Derek Holton

Vol. 1 Mathematical Olympiad Series

A First Step to Mathematical Olympiad Problems

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Vol. 1 A First Step to Mathematical Olympiad Problems *by Derek Holton (University of Otago, New Zealand)*

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To Marilyn, for all her help and encouragement

Foreword

The material in this book was first written for students in New Zealand who were preparing to compete for the six positions in New Zealand's International Mathematical Olympiad (IMO) team. At that stage there was very little mathematical writing available for students who were good at high school mathematics but not yet competent to tackle IMO problems. The aim of the material here then was to give those students sufficient background in areas of mathematics that are commonly the subject of IMO questions so that they were ready for IMO standard work.

This book covers discrete mathematics, number theory and geometry with a final chapter on some IMO problems.

So this book can provide a basis for the initial training of potential IMO students, either with students in a group or for students by themselves. However, I take the approach that solving problems is what mathematics is all about and my second aim is to introduce the reader to what I believe is the essence of mathematics. In many classrooms in many countries, mathematics is presented as a collection of techniques that have to be learnt, often just to be reproduced in examinations. Here I try to present the other, creative, side of the mathematical coin. This is a side that I believe to be far more interesting and exciting. It is also the side that enables students to get some idea of the way that research mathematicians approach their work.

So this book can be used to start students on the trail towards the IMO but its broader aim is to start students on a trail to understanding what mathematics really is and then possibly to taking that understanding and using it in later life, both inside mathematics and outside it.

I would like to thank Irene Goodwin, Leanne Kirk, Lenette Grant, Lee Peng Yee and Zhang Ji for all of their assistance in the preparation of this book.

Contents

Chapter 1
Jugs and Stamps: How To Solve Problems

1.1. Introduction

In this chapter I look at some number problems associated with jugs, consecutive numbers and stamps. I extend and develop these problems in the way that a research mathematician might. At the same time as this is being done, I develop skills of problem solving and introduce some basic mathematical theory, especially about a basic fact of relatively prime numbers.

Whether you are reading this book as a prelude to IMO training or out of interest and curiosity, you should know from the start that mathematics is all about solving problems. Hence the book concentrates on problem solving. Now a problem is only something that, at first sight, you have no idea how to solve. This doesn't mean that a problem is a problem for everyone. Indeed, after you have solved it, it isn't a problem for you any more either. But what I am trying to do here is to both introduce you to some new mathematics and at the same time show you how to tackle a problem that you have no idea at first how to solve.

This book tackles areas of mathematics that are usually not covered in most regular school syllabuses. Sometimes some background is required before getting started but the goal is to show how mathematics is created and how mathematicians solve problems. In the process I hope you, the reader, get a great deal of pleasure out of the work involved in this book.

I have tried to design the material so that it can be worked through by individuals in the privacy of their own brains. But mathematics, like other human pursuits is more fun when engaged in by a group. So let me encourage you to rope in a friend or two to work with you. Friends are also good for talking to about mathematics even if they know nothing about the subject. It's amazing how answers to problems appear when you say your problem out loud.

Now I expect the geniuses amongst you will be able to work through all this book from cover to cover without a break. The mere mortals, however, will most likely read, get stuck somewhere, put the book down (or throw it away) and hopefully go back to it later. Sometimes you'll skim over a difficulty and go back later (maybe much later). But I hope that you will all get some enjoyment out of solving the problems here.

1.2. A Drinking Problem

No problem solving can be done without a problem, so here is the first of many.

Problem 1. *Given a* 3 *litre jug and a* 5 *litre jug can I measure exactly* 7 *litres of water?*

Discussion. You've probably seen this question or one like it before but even if you haven't you can most likely solve it very quickly. Being older and more senile than most of you, bear with me while I slog through it.

I can't see how to get 7. So I'll doodle a while. Hmm. I can make 3, 6 or 9 litres just using the 3 litre jug and 5, 10 or 15 litres with the 5 litre jug. It's obvious, from those calculations that I'm going to have to use both jugs.

Well, it's also pretty clear that $7 \neq 3a + 56$ if I keep *a* and *b* positive or zero. So I can't get 7 by just adding water from the two jugs in some combination. So what if I pour water from one jug into another?

