI = PK$ and $PE = PH$.
This completes the proof of (*).

Now it is easy to see that *AMCN* is a parallelogram (Exercise 1.13).

Chapter 4

4.1 Draw the common tangent of $\odot O$ and $\odot P$ at *C*. By applying Theorem 3.2.10 repeatedly, we have $\angle A = \angle 1 = \angle 2 = \angle D$. Hence, $AB \,//\, DE$. Since *A*, *B*, *D*, *E* are concyclic, we have $\angle A = \angle E$ (angles in the same arc).
Now $\angle D = \angle E$ and since $AB \,//\, DE$, $\angle B = \angle E = \angle A$. It is easy to see that *ABDE* is an isosceles trapezium and $\Delta ABC \sim \Delta EDC$.



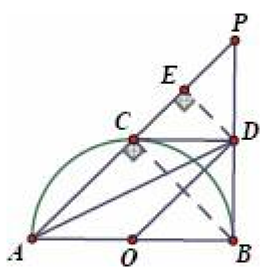
We have $\frac{AC}{OC} = \frac{EC}{PC}$ since they are corresponding line segments. The conclusion follows.

Note: One may also see that $\frac{AC}{OC} = 2\sin\angle B$ and $\frac{EC}{PC} = 2\sin\angle D$ Sine Rule. Since $\angle B = \angle D$, we must have $\frac{AC}{OC} = \frac{EC}{PC}$.

4.2 We are given that *ACDO* is a parallelogram, i.e., $CD \parallel AB$. Since *D* is the midpoint of *BP*, we must have $AC = CP$.

Connect *BC*. Since *AB* is the diameter, we have $BC \perp AP$. Since $AC = CP$, one sees that ΔABP is a right angled isosceles triangle where $AB = BP$ (because $\Delta ABC \cong \Delta PBC$).

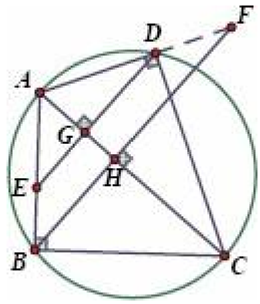
Draw $DE \perp AP$ at *E*. We have $DE \parallel BC$ Since *D* is the midpoint of *BP*, we must have $CE = PE = \frac{1}{4}AP$ by the Intercept Theorem.



Let $OA = 1$. It is easy to see that $AD = \sqrt{AB^2 + BD^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$ and $DE = \frac{1}{2}BC = \frac{\sqrt{2}}{2}$. Now $\sin\angle PAD = \frac{DE}{AD} = \frac{\sqrt{10}}{10}$.

4.3 Let *AC* intersect *DE, BF* at *G, H* respectively. Since $\angle B = 90°$ and *ABCD* is cyclic, we must have $\angle ADC = 90°$.
Since $DE \perp AC$, we have $AB^2 = AH \cdot AC$.

(Example 2.3.1). Similarly, $AD^2 = AG \cdot AC$.



It follows that $\dfrac{AB^2}{AD^2} = \dfrac{AH}{AG}$. (1)

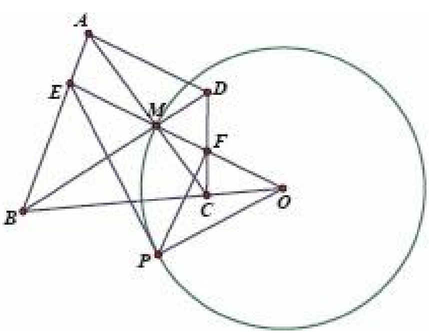
Since *DE* // *EF*, we have $\dfrac{AE}{AB} = \dfrac{AG}{AH} = \dfrac{AD}{AF}$.

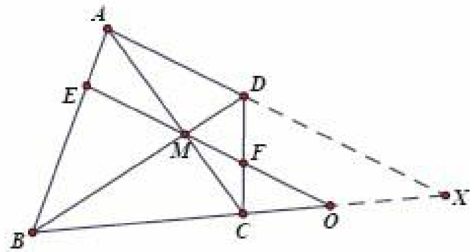
By (1), $\dfrac{AE}{AB} = \dfrac{AD^2}{AB^2} = \dfrac{AD}{AF}$. (2)

Now $\dfrac{AE}{AF} = \dfrac{AE}{AB} \cdot \dfrac{AB}{AD} \cdot \dfrac{AD}{AF} = \left(\dfrac{AD}{AB}\right)^2 \cdot \dfrac{AB}{AD} \cdot \left(\dfrac{AD}{AB}\right)^2 = \dfrac{AD^3}{AB^3}$.

Note: There are many ways to derive the conclusion from (2). For example, one may write $AE = \dfrac{AD^2}{AB}$, $AF = \dfrac{AB^2}{AD}$ and hence obtain $\dfrac{AE}{AF}$.

4.4 Refer to the left diagram below. We claim that $\dfrac{OF}{OP} = \dfrac{OP}{OE}$. Since *OP* = *OM*, it suffices to show $\dfrac{OF}{OM} = \dfrac{OM}{OE}$.





Refer to the right diagram above. Let *AD* extended intersect *BC* extended at

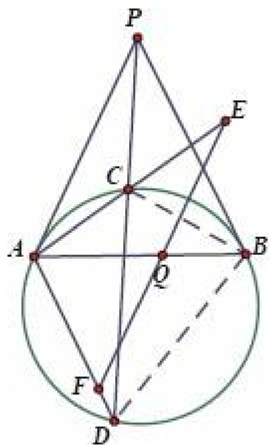
X. We have $\frac{OF}{DX}=\frac{CO}{CX}=\frac{OM}{AX}$, i.e., $\frac{OF}{OM}=\frac{DX}{AX}$.

Similarly, $\frac{OM}{OE}=\frac{DX}{AX}$. We conclude that $\frac{OF}{OM}=\frac{OM}{OE}$, or equivalently, $\frac{OF}{OP}=\frac{OP}{OE}$. It follows that $\Delta OFP \sim \Delta OPE$ and hence, $\angle OPF = \angle OEP$.

4.5 Notice that $\angle ABC = \angle PAE = \angle E$, which implies $\Delta ABC \sim \Delta AEQ$. Hence, $\frac{QE}{AQ}=\frac{BC}{AC}$.

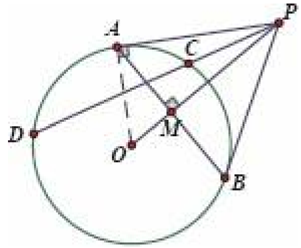
It follows that $QE=\frac{BC\cdot AQ}{AC}$.

Similarly, $\Delta ABD \sim \Delta AFQ$ and $QF=\frac{BD\cdot AQ}{AD}$.

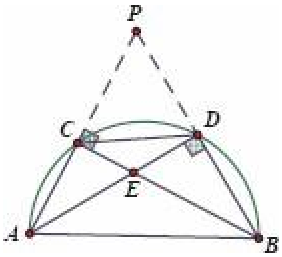


Now it suffices to show that $\frac{BC}{AC}=\frac{BD}{AD}$, but this is by Example 4.1.1.

4.6 Connect OA. In the right angled triangle ΔAOP, $PA^2 = PO\cdot PM$ (Example 2.3.1). We also have $PA^2 = PC\cdot PD$ by the Tangent Secant Theorem. Hence, $PC\cdot PD = PO\cdot PM$ and the conclusion follows.



4.7 Let AC extended and BD extended intersect at P. One sees that AD bisects $\angle BAC$ (Corollary 3.3.3). Since AB is the diameter, we have $AD \perp BP$ and hence, ΔABP is an isosceles triangle where $AB = AP$ (because $\Delta ABD \cong \Delta APD$). Now $BP = 2BD = 4\sqrt{5}$.



It is also easy to see that $\Delta BDE \sim \Delta BCP$ since both are right angled triangles. Hence, we have $BE \cdot BC = BD \cdot BP$ (One may also see this by the Tangent Secant Theorem because C, E, D, P are concyclic.) It follows that $BE \cdot (BE+3) = 2\sqrt{5} \cdot 4\sqrt{5}$, solving which gives $BE = 5$.
Hence, $BC = 8$ and by Pythagoras' Theorem,

$$CP = \sqrt{BP^2 - BC^2} = \sqrt{\left(4\sqrt{5}\right)^2 - 8^2} = 4.$$

Since $PA \cdot PC = PB \cdot PD$ by the Tangent Secant Theorem, we must have $PA \cdot 4 = 2\sqrt{5} \cdot 4\sqrt{5}$. We conclude that $AB = PA = 10$.

4.8 Since $\angle BCF = \angle BAC$ (Theorem 3.2.10), we have $\Delta BCF \sim \Delta CAF$. Hence, $\frac{AF}{CF} = \frac{CF}{BF} = \frac{AC}{BC}$ and we have $\frac{AF}{BF} = \frac{AF}{CF} \cdot \frac{CF}{BF} = \left(\frac{AC}{BC}\right)^2$.

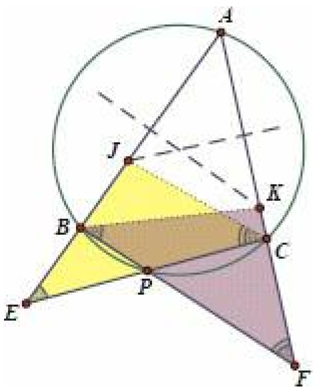
Similarly, $\frac{BD}{CD} = \left(\frac{AB}{AC}\right)^2$ and $\frac{CE}{AE} = \left(\frac{BC}{AB}\right)^2$.

It follows that $\frac{AF}{BF} \cdot \frac{BD}{CD} \cdot \frac{CE}{AE} = \left(\frac{AC}{BC}\right)^2 \cdot \left(\frac{AB}{AC}\right)^2 \cdot \left(\frac{BC}{AB}\right)^2 = 1$ and by Menelaus' Theorem, D, E, F are collinear.

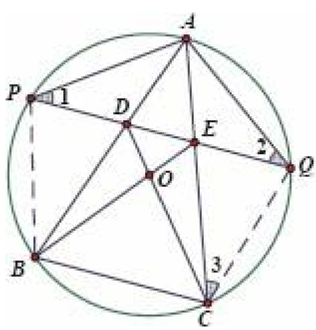
Note: This is an example of Menelaus' Theorem where the line does not intersect the triangle, but the division points are on the extension of the sides instead. In this case, writing down the equation mechanically could be easier than referring to the diagram, especially for beginners.

4.9 Refer to the diagram below. Connect CJ, BK. It is easy to see that $CJ = AJ$ and $BK = AK$. (*)

Notice that $\angle E = \angle ABF - \angle BPE$, while $\angle BPE = \angle A$ (Corollary 3.1.5) = $\angle ABK$ (since $AK = BK$).
Hence, $\angle E = \angle ABF - \angle ABK = \angle FBK$.
Similarly, $\angle F = \angle ECJ$.
It follows that $\Delta CEJ \sim \Delta FBK$.
Now $\frac{CE}{BF} = \frac{JE}{BK} = \frac{CJ}{KF}$.
Hence, $\frac{CE^2}{BF^2} = \frac{JE}{BK} \cdot \frac{CJ}{KF} = \frac{AJ \cdot JE}{AK \cdot KF}$ by (*).



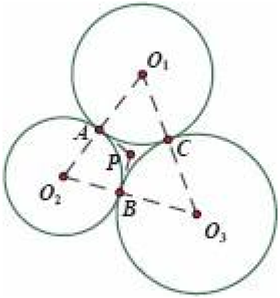
4.10 Connect BP, CP. Since $AP = AQ$, we have $\angle 2 = \angle 1 = \angle 3$ (angles in the same arc). Now $\Delta ADQ \sim \Delta AQC$, which implies $\frac{AQ}{AC} = \frac{AD}{AQ}$, or $AQ^2 = AC \cdot AD$.



Similarly, $AP^2 = AB \cdot AE$. Since $AP = AQ$, we have $AC \cdot AD = AB \cdot AE$.
It follows that B, C, D, E are concyclic and hence, $\angle ABO = \angle ACO$. Notice that $\Delta OBE \sim \Delta OCD$. Since $OB = OC$, we have $\Delta OBE \cong \Delta OCD$ (A.A.S.) and hence, $OD = OE$. Now $\Delta OBC \sim \Delta ODE$ since both are isosceles triangles. Hence, $\angle OBC = \angle ODE$, which implies $DE \,//\, BC$.

Note: Since B, C, D, E concyclic, one sees that ΔABC and ΔADE are isosceles triangles where $AB = AC$ and $AD = AE$.

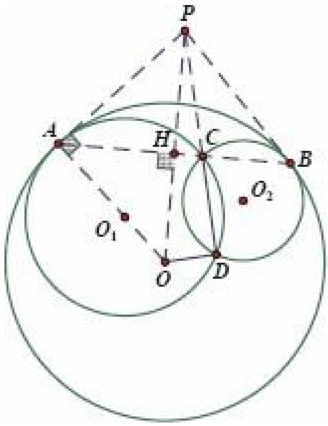
4.11 Let the common tangents passing through A and B intersect at P, i.e., $PA \perp O_1O_2$ and $PB \perp O_2O_3$. Notice that $PA = PB$ (equal tangent segments). Refer to the diagram below. We claim that PC must be a common tangent of $\odot O_1$ and $\odot O_3$.



Suppose otherwise, say the line PC intersects $\odot O_1$ at C and D. By the Tangent Secant Theorem, $PB^2 = PA^2 = PC \cdot PD$. If the line PC touches $\odot O_3$ at C, we have $PB = PC$. This is only possible if PC is a common tangent of $\odot O_1$ and $\odot O_3$, i.e., C, D coincide. If the line PC intersects $\odot O_3$ at C and E, we have $PB^2 = PC \cdot PE$ and hence, D and E coincide. Since $\odot O_1$ and $\odot O_3$ are tangent to each other at C, this implies C, D, E coincide and hence, PC is a common tangent of $\odot O_1$ and $\odot O_3$.

In conclusion, PC is the radical axis of $\odot O_1$ and $\odot O_3$. Hence, ℓ_1, ℓ_2, ℓ_3 are concurrent at P.

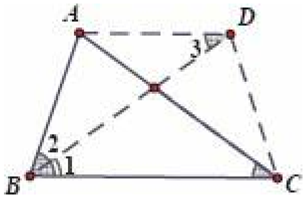
4.12 Draw PA, PB tangent to $\odot O$ at A, B respectively. Notice that AP is the common tangent of $\odot O$ and $\odot O_1$ and hence, the powers of P with respect to $\odot O$ and $\odot O_1$ are the same. Similarly, the power of P with respect to $\odot O$ and $\odot O_2$ are the same. It follows that P lies on the radical axis of $\odot O_1$ and $\odot O_2$ (Theorem 4.3.6), i.e., P lies on the line CD.



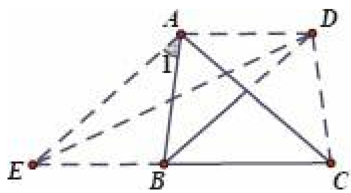
Let OP intersect AB at H. Clearly, OP is the perpendicular bisector of AB. Hence, we have $PA^2 = PH \cdot OP$ (Example 2.3.1).

Since $PA^2 = PC \cdot PD$, we must have $PC \cdot PD = PH \cdot PO$. This implies C, D, O, H are concyclic. It follows that $\angle ODC = \angle OHC = 90°$.

4.13 Let D be the point on the angle bisector of $\angle B$ such that AD // BC. Since $\angle B = 2\angle C$, we have $\angle 1 = \angle 2 = \angle C$.
Since AD // BC, we have $\angle 1 = \angle 3 = \angle C$. It follows that $ABCD$ is an isosceles trapezium which is obviously cyclic.
By Ptolemy's Theorem, $AC \cdot BD = AD \cdot BC + AB \cdot CD$.
Since $AC = BD$ and $CD = AB = AD$ (because $\angle 2 = \angle 3$), we have $AC^2 = AB \cdot BC + AB^2 = AB \cdot (AB + BC)$.



Note: Once we have $AB = AD = CD$, one may also show the conclusion by the area method. Refer to the diagram below. Extend CB to E such that $BE = AB$. It is easy to see that $AE = BD = AC$.



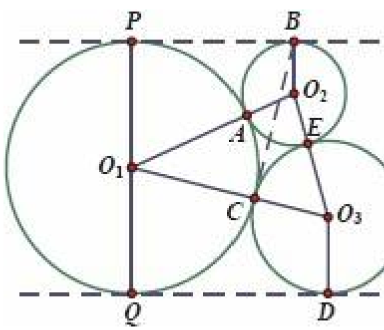
Hence, $[\Delta ACE]=\frac{1}{2}AC\cdot AE\sin\angle CAE=\frac{1}{2}AC^2\sin\angle CAE$ and $[\Delta CDE]=\frac{1}{2}CD\cdot CE\sin\angle BCD=\frac{1}{2}AB(AB+BC)\sin\angle BCD$.

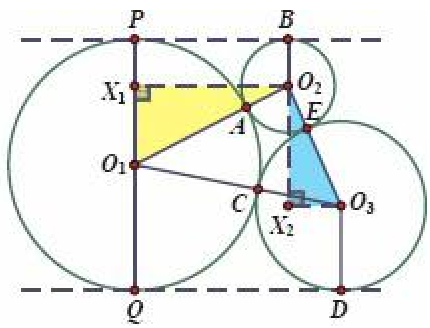
Since $\angle CAE = \angle BAC + \angle 1$ and $\angle 1 = \angle AEB = \angle ACB$ by isosceles triangles, we have $\angle CAE = \angle BAC + \angle ACB = 180^\circ - \angle ABC$. It follows that $\sin\angle CAE = \sin\angle ABC = \sin\angle BCD$.

Since $[\Delta ACE] = [\Delta CDE]$, we must have $AC^2 = AB\cdot(AB+BC)$.

One may notice that applying Ptolemy's Theorem is much faster.

4.14 Refer to the left diagram below. Let $\odot O_1$ touch ℓ_1, ℓ_2 at P, Q respectively. It is easy to see that PQ is a diameter of $\odot O_1$ and $O_2B \,//\, O_3D \,//\, PQ$. Let I be the circumcenter of ΔACE.