Let's fill up the 3 litre jug, then pour the water into the 5 litre jug. I can then fill up the 3 litre jug and pour into the bigger jug again until it's full. That leaves 1 litre in the 3 litre jug. Now if I drink the 5 litres of water from the larger jug I could pour 1 litre of water into some container.

So it's easy. Repeat the performance seven times and we've got a container with 7 litres of water!

Exercises

1. Drink 35 litres of water.
2. Find a more efficient way of producing 7 litres.

   What does it mean by “more efficient”? Does it mean you'll have to drink less or you'll use less water or what?

1.3. About Solving Problems

Now we've seen a problem and worked out a solution, however rough, let's look at the whole business of problem solving. There is no way that at the first reading I can expect you to grasp all the infinite subtleties of the following discussion. So read it a couple of times and move on. But do come back to it from time to time. Hopefully you'll make more sense of it all as time goes on.

Welcome to the Holton analysis of solving problems.

(a) *First take one problem.* Problem solving differs in only one or two respects to mathematical research. The difference is simply that most problems are precisely stated and there is a definite answer (which is known to someone else at the outset). All the steps in between problem and solution are common to both problem solving and research. The extra skill of a research mathematician is learning to pose problems precisely. Of course he/she has more mathematical techniques to hand too.

(b) *Read and understand.* It is often necessary to read a problem through several times. You will probably initially need to read it through two or three times just to get a feel for what's needed. Almost certainly you will need to remind yourself of some details in mid solution. You will definitely need to read it again at the end to make sure you have answered the problem that was actually posed and not something similar that you invented along the way because you could solve the something similar.

(c) *Important words.* What are the key words in a problem? This is often a difficult question to answer, especially on the first reading. However, here is one useful tip. Change a word or a phrase in the problem. If this changes the problem then the word or phrase is important. Usually numbers are important. In the problem of the last section, “jug” is only partially important. Clearly if “jug” was changed to “vase” everywhere, the problem is essentially not changed. However “3” can't be changed to “7” without affecting the problem.

Now you've come this far restate the problem in your own words.

(d) *Panic!* At this stage it's often totally unclear as to what to do next. So, doodle, try some examples, think “have I seen a problem like this before?”. Don't be afraid to think “I'll never solve this (expletives deleted) problem”. Hopefully you'll get inspiration somewhere. Try another problem. Keep coming back to the one you're stuck on and keep giving it another go. If, after a week, you're still without inspiration, then talk to a friend. Even mothers (who may know nothing about the problem) are marvellous sounding boards. Often the mere act of explaining your difficulties produces an idea or two. However, if you've hit a real toughie, then get in touch with your teacher — that's why they exist. Even then don't ask for a solution. Explain your difficulty and ask for a hint.

(e) *System.* At the doodling stage and later, it's important to bring some system into your work. Tables, charts, graphs, diagrams are all valuable tools. Never throw any of this initial material away. Just as soon as you get rid of it you're bound to want to use it.

Oh, and if you're using a diagram make sure it's a big one. Pokey little diagrams are often worse than no diagram.

And also make sure your diagram covers all possibilities. Sometimes a diagram can lead you to consider only part of a problem.

(f) *Patterns.* Among your doodles, tables and so forth look for patterns. The exploitation of pattern is fundamental to mathematics and is one of its basic powers.

(g) *Guess.* Yes, guess! Don't be afraid to guess at an answer. You'll have to check your guess against the data of the problem or examples you've generated yourself but guesses are the lifeblood of mathematics. OK so mathematicians call their guesses “conjectures”. It may sound more sophisticated but it comes down to the same thing in the long run. Mathematical research stumbles from one conjecture (which may or may not be true) to the next.

(h) *Mathematical technique.* As you get deeper into the problem you'll know that you want to use algebraic, trigonometric or whatever techniques. Use what methods you have to. Don't be surprised though, if someone else solves the same problem using some quite different area of mathematics.

(i) *Explanations.* Now you've solved the problem *write out your solution.* This very act often exposes some case you hadn't considered or even a fundamental flaw. When you're happy with your *written* solution, test it out on a friend. Does your solution cover all their objections? If so, try it on your teacher. If not, rewrite it.