Example 4.3.3 states that the circumcircle of ΔACE is the incircle of $\Delta O_1O_2O_3$. In particular, $IC \perp O_1O_3$. We claim that $BC \perp O_1O_3$.

Let the radii of $\odot O_1$, $\odot O_2$ and $\odot O_3$ be r_1, r_2, r_3 respectively. Refer to the right diagram above. Draw $O_2X_1 \perp PQ$ at X_1 and $O_3X_2 \perp BO_2$ at X_2. Pythagoras' Theorem gives us $BP=O_2X_1=\sqrt{O_1O_2^{\,2}-O_1X_1^{\,2}}=\sqrt{(r_1+r_2)^2-(r_1-r_2)^2}=2\sqrt{r_1r_2}$. Similarly, $DQ=2\sqrt{r_1r_3}$.

In the right angled triangle $\Delta O_2O_3X_2$, $O_2O_3^{\,2}=O_2X_2^{\,2}+O_3X_2^{\,2}$. Observe that $O_2O_3=r_2+r_3$, $O_2X_2=PQ-BO_2-DO_3=2r_1-r_2-r_3$ and $O_3X_2=DQ-BP=2\left(\sqrt{r_1r_3}-\sqrt{r_1r_2}\right)$. Hence, $(r_2+r_3)^2=(2r_1-r_2-r_3)^2+4\left(\sqrt{r_1r_3}-\sqrt{r_1r_2}\right)^2$, the simplification of which gives $r_1=2\sqrt{r_2r_3}$. (*)

Now $BO_3^{\,2}-CO_3^{\,2}=BX_2^{\,2}+O_3X_2^{\,2}-r_3^{\,2}$

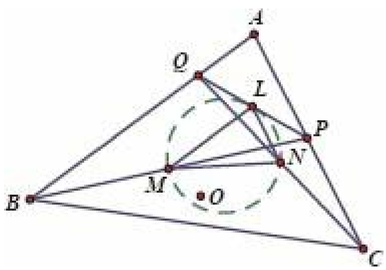
$= (2r_1 - r_3)^2 + 4\left(\sqrt{r_1r_3} - \sqrt{r_1r_2}\right)^2 - r_3^2 = 4r_1^2 + 4r_1r_2 - 8r_1\sqrt{r_2r_3}$

$= 4r_1r_2$ by (*).

This implies $BO_3^2 - CO_3^2 = BP^2 = BO_1^2 - O_1P^2 = BO_1^2 - O_1C^2$.

It follows that $BC \perp O_1O_3$ by Theorem 2.1.9. Since $IC \perp O_1O_3$, B, I, C are collinear. Similarly, A, I, D are collinear. The conclusion follows.

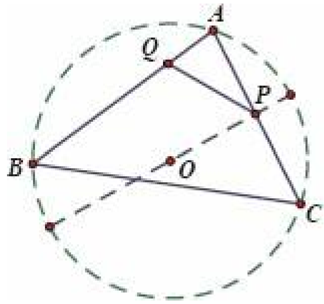
4.15 Since PQ is tangent to the circumcircle of ΔMNL, i.e., PQ touches the circle exactly once, the point of tangency must be L. It is easy to see AB // ML and AC // NL because M, N, L are midpoints.



Clearly, $\angle BAC = \angle MLN$. Notice that $\angle APQ = \angle PLN = \angle LMN$ and similarly, $\angle LMN = \angle AQP$. It follows that $\Delta LMN \sim \Delta APQ$ and hence, $\frac{LM}{LN} = \frac{AP}{AQ}$. Since $PL = QL$, we must have $\frac{LM}{BQ} = \frac{PL}{PQ} = \frac{QL}{PQ} = \frac{LN}{CP}$, i.e., $\frac{LM}{LN} = \frac{BQ}{CP}$.

Now we have $\frac{AP}{AQ} = \frac{BQ}{CP}$, or equivalently, $AP \cdot CP = AQ \cdot BQ$.

Let $\odot O$ denote the circumcircle of ΔABC. Consider the power of point P with respect to $\odot O$. We see that $OP^2 - r^2 = -AP \cdot CP$ where r is the circumradius of ΔABC. (Refer to Definition 4.3.5.) Similarly, we have $OQ^2 - r^2 = -AQ \cdot BQ$.

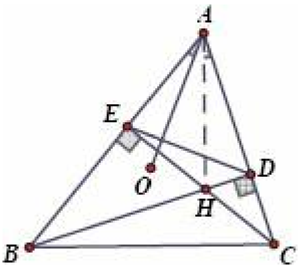


Since $AP \cdot CP = AQ \cdot BQ$, we obtain $OP^2 = OQ^2$, i.e., $OP = OQ$.

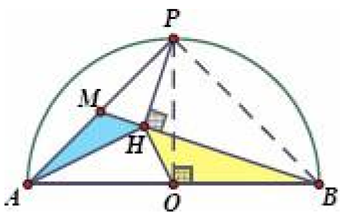
Note: It is easier to write down the expression for the power of a point *without* referring to the diagram. Indeed, those irrelevant lines in the diagram could be very confusing.

Chapter 5

5.1 Let *BD*, *CE* intersect at *H*, the orthocenter of ΔABC. By Example 3.4.1, $\angle BAO = \angle CAH = 90° - \angle C$. It is easy to see that *B*, *C*, *D*, *E* are concyclic. Hence, $\angle C = \angle AED$. It follows that $\angle BAO + \angle AED = 90°$, i.e., $AO \perp DE$.



5.2 Connect *BP, OP*. It is easy to see that ΔPAB is a right angled isosceles triangle where $\angle APB = 90°$ and $\angle PAB = \angle PBA = 45°$.



In the right angled triangle ΔPBM, we have $\begin{cases} PM^2 = MH \cdot BM & (1) \\ PH^2 = MH \cdot BH & (2) \end{cases}$

Since $AM = PM$, (1) gives $AM^2 = MH \cdot BM$, or $\dfrac{AM}{MH} = \dfrac{BM}{AM}$.

It follows that $\Delta AHM \sim \Delta BAM$. Hence, $\angle MHA = \angle MAB = 45°$ and $\angle MAH = \angle MBA$

On the other hand, since $\angle BHP = \angle BOP = 90°$, *B*, *O*, *H*, *P* are concyclic, which implies $\angle BHO = \angle BPO = 45° = \angle MHA$.

Now we have $\Delta AHM \sim \Delta BHO$ and hence, $\dfrac{MH}{AH} = \dfrac{OH}{BH}$, or

$AH \cdot OH = MH \cdot BH = PH^2$ by (2). This completes the proof.

Note: One sees the conclusion is essentially a property of the right angled isosceles triangle ΔPAB where only medians and perpendicular lines are introduced. Hence, one may solve it by brute force, i.e., calculating *PH, AH* and *OH*.

Let $AO = BO = OP = 1$. We have $PA = PB = \sqrt{2}$. Notice that ΔPBM is a right angled triangle whose sides are of the ratio $1:2:\sqrt{5}$ (Pythagoras' Theorem). Hence, $PH = \frac{PM \cdot PB}{BM} = \frac{\sqrt{2}}{2} \cdot \frac{1\times 2}{\sqrt{5}} = \frac{\sqrt{10}}{5}$.

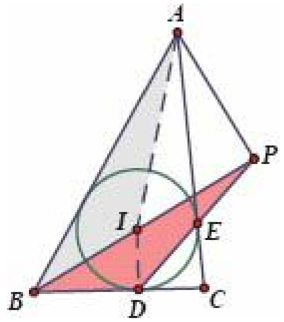
Notice that $\frac{MH}{PH} = \frac{1}{2}$, i.e., $MH = \frac{\sqrt{10}}{10}$. Since $AM = \frac{\sqrt{2}}{2}$, we have $AH^2 = AM^2 + MH^2 - 2AM \cdot MH\cos\angle AMB$ by Cosine Rule, where $\cos\angle AMB = -\cos\angle PMB = -\frac{1}{\sqrt{5}}$.

Hence, $AH^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{10}}{10}\right)^2 + 2\cdot\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{10}}{10} = \frac{4}{5}$.

Notice that *OH* is a median of ΔABH, where $BH = 2PH = \frac{2\sqrt{10}}{5}$.

Hence, $OH^2 = \frac{1}{2}AH^2 + \frac{1}{2}BH^2 - \frac{1}{4}AB^2 = \frac{1}{2}\cdot\frac{4}{5} + \frac{1}{2}\left(\frac{2\sqrt{10}}{5}\right)^2 - \frac{1}{4}\times 2^2$ $= \frac{1}{5}$. It follows that $AH \cdot OH = \frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}} = \frac{2}{5} = PH^2$.

5.3 Recall that $\angle AIB = 90° + \frac{1}{2}\angle C$. Since $CD = CE$, we have $\angle CDE = \frac{1}{2}(180° - \angle C)$ and hence, $\angle BDP = 90° + \frac{1}{2}\angle C = \angle AIB$. Since $\angle ABI = \angle PBC$, we must have $\Delta ABI \sim \Delta PBD$.

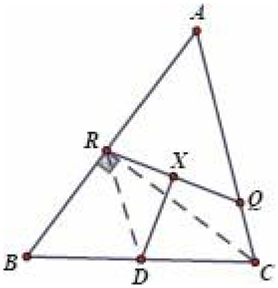


Hence, $\frac{AB}{BP}=\frac{IB}{BD}$ and we conclude that $\Delta ABP \sim \Delta IBD$.

It follows that $\angle APB = \angle IDB = 90°$.

Note: One may also show that $\angle API = \angle AEI = 90°$. In fact, once we obtain $\angle AIP = 90° - \frac{1}{2}\angle C = \angle CED = \angle AEP$, one immediately sees that A, I, E, P are concyclic and hence the conclusion.

5.4 Notice that in the right angled triangle ΔBCR, $BD = CD = DR = \frac{1}{2}BC$.



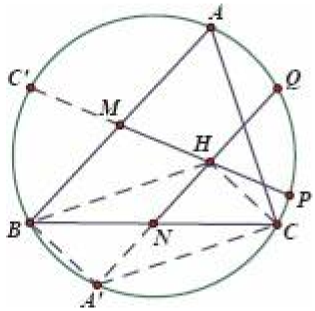
Similarly, $DQ = \frac{1}{2}BC = DR$.

It follows that DX is the perpendicular bisector of QR. Similarly, EY, FZ are the perpendicular bisectors of PR, PQ respectively. Hence, DX, EY, FZ are concurrent at the circumcenter of ΔPQR.

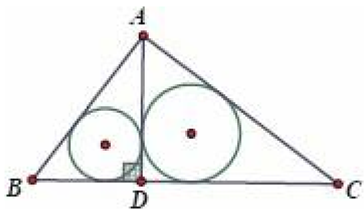
5.5 Let AA' be a diameter of $\odot O$. By Example 3.4.4, $A'BHC$ is a parallelogram and N is the midpoint of $A'H$. Notice that A', H, Q are collinear. Let CC' be a diameter of $\odot O$. Similarly, we have C', H, P collinear and M is the midpoint of $C'H$.

By the Intersecting Chords Theorem, $A'H \cdot HQ = C'H \cdot HP$, which implies $MH \cdot HQ = NH \cdot HP$ since $A'H = 2MH$ and $C'H = 2NH$.

It follows that M, N, P, Q are concyclic.



Note: One may also notice that MN // $A'C'$ (Midpoint Theorem) and hence, A', C', Q, P concyclic implies M, N, Q, P are concyclic (Example 3.1.7).

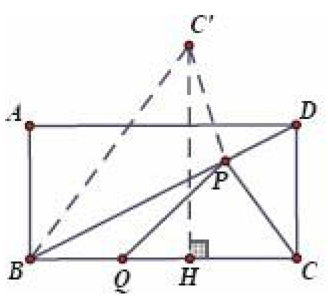


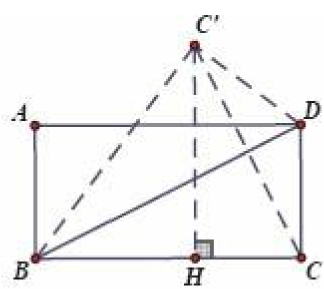
5.6 Notice ΔABC is a right angled triangle. We have $r=\frac{1}{2}(AB+AC-BC)$.

Similarly, $r_1=\frac{1}{2}(AD+BD-AB)$ and $r_2=\frac{1}{2}(AD+CD-AC)$.

It follows that $r+r_1+r_2=AD+\frac{1}{2}(BD+CD-BC)=AD$.

5.7 Let C' be the reflection of C about BD. Draw $C'H \perp BC$ at H. One sees that $CP+PQ=C'P+PQ\geq C'Q\geq C'H$. Refer to the left diagram below.



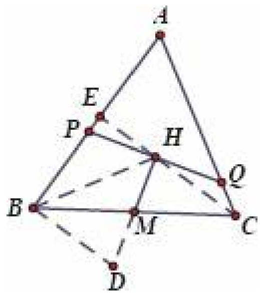


It is easy to see that $CC' \perp BD$ and hence, $\Delta CC'H \sim \Delta DBC$.

We have $\frac{C'H}{CC'}=\frac{BC}{BD}=\frac{2}{\sqrt{5}}$. Refer to the right diagram above.

Notice that $BC \cdot CD = 2[\Delta BCD] = BD \cdot \frac{1}{2}CC'$. Hence, $CC' = 2\frac{BC \cdot CD}{BD}$ $= \frac{4}{\sqrt{5}}$. It follows that $C'H = \frac{2}{\sqrt{5}}CC' = \frac{8}{5}$.

In conclusion, the smallest value of $CP + PQ$ is $\frac{8}{5}$, where *C'Q* ⊥ *BC* at *Q* and *C'Q* intersects *BD* at *P*.



5.8 Extend *HM* to *D* such that *HM* = *DM*. Clearly, *BDCH* is a parallelogram where ∠*DBH* = 180° – ∠*BHC* = ∠*A* because *H* is the orthocenter. (1)
Let *CE* be a height. Notice that ∠*APQ* = 90° – ∠*PHE* = 90° – ∠*CHQ* = ∠*CHM* = ∠*BDH*. (2)

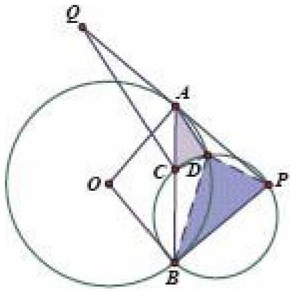
(1) and (2) imply that Δ*APQ* ~ Δ*BDH*. Since ∠*HBM* = 90° – ∠*C* = ∠*CAH*, we conclude that *H* and *M* are corresponding points, i.e., $\frac{PH}{QH} = \frac{DM}{HM} = 1$. This completes the proof.

Note:

(1) One may also see that Δ*APH* ~ Δ*CHM* and Δ*AGH* ~ Δ*BHM*.
Now $\frac{PH}{AH} = \frac{MH}{CM} = \frac{MH}{BM} = \frac{QH}{AH}$ leads to the conclusion.

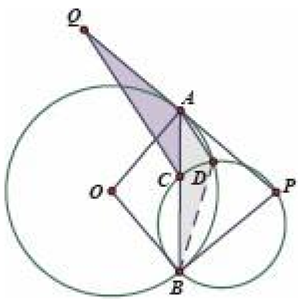
(2) Notice that the diagram of this question is similar to Exercise 5.4. However, the techniques used are entirely different. In fact, this question is more closely related to Example 5.2.6. Can you see that Δ*APQ* and Δ*BCH* are related in a similar way as Δ*ABC* and Δ*EFO* in that example?

5.9 Refer to the diagram below. Since *PB* is tangent to ⊙*O*, we have ∠*PBD* = ∠*BAD*. Since *B, C, D, P* are concyclic, we have ∠*ACD* = ∠*BPD*. It follows that Δ*ACD* ~ Δ*BPD*.



Hence, $\frac{BP}{AC} = \frac{BD}{AD}$.   (1)

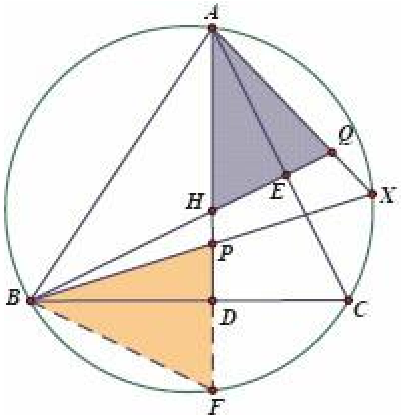
Since *OA* is the perpendicular bisector of *PQ*, we have *AQ* = *AP* = *BP*.

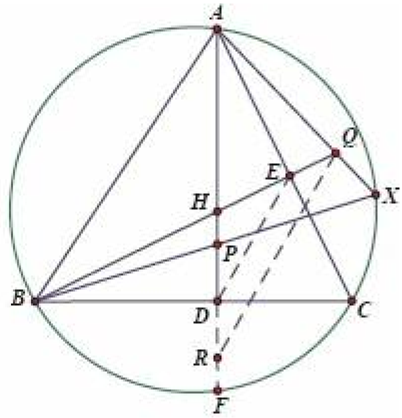


By (1), $\frac{AQ}{AC} = \frac{BD}{AD}$. Refer to the diagram below. Notice that $\angle BAQ = \angle ADB$ (because *AQ* is tangent to ⊙*O*). We conclude that $\Delta ACQ \sim \Delta DAB$. Now $\angle DAB = \angle ACQ$ and we must have *AD* // *CQ*.

5.10 Let *H* be the orthocenter of ΔABC and *AD* extended intersect the circumcircle of ΔABC at *F*. It is easy to see that *DH* = *DF* and $\angle BFH = \angle BHF = \angle AHQ$.

On the other hand, we have $\angle QAH = \angle PBF$ (angles in the same arc). It follows that $\Delta AHQ \sim \Delta BFP$. Refer to the left diagram below. Since $\angle CAF = \angle CBF$, *D* and *E* are corresponding points and $\frac{EQ}{EH} = \frac{PD}{DF}$.   (1)



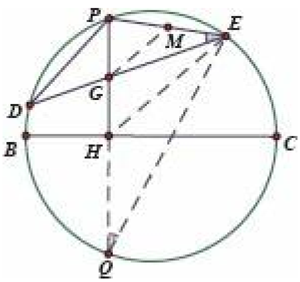


Draw *QR* // *DE*, intersecting *AF* at *R*. Refer to the right diagram above.