My research experience tells me that, at this point, you'll often find a much nicer, shorter, more elegant solution. Somehow the more you work on a problem the more you see through it. It also is a matter of professional pride to find a neat solution.

(j) *Generalisation.* So you may have solved the original problem but now and then you may only have exposed the tip of the iceberg. There may be a much bigger problem lurking around waiting to be solved. Solving big problems is more satisfying than solving little ones. It's also potentially more useful. Have a crack at some generalisations.

In conclusion though, problem solving is like football or chess or almost anything worthwhile. Most of us start off with more or less talent but to be really good you have to practice, practice, practice.

Exercise

3. Look at the steps (a) to (i) and see which of them we went through in the last section with the 3 and 5 litre jugs.

1.4. Rethinking Drinking

How did you go with your 35 litre glug?

Apart from the drinking, there's the question of the unnecessary energy expended.

$$1=2 \times 3 - 1 \times 5.$$

Looking at this equation we can interpret it as “fill the 3 litre jug two times and throw away one lot of 5 litres”. “fill” because 2 is positive and “throw away” because —1 is negative.

So

$$7 = 14 \times 3 - 7 \times 5.$$

This means we have to fill the 3 litre jug 14 times and throw away 7 lots of 5 litres! Surely there's a more efficient way? Stop and find one — if you haven't done so already.

OK if you do things the opposite way it's more efficient. Take and fill the 5 litre jug and pour the contents, as far as possible, into the 3 litre jug. Left in the 5 litre jug is a measured 2 litres which you can put into your container. Now fill the 5 litre jug again and add the contents to the container. This gives the 7 litres we wanted and means you only have to drink 3 litres of water.

$$7 = 2 \times 5 - 1 \times 3.$$

With satisfaction you start to move off to another problem. But stop. We've started to see what I was talking about in (i) in the last section. Here we've not just been satisfied with finding a solution. We have been looking for a better solution. Have we found the *best* solution? Think.

Remember $7 = 14 \times 3 - 7 \times 5$.

Notice that $14 = 5 + 9$ and $7 = 3 + 4$. So

$$14 \times 3 - 7 \times 5 = (5 + 9) \times 3 - (3 + 4) \times 5 = 9 \times 3 - 4 \times 5.$$

Filling up the 3 litre jug 9 times is an improvement on our first effort but not as good as our filling up the 5 litre jug twice.

$$\begin{aligned} 9 \times 3 - 4 \times 5 &= (5+4) \times 3 - (3+1) \times 5 \\ &= 4 \times 3 - 1 \times 5 \quad \text{(another improvement)} \\ &= (5-1) \times 3 - (3-2) \times 5 \\ &= 2 \times 5 - 1 \times 3 \quad \text{(our best so far)} \\ &= (3+2) \times 5 - (1+5) \times 3 \\ &= 5 \times 5 - 6 \times 3 \quad \text{(now it's getting worse)} \end{aligned}$$

It's becoming clear that we probably do have the best solution but it will take a little work to prove it.

Let's follow up (j) for a minute. Why stop at 7 litres? Can we produce *m* litres in the container for any positive integer m? That's too easy.

What if we had 3 and 7 litre jugs? Can we put *m* litres of water in our container? What about 3 and 8? What about 3 and *s*? What about *r* and s?

Go on thinking. In the meantime here's a little result in number theory that you should know.

Theorem 1. *Let c and d be positive integers which have no common factors. Then there exist integers a and b such that* $ac + bd = 1$.

In our example with the water we had $c = 3$ and $d = 5$ and we found that $a = 2$ and $b = -1$. But, of course, there are lots of other possible values for *a* and b, so given *c* and d, *a* and *b* are not unique.

Exercises

4. (a) In Theorem 1, let $c = 3$ and $d = 7$. Find possible values for a and b. Can you find *all* possible values for *a* and b?
   (b) Repeat (a) with $c = 4$ and $d = 5$.
5. Given *c* and d, where (c, d) = 1 (c and *d* have no common factor), find all possible *a* and *b* which satisfy the equation $ac + bd = 1$.
6. Given a 3 litre jug and a 5 litre jug what is the best possible way to measure 73 litres into a container? (What do you mean by "best"? Minimum water wasted or minimum number of uses of jugs?)
7. What is the best possible way to get 11 litres of water using only a 3 litre and a 7 litre jug?
8. Show that it is possible to measure any integral number of litres using only a 3 litre and a 7 litre jug.
9. Repeat Exercises 7 and 8 using 4 litre and 13 litre jugs.
10. Is it true that given *r* and s litre jugs, *m* litres of water can be measured for any positive integer m? (Assume *r* and s are both integers.) Can a best possible solution be found for this problem?

1.5. Summing It Up

Problem 2. *Is it possible to find a sequence of consecutive whole numbers which add up to* 1000? *If so, is the sequence unique?*

Discussion. So we've landed at step (a) again. We've got ourselves another problem.

Working on to (b), what the question asks is can we find numbers *a*, a+1, *a*+2 and so on, up to say *a*+*k*, so that *a* +(a+1) + (a+2) + - ∠ -+(a+*k)* equals 1000? When we've done that it wants to know if there's more than one set of consecutive numbers whose sum is 1000.

Moving to step (c) we play "hunt the key words". Well, this question has "consecutive numbers", "add" and "1000". Changing any of these changes the problem. In the follow-up question "unique" is important.

So I understand the problem. Help! I see no obvious way of tackling this at the moment. The solution doesn't appear obvious. Hmm…

Let's see what we can do. Clearly 1000 is too large to handle. Let's get some insight into things by trying for 10 instead.

Well I can do it with *one* consecutive number. Clearly 10 adds up to 10! But I doubt that's what the question is all about. In fact, because it says "numbers" I think it really rules out *one* consecutive number. So we'll work on two or more numbers.

Can we get 10 with two consecutive numbers? Can $a + (a + 1) = 10$? That would mean that $2a + 1 = 10$. Hence $2a = 9$, so $a = 9/2$. But *a* was supposed to be a whole number, so it can't be a fraction.

Hang on. One of *a* and *a* + 1 is even while the other is odd. Since the sum of an even and an odd number is odd then we should have known that two consecutive numbers couldn't possibly add up to 10, an even number. (Hmm. Ditto for 1000.)

So what about three numbers? $a + (a + 1) + (a + 2) = 10$ gives $3a + 3 = 10$…No solutions folks.

Four numbers? $a + (a + 1) + (a + 2) + (a + 3) = 4a + 6 = 10$. Ah, $a = 1$. Yes, 1, 2, 3, 4 do add up to 10.

Five numbers? $a + (a + 1) + (a + 2) + (a + 3) + (a + 4) = 10$ gives…Yes, 0, 1, 2, 3, 4 add up to 10.

Six or more numbers clearly won't work. So we see that there are two answers for 10. Will the same thing happen for 1000?

Before you go on you might like to search for "whole numbers" on the web. You'll find that some people accept 0 as a whole number but it doesn't seem to make much sense in this problem. It would be nice not to have both 0+1 + 2 + 3 + 4 and 1 + 2 + 3 + 4. So let's not count 0 among the whole numbers in this book. This also has the virtue of giving a *unique* set of consecutive whole numbers that add up to 10.

Exercise

11. Try Problem 2 with 1000 replaced by (*a*) 20; (b) 30; (c) 40; (d) 100.

Skipping to step (h), I've got the feeling that a little algebra might be useful. We want to find all possible a and k such that

$$a + (a + 1) + (a + 2) + +(a + k) = 1000. \qquad (1)$$

Trial and error is a possibility. We could try $k = 1$ (two consecutive numbers)…Oh no. We know that two consecutive numbers add up to an odd sum.

Sorry, we could try $k = 2$, then $k = 3$, and so on till we've exhausted all possibilities. But,…I know how to add up the left side of equation (1).

$$a + (a + 1) + (a + 2) + \cdots + (a + k) = \frac{1}{2}(2a + k)(k + 1).^{a}$$

So then has to be solved for a and k. Has that really made things any easier?

$$(2a + k)(k + 1) = 2000 \qquad (2)$$

Wait a minute. Since $k + 1$ is a factor of the left-hand side of equation (2), it must be a factor of the right-hand side. So $k + 1 = 1, 2, 4, 5, 8, 10, \ldots$ Yuk! There seem to be an awful lot of cases.

Of course $k + 1$ is the *number* of consecutive numbers. So we know that $k + 1$ isn't 1 or 2. I suppose that cuts things down a bit.