Now $\dfrac{EQ}{EH} = \dfrac{DR}{DH}$. (2)

Since *DH* = *DF*, (1) and (2) imply *PD* = *DR* By the Intercept Theorem, *DE* must pass through the midpoint of *PQ*.

Note: This is not an easy problem. Recognizing similar triangles Δ*AHQ*, Δ*BFP* and Δ*BHR* is the key step, even though Δ*BHR* is not drawn explicitly in the proof. Indeed, one may see this problem as an extension of Example 3.4.3.



5.11 Since ∠*A* = 90°, one sees that *BC* is a diameter of Γ. Let *DE* intersect *PH* at *G*. Let *M* be the midpoint of *PE*. Let *PH* extended intersect Γ at *Q*.
We have ∠*PED* = ∠*D* = ∠*Q* (angles in the same arc). It follows that Δ*PGE* ~ Δ*PEQ*.

Clearly, *H* is the midpoint of *PQ*. Since *M* is the midpoint of *PE*, we have Δ*PGM* ~ Δ*PEH*. Now $\dfrac{PG}{PM} = \dfrac{PE}{PH} = 1$, i.e., *PG* = *PM*.

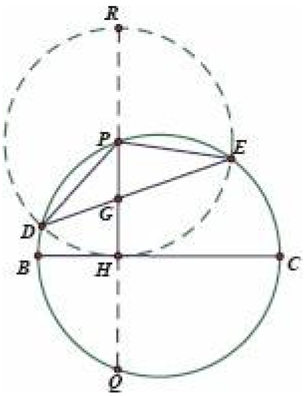
It follows that *MG* // *EH*, and hence, *G* is the midpoint of *PH*.

Note:

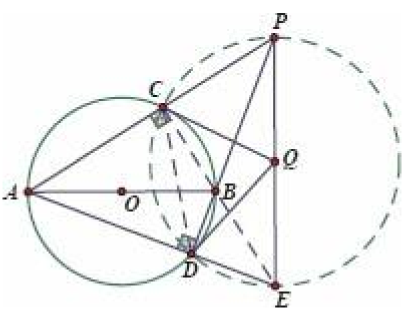
(1) We introduced the midpoint M of PE instead of explicitly drawing a perpendicular from the center of Γ to the chord PE. Nevertheless, the motivation still comes from this technique.

(2) One may also connect MH and see that $MH // EQ$. Since $PE = PH$, we have $\Delta PMH \sim \Delta PEQ$ and $\Delta PMH \cong \Delta PGE$ (A.A.S.). Now $EHGM$ is an isosceles trapezium and the conclusion follows.

(3) There is an alternative solution by the Intersecting Chords Theorem. Refer to the diagram below. Draw the circumcircle of ΔDEH.
Since $PD = PE = PH$, P is the circumcenter of ΔDEH. It is easy to see that $PH = HQ = PR$. Let $PH = r$, $PG = a$ and $GH = r - a$.
By the Intersecting Chords Theorem, $PG \cdot GQ = DG \cdot GE = GH \cdot GR$.
Hence, $a(2r - a) = (r - a)(r + a)$, i.e., $2ra = r^2$.
It follows that $a = \frac{r}{2}$ and G is the midpoint of PH.



5.12 By considering the isosceles triangle ΔCDQ, one sees that $\angle CDQ = \angle DCQ = \angle CAD$ and hence, $\angle CQD = 180° - 2\angle CAD$.



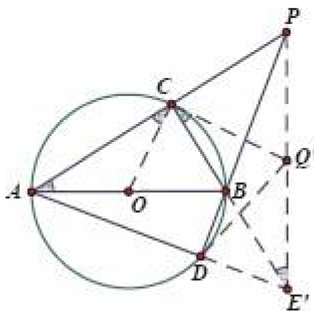
Draw $\odot Q$ with radius CQ. Since AB is a diameter, $PD \perp AD$ and hence, $\angle CPD = 90° - \angle CAD = \frac{1}{2}\angle CQD$. It follows that P lies on $\odot Q$.

Hence, $PQ = CQ$ and we have $\angle CPQ = \angle PCQ$.
Since $AC \perp BC$, we have $\angle PCQ = 90° - \angle BCQ = 90° - \angle BAC$. This implies that $\angle CPQ + \angle BAC = 90°$, i.e., $AB \perp PQ$.
Since $PD \perp AE$, B must be the orthocenter of ΔAEP. Now $BE \perp AC$, which implies B, C, E are collinear.

Note: One sees from the proof that $PQ = CQ = DQ = EQ$, *i.e.,* Q is the midpoint of PE. Indeed, one may find an alternative solution as follows. Suppose the lines AD and BC intersect at E' where Q' is the midpoint of PE'. Connect $Q'C$ and $Q'D$. Refer to the diagram below.

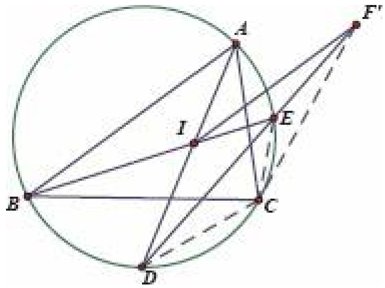
Since $\Delta CPE'$ and $\Delta DE'P$ are right angled triangles sharing a common hypotenuse, we have $Q'C = \frac{1}{2}PE' = Q'D$.

Now it suffices to show that $Q'C$ and $Q'D$ are tangent to $\odot O$, or equivalently, $OC \perp Q'C$.

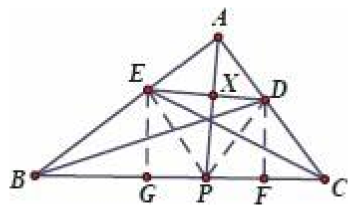


Since $AC \perp BC$, it suffices to show $\angle ACO = \angle BCQ'$. Notice that B is the orthocenter of $\Delta APE'$, i.e., $AB \perp PE'$. We have $\angle BCQ' = \angle CE'Q' = 90° - \angle APE' = \angle CAO = \angle ACO$. This implies $OC \perp Q'C$ and similarly, $OD \perp Q'D$. Hence, Q' coincides with Q and we conclude that E' coincides with E. This completes the proof because E' lies on BC.

5.13 Since I is the incenter of ΔABC, we know that $DC = DI$ and $EC = EI$. Hence, DE is the perpendicular bisector of CI. Let the line DE and ℓ_1 intersect at F'.
It is easy to see that $\Delta EF'I \cong \Delta EF'C$ (S.S.S.) and hence, $\angle ECF' = \angle EIF'$
Since $IF'AB$, we have $\angle EIF' = \angle ABI = \angle CBI$. It follows that $\angle CBI = \angle ECF'$ which implies CF' is tangent to $\odot O$ at C. In conclusion, F and F' coincide. This completes the proof.



5.14 Draw $DF \perp BC$ at F and $EG \perp BC$ at G. Since BD bisects $\angle ABC$ and $\angle BAD = \angle BFD = 90°$, we must have $\Delta ABD \cong \Delta FBD$ (A.A.S.). Hence, $AB = BI$ and $AD = DF$.
Similarly, $AC = CG$ and $AE = EG$. We claim that $FP = GP$.



Let AP and DE intersect at X. By applying Pythagoras' Theorem repeatedly, we have $FP^2 = PD^2 - DF^2 = PD^2 - AD^2$

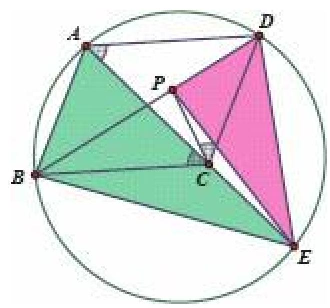
$= (PX^2 + DX^2) - (DX^2 + AX^2) = PX^2 - AX^2$. (1)

Similarly, $GP^2 = PE^2 - EG^2 = PE^2 - AE^2$

$= (PX^2 + EX^2) - (EX^2 + AX^2) = PX^2 - AX^2$. (2)

(1) and (2) imply that $FP = GP$.
Now $AB - AC = BF - CG = (BP + FP) - (CP + GP) = BP - CP$.

5.15 Since $\angle ACD = \angle BCP$, one sees that $\angle PCD = \angle ACB = \angle CAD$ because AD // BC. Since AB // CD, we have $\angle PDC = \angle ABD = \angle AED$ (angles in the same arc). It follows that $\Delta ADE \sim \Delta CPD$ and hence, $\frac{PD}{CD} = \frac{DE}{AE}$.



Since $CD = AB$, we have $\frac{PD}{AB} = \frac{DE}{AE}$.

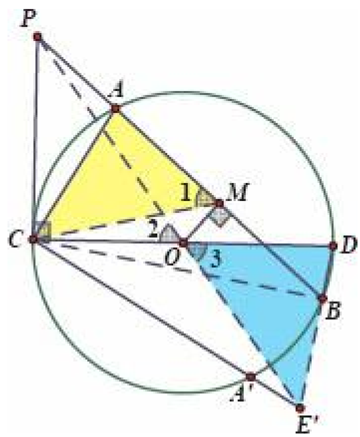
Clearly, $\angle BAE = \angle PDE$ (angles in the same arc).

We conclude that $\Delta ABE \sim \Delta DPE$. It follows that $\angle AEB = \angle DEP$, or equivalently, $\angle AED = \angle BEP$.

Note: We applied the technique of similar triangles sharing a common vertex to show $\Delta ABE \sim \Delta DPE$, where the "common" vertex is not *E*, but *A* and *D*: although these are different points, the corresponding angles at the vertices are the same due to the concyclicity.

5.16 Let *A*' be the point symmetric to *A* about *O*. Let the lines *CA*' and *BD* intersect at *E*'. Since *A*'*A* is a diameter of ⊙*O*, we must have $AC \perp CE'$. Now it suffices to show that *P, O, E*' are collinear.

Connect *BC*. Notice that $\angle ABC = \angle A'CD$ because they correspond to equal arcs (i.e., $\overset{\frown}{AC} = \overset{\frown}{A'D}$ by symmetry).



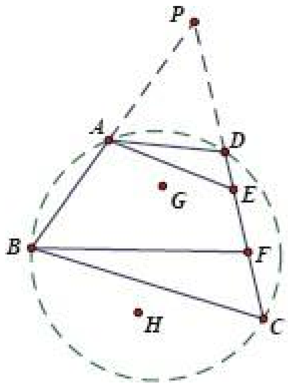
Clearly, $\angle CAB = \angle CDB$. We must have $\Delta ACB \sim \Delta DE'C$. Draw $OM \perp AB$ at *M* We see that *M* is the midpoint of *AB*. Connect *CM, OE*' Since *O* is the midpoint of *CD*, we have $\Delta ACM \sim \Delta DE'O$.

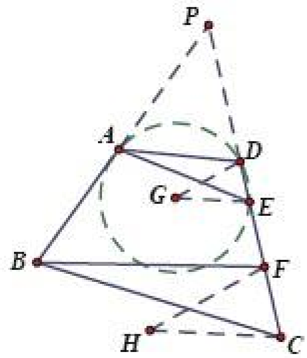
It follows that $\angle 1 = \angle 3$. Connect *OP*. Since $\angle OMP = \angle OCP = 90°$, *P, C, O, M* are concyclic and we have $\angle 1 = \angle 2$. Now $\angle 2 = \angle 3$, which implies *P*, *O*, *E*' are collinear.

Note: One may also re-write the proof in a direct approach: upon drawing $OM \perp AB$ at *M*, we show that $\Delta ACM \sim \Delta DEO$ and hence, $\Delta ACB \sim \Delta DEC$. Now the angles extended by $\overset{\frown}{AC}$ and $\overset{\frown}{A'D}$ on ⊙*O* are the same (where *CE* intersects ⊙*O* at *A*'). Hence, *A* and *A*' are symmetric about *O*. We conclude that *AA*' is a diameter of ⊙*O* and hence, $AC \perp CE$.

5.17 Notice that $\angle DAE = \angle BAD - \angle BAE$ where $\angle BAD = 180° - \angle C$ and

$\angle BAE = \angle BFC$ (because A, B, C, D and A, B, F, E are concyclic).
It follows that $\angle DAE = 180° - \angle C - \angle BFC = \angle CBF$. (1)
Refer to the left diagram below.





Since G is the circumcenter of ΔADE, we have $\angle DGE = 2\angle DAE$.
Similarly, $\angle CHF = 2\angle CBF$. Refer to the right diagram above.
Now (1) implies $\angle DGE = \angle CHF$ and hence, the isosceles triangles ΔDEG and ΔCFH are similar. In particular, $DG \,//\, FH$.

Consider ΔAPE.

Sine Rule gives $\dfrac{PD}{AP} = \dfrac{\sin\angle PAD}{\sin\angle PDA}$ and $\dfrac{DE}{AE} = \dfrac{\sin\angle DAE}{\sin\angle ADE}$.

Since $\sin\angle PDA = \sin\angle ADE$, we have $\dfrac{PD}{DE} = \dfrac{AP\sin\angle PAD}{AE\sin\angle DAE}$.

By Sine Rule, $\dfrac{AP}{AE} = \dfrac{\sin\angle AEP}{\sin\angle P}$. Now $\dfrac{PD}{DE} = \dfrac{\sin\angle AEP\cdot\sin\angle PAD}{\sin\angle P\cdot\sin\angle DAE}$. (2)

A similar argument applies in ΔBPC, which gives

$$\frac{PF}{CF} = \frac{\sin\angle BCP\cdot\sin\angle PBF}{\sin\angle P\cdot\sin\angle CBF}. \quad (3)$$

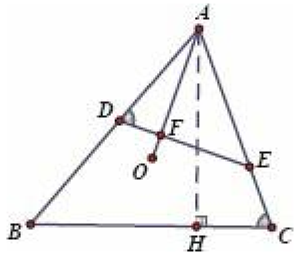
Notice that $\angle AEP = \angle PBF$ and $\angle BCP = \angle PAD$ by concyclicity. We also have $\angle DAE = \angle CBF$ by (1).

Now (2) and (3) implies that $\dfrac{PD}{DE} = \dfrac{PF}{CF}$. (4)

Since $\Delta DEG \sim \Delta CFH$, we have $\dfrac{DG}{DE} = \dfrac{FH}{CF}$. (5)

(4) and (5) give $\dfrac{PD}{DG} = \dfrac{PF}{FH}$ and hence, $\Delta PDG \sim \Delta PFH$.

Now $\angle DPG = \angle FPH$ and it follows that P, G, H are collinear.



Chapter 6

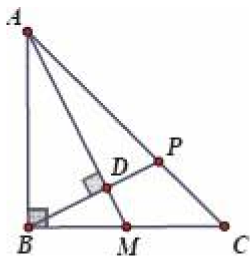
6.1 Draw $AH \perp BC$ at H. Let OA and DE intersect at F. It is well-known (Example 3.4.1) that $\angle OAD = \angle CAH$.

If B, C, E, D are concyclic, we must have $\angle ADE = \angle C$.

Now $\angle ADE + \angle OAD = \angle C + \angle CAH = 90°$, i.e., $DE \perp OA$.

On the other hand, if $DE \perp OA$, we have $\angle ADE = 90° - \angle OAD = 90° - \angle CAH = \angle C$ and hence, B, C, E, D are concyclic.

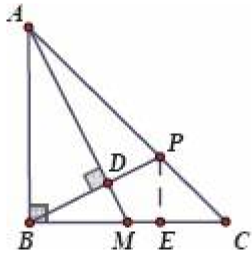
Note: Exercise 5.1 is a special case of this problem.



6.2 Let AM and BP intersect at D. It is easy to see that $\frac{AB}{BM} = 2$ and hence, in the right angled triangle ΔABM, $\frac{AD}{DM} = \left(\frac{AB}{BM}\right)^2 = 4$.

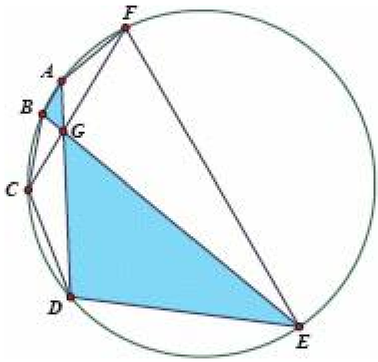
Apply Menelaus' Theorem when ΔACM is intercepted by the line BP. We have $\frac{AD}{DM} \cdot \frac{BM}{BC} \cdot \frac{CP}{AP} = 1$. It follows that $\frac{CP}{AP} = \frac{1}{2}$.

Now $AP = 2\sqrt{2}$ and $AC = 3\sqrt{2}$. Hence, $AB = 3$.



Note: One may also draw $PE \perp BC$ at E. It is easy to see that $\Delta ABM \sim \Delta BEP$ and we have $PE = CE = \frac{1}{2}BE$, i.e., $CE = \frac{1}{3}BC$.

Since $CP = \sqrt{2}$, we have $CE = 1$ and $AB = 3$.



6.3 Refer to the diagram below. It is easy to see that $\Delta ABG \sim \Delta EDG$.

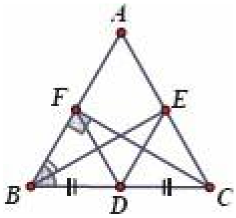
Hence, $\frac{AG}{EG} = \frac{BG}{DG} = \frac{AB}{DE} = \frac{1}{4}$. (1)

Similarly, $\Delta BCG \sim \Delta FEG$, which implies $\frac{BG}{FG} = \frac{CG}{EG} = \frac{BC}{EF} = \frac{2}{5}$. (2)

By (1) and (2), $AG = \frac{1}{4}EG$ and $CG = \frac{2}{5}EG = \frac{8}{5}AG$.

Since $\Delta CDG \sim \Delta AFG$, $\frac{AF}{CD} = \frac{AG}{CG} = \frac{5}{8}$. It follows that $AF = \frac{5}{8}CD = \frac{15}{8}$.

6.4 In the right angled triangle ΔBCD, $DF = \frac{1}{2}BC$ because D is the midpoint of BC. Since $DE = DF$, $DF = \frac{1}{2}BC$, which implies $\angle BEC = 90°$.

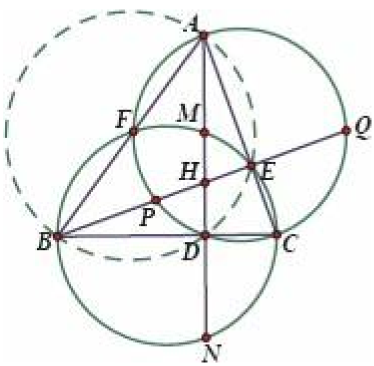


Since BE bisects $\angle ABC$, we have $\Delta ABE \cong \Delta CBE$ (A.A.S.), which implies $AB = BC$ and E is the midpoint of AC.

Hence, in the right angled triangle ΔACF, $EF = \frac{1}{2}AC$.

Since $DE = EF$, we have $AB = BC = 2DE = 2EF = AC$. This completes the proof.

6.5 Refer to the diagram below. Let AD, BE intersect at H, the orthocenter of ΔABC. It is easy to see that A, B, D, E are concyclic.
Apply the Intersecting Chords Theorem repeatedly: $PH \cdot QH = AH \cdot DH = BH \cdot EH = MH \cdot NH$.
It follows that M, P, N, Q are concyclic.

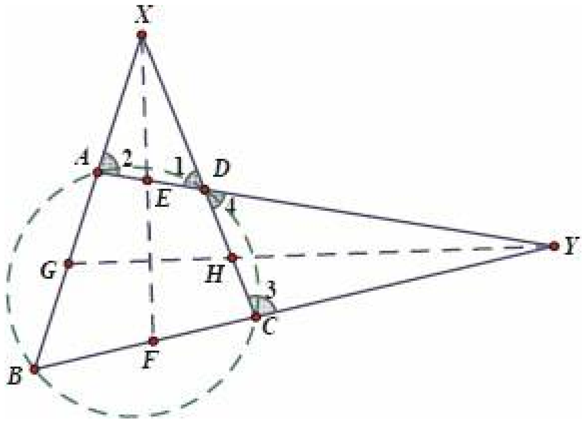


Note:

(1) One may notice that M, P, N, Q lie on a circle centered at C. In fact since BC is the perpendicular bisector of MN, we have $CM = CN$ and similarly, $CP = CQ$. We claim that $CM = CP$.
Since BC is a diameter, $\angle BMC = 90°$ and hence, $CM^2 = CD \cdot BC$(Example 2.3.1). Similarly, we have $CP^2 = CE \cdot AC$. By the Tangent Secant Theorem, $CD \cdot BC = CE \cdot AC$. Hence, $CM = CP$ and M, P, N, Q lie on the circle centered at C with the radius CM.

(2) One may also draw $CF \perp AB$. at F. Since AC, BC are diameters, F lies on the circumcircles of ΔACD and ΔBCE. By the Intersecting Chords Theorem, $PH \cdot QH = CH \cdot FH = MH \cdot NH$ and hence the conclusion.

6.6 Refer to the diagram below. Apply Sine Rule to ΔADX and ΔCDY.



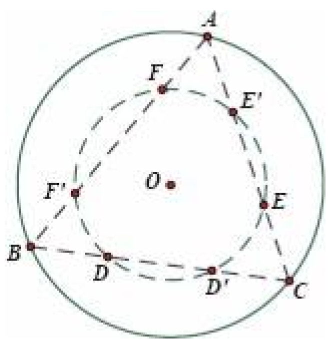
We have $\frac{AX}{DX}=\frac{\sin\angle 1}{\sin\angle 2}$ and $\frac{CY}{DY}=\frac{\sin\angle 4}{\sin\angle 3}$. Notice that $\angle 1=\angle 4$ and $\angle 2+\angle 3=180°$, i.e., $\sin\angle 2=\sin\angle 3$. It follows that $\frac{AX}{DX}=\frac{CY}{DY}$.

We have $\frac{AE}{DE}=\frac{AX}{DX}$ and $\frac{CH}{DH}=\frac{CY}{DY}$ by the Angle Bisector Theorem.

Hence, $\frac{AE}{DE}=\frac{CH}{DH}$, which implies *AC // EH*.

Similarly, *FG // AC*. (**Hint**: $\frac{AG}{BG}=\frac{AY}{BY}=\frac{CX}{BX}=\frac{CF}{BF}$.)

We conclude that *EH // FG*. Similarly *EG // BD // FH* It follows that *EGFH* is a parallelogram.



6.7 Let the circumcircle of Δ*DEF* intersect *BC* at *D, D*', *AC* at *E, E*' and *AB* at *F*, *F*'. Notice that the midpoints of *BC* and *DD*' coincide, i.e., *D* and *D*' are symmetric about the midpoint of *BC*.

Let $\frac{BD}{CD}=\frac{CE}{AE}=\frac{AF}{BF}=k$.

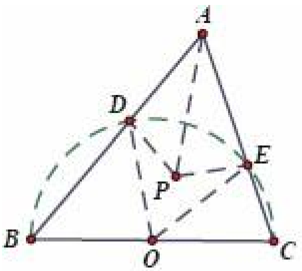
We have $BD = \frac{k}{k+1}BC$ and $BD' = CD = \frac{1}{k+1}BC$.

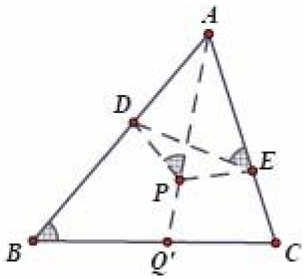
Similarly, $BF = \frac{1}{k+1}AB$ and $BF' = AF = \frac{k}{k+1}AB$.

We have $BD \cdot BD' = \frac{k}{(k+1)^2}BC^2$ and $BF \cdot BF' = \frac{k}{(k+1)^2}AB^2$.

Since $BD \cdot BD' = BF \cdot BF'$ (Tangent Secant Theorem), we must have $AB^2 = BC^2$, i.e., $AB = AC$.
Similarly, $BC = AC$ and the conclusion follows.

6.8 Refer to the left diagram below. Since $OD = OE$ and OP bisects $\angle DOE$, we must have $PD = PE$ (because $\Delta OPD \cong \Delta OPE$).
Clearly, $AD \neq AE$ because ΔABC is non-isosceles. Since AP bisects $\angle A$, we must have A, D, P, E concyclic (Example 3.1.11).
It follows that $\angle AED = \angle APD$.

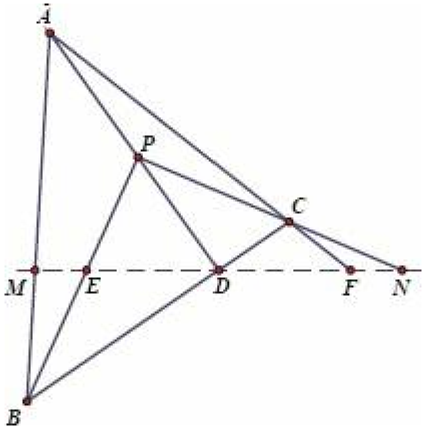




Refer to the right diagram above. Let AP extended intersect BC at Q'. Since B, C, D, E are concyclic, we must have $\angle B = \angle AED = \angle APD$.
Hence, B, D, P, Q' are concyclic.
Similarly, C, E, P, Q' are concyclic. It follows that the circumcircles of ΔBPD and ΔCPE intersect at P and Q', i.e., Q and Q' coincide. This completes the proof.

Note:
(1) It is easy to see that BE, CD are the heights of ΔABC, but this is not important when solving this problem.
(2) Recognizing A, D, P, E concyclic is the key step. This is the conclusion of Example 3.1.11, a commonly used fact.

6.9 Apply Menelaus' Theorem to ΔAMD intersected by BP, ΔAMF intersected by BC and ΔADF intersected by PN:



$$\frac{AB}{MB}\cdot\frac{DP}{AP}\cdot\frac{ME}{DE}=1 \quad (1)$$

$$\frac{MB}{AB}\cdot\frac{FD}{MD}\cdot\frac{AC}{FC}=1 \quad (2)$$

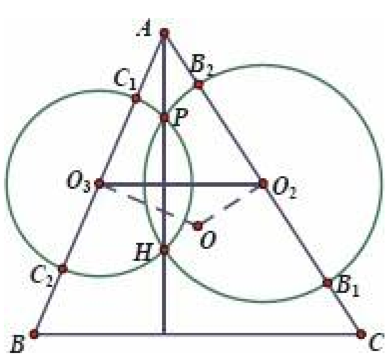
$$\frac{AP}{DP}\cdot\frac{DN}{FN}\cdot\frac{FC}{AC}=1 \quad (3)$$

Multiplying (1), (2), (3) gives $\frac{ME\cdot FD\cdot DN}{DE\cdot MD\cdot FN}=1$.

Since $DE = DF$, we have $\frac{DM}{EM}=\frac{DN}{FN}$, i.e., $\frac{DE}{EM}+1=\frac{DF}{FN}+1$.

It follows that $EM = FN$ and hence, $DM = DN$.

Note: Multiplying (1), (2), (3) is a quick way to cancel out the terms. Of course, one may also manipulate each equation by moving the desired terms (*DE, DF, MD, ME*, etc.) to one side and the rest to the other side. This is a basic technique when applying Menelaus' Theorem.



6.10 Let $\odot O_2$ and $\odot O_3$ intersect at P and H. We have $PH \perp O_2O_3$ and O_2O_3 // BC (Midpoint Theorem).

Hence, $PH \perp BC$, which implies A, P, H are collinear. By the Tangent Secant Theorem, we have

$AC_1 \cdot AC_2 = AP \cdot AH = AB_1 \cdot AB_2$.

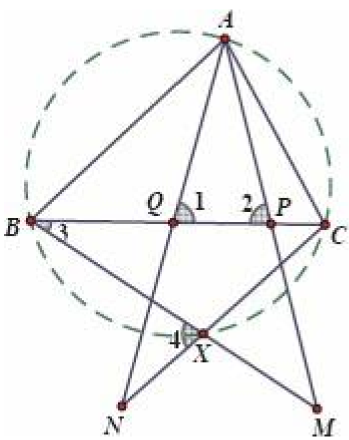
Hence, B_1, B_2, C_1, C_2 are concyclic.

Let the perpendicular bisectors of BB_1, CC_1 intersect at O. Notice that OO_2, OO_3 are also the perpendicular bisectors of AC, AB respectively.

Hence, O is the circumcenter of ΔABC, i.e., B_1, B_2, C_1, C_2 lie on $\odot O$ whose radius is OB_1.

A similar argument gives that A_1, A_2, B_1, B_2 also lie on $\odot O$. It follows that $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic on $\odot O$.

6.11 Let BM and CN intersect at X. Since $\angle C = \angle PAB$, we have $\Delta ABC \sim \Delta PBA$. Similarly, $\Delta ABC \sim \Delta QAC$.



Hence, $\angle 1 = \angle 2 = \angle BAC$ and we also have $\angle BPM = \angle NQC$.

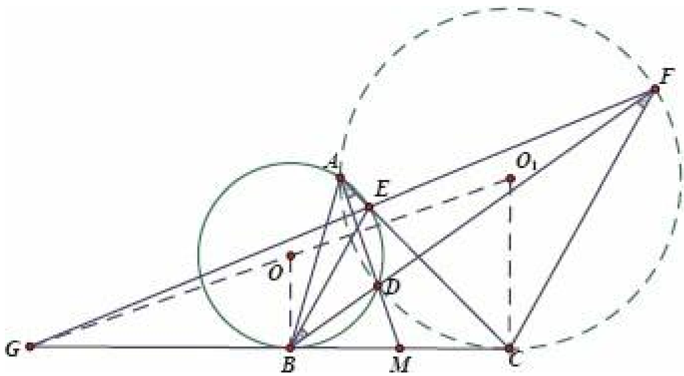
Consider ΔBPM and ΔNQC. Since P is the midpoint of AM, we have $\frac{BP}{PM} = \frac{BP}{AP} = \frac{AB}{AC}$ because $\Delta ABC \sim \Delta PBA$.

Similarly, $\frac{NQ}{CQ} = \frac{AQ}{CQ} = \frac{AB}{AC}$.

It follows that $\frac{BP}{PM} = \frac{NQ}{CQ}$ and hence, $\Delta BPM \sim \Delta NQC$.

Now $\angle 3 = \angle N$ and we must have $\angle 4 = \angle BQN = \angle 1 = \angle BAC$. It follows that A, B, X, C are concyclic.

6.12 Since $BE \,//\, CF$, we have $\angle BFC = \angle EBF = \angle CAD$ (angles in the same arc), which implies A, D, C, F are concyclic, say on $\odot O_1$.
Since M is the midpoint of BC, by the Tangent Secant Theorem, $AM \cdot DM = BM^2 = CM^2$, which implies BC is tangent to $\odot O_1$.
Since $\Delta BEG \sim \Delta CFG$, it follows that O and O_1 are corresponding points.



It follows that $\angle BGO = \angle CGO_1$ because they are corresponding angles in ΔBEG and ΔCFG respectively. This implies G lies on the line OO_1 Since OO_1 is the perpendicular bisector AD, we have $AG = DG$.

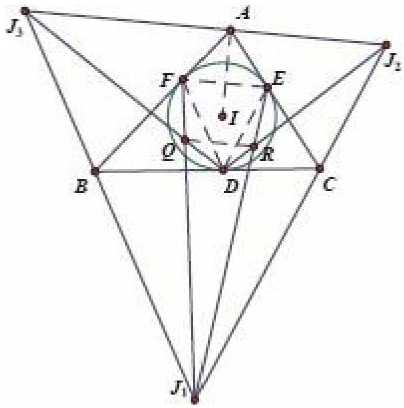
Note: One may also show G, O, O_1 collinear via $\Delta OBE \sim \Delta O_1CF$ and hence, $\frac{OB}{O_1C} = \frac{BE}{CF} = \frac{BG}{CG}$. Now $\Delta OBG \sim \Delta O_1CG$ and $\angle BGO = \angle CGO_1$.

6.13 Recall that $J_2J_3 \,//\, EF$ because both are perpendicular to AI (Exercise 1.5). Similarly, $J_1J_2 \,//\, DE$ and $J_1J_3 \,//\, DF$. It follows that $\Delta DEF \sim \Delta J_1J_2J_3$.

Now $\frac{J_1Q}{FQ} = \frac{DF}{J_1J_3}$ (since $J_1J_3 \,//\, DF$)

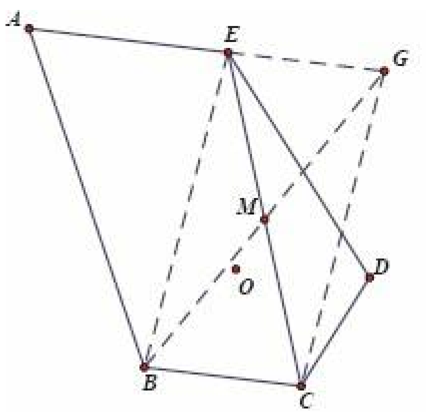
$= \frac{DE}{J_1J_2}$ (since $\Delta DEF \sim \Delta J_1J_2J_3$)

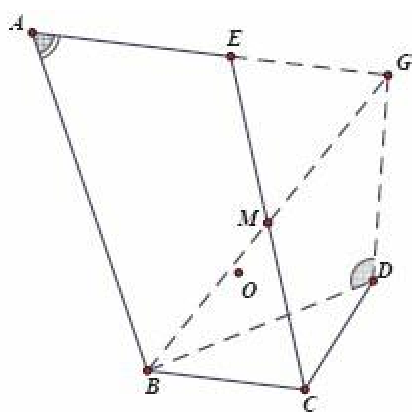
$= \frac{J_1R}{ER}$ (since $DE \,//\, J_1J_2$)



Hence, QR // EF. Notice that AJ_1 is the perpendicular bisector of EF and hence, $J_1E = J_1F$. It follows that AJ_1 is also the perpendicular bisector of QR. Since I lies on AJ_1, we must have $QI = RI$.
Similarly, $PI = QI$ and the conclusion follows.

6.14 Refer to the left diagram below. Extend AE to G such that $BC = EG$. Since $AB = BC + AE$, we have $AB = AG$.



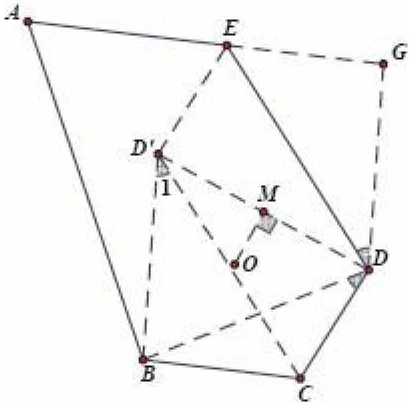


Now $\angle ABG = \angle AGB = \angle CBG$ (because AE // BC), i.e., BG bisects $\angle ABC$. It is also easy to see that $BCGE$ is a parallelogram where M is the center.

We claim that A, B, D, G are coney clic. (1)

Notice that (1) would imply that $\angle ADB = \angle AGB$, which leads to the conclusion because $\angle AGB = \angle CBG = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE$.

Refer to the right diagram above. It suffices to show that $\angle BDG = 180° - \angle A$, where $180° - \angle A = \angle ABC = \angle CDE$. Hence, it suffices to show $\angle BDG = \angle CDE$, or $\angle BDC = \angle EDG$. (2)



Let *D*' be the reflection of *D* about *OM*. Refer to the diagram below. Since *OD* = *OD*', *D*' must lie on ⊙*O* whose radius is *OD*. Notice that ⊙*O* is exactly the circumcircle of Δ*BCD*, i.e., *B, C, D, D*' are concyclic.

Now ∠*BDC* = ∠1. (3)

On the other hand, one sees that *CDED*' is a parallelogram because *DD*' and *CE* bisect each other at *M*

It follows that *CD*' = *DE*' and *CD*' // *DE*. Now it is easy to see that Δ*BCD*' ≅ Δ*GED* (S.A.S.). We conclude that ∠*EDG* = ∠1. (4).

(3) and (4) imply (2), which completes the proof.