Exercise

12. Use equation (2) to try Problem 2 with 1000 replaced by (*a*) 50; (b) 80; (c) 100; (d) 200.
    See if there are ways of reducing the number of cases we need to try for k + 1.

Well, I'm not really sure that any of that helped. All we've seen is that some numbers have unique consecutive sets and others have more than one.

But there do seem to be two reasons why we can't solve the $2a + k$ equation. Either $2a + k$ is odd and the thing we're equating it to is even or $2a + k$ is too big for the right side of its equation. When do those cases occur for our original problem?

Now if $k + 1$ is even, then both k and $2a + k$ are odd. Does 2000 have any *odd* factors? Apart from 1, only 5, 25 and 125. If $2a + k = 5$, then $k + 1 = 400$. Clearly there's no value for a there. If $2a + k = 25$, then $k + 1 = 80$. Again no solution for a. If 2a

+ k = 125, then k + 1 = 16. Ah! Here a = 55. This means we get 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70.

But what if k + 1 is odd? Then k +1 = 5, 2a + 4 = 400 and we get 198, 199, 200, 201, 202 or k + 1 = 25, $2a$ + 24 = 80 and we get 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52 or k + 1 = 125, $2a$ + 124 = 16 and we don't get anything.

Ah, that's the key! If k + 1 is odd, then $2a$ + k is even, while if k + 1 is even, then $2a$ + k is odd. So we have to find the odd factors of 2000 and we also have to find the even factor where the other factor is odd. Once we've worked out that arithmetic then it's all downhill.

Exercises

13. Collect all the solutions to Problem 2.
14. Generalise. (Have a look at odd numbers. Also see if you can find which numbers are the sum of a unique set of consecutive numbers. Are there any numbers that are not the sum of *any* set of consecutive numbers?)

1.6. Licking a Stamp Problem

Problem 3. *The Otohaihai Post Office is in a predicament. It has oodles of* 3 c *and* 5 c *stamps but it has no other stamps at all. What amounts of postage can the Otohaihai post office sell?*

Discussion. Let's look at this problem in the light of the problem solving steps I suggested in Section 1.3.

Well, yes, (*a*) we have a problem. So go to (b) and understand what the question is asking. Isn't this the 3 litre and 5 litre jug problem in disguise?

Follow this idea up in (c). Does it really change the essential nature of Problem 2 if we exchange pence for litres and stamps for jugs? Is there any mathematical difference between stamps and jugs?

Somehow with jugs of water we could “take away”. With stamps we can only “add on” or “stick on”. If we are trying to get 7 ¢. worth of postage we would need to be able to solve

$$7 = 3a + 5b,$$

where neither a nor b was ever negative.

There is then, an essential, a mathematically essential, difference between jugs and stamps. We could certainly pour out 7 litres. We certainly *can't* stick down 7 worth.

We may well have reached step (d). If so you may like to go kick a ball, turn on the iPod, watch TV or make yourself a snack. When you've gathered strength move right along to (e).

(Incidentally, this avoidance strategy is well known to mathematicians. We all fervently believe that if we con our brains into thinking they're having a rest, then they mysteriously churn out great new thoughts and theorems. Many of us have woken up in the morning with a pr oblem solved.

It was probably the avoidance strategy of coffee drinking, coupled with the conned-brain syndrome, which prompted Erdös — one of the most prolific mathematicians of the 20th Century[b] — to define a mathematician as someone who turns coffee into theorems.)

OK so back to (e). Rather than using a scattergun approach let's be systematic. It's probably useful to draw up a table at this stage.

Table 1.

amount of postage	1	2	3	4	5	6	7	8	9	10
can be made (√) can't be made (x)	x	x	√	x	√					

Copy and complete the table above. Take the amount of postage up to 25 ¢.

Are there any patterns? We're up to (f) now. Well, of course we can get all multiples of 3 and 5 but we can get a lot of other values too. Obviously 8, 13, 18, etc. can be obtained.

Now to (g). From the data you've compiled what guesses can you make about the amounts of postage you can produce with 3 ¢ and 5 ¢ stamps? If you're arithmetic is correct you should have found that the last cross you have is at 7. From 8 onwards *every* number is ticked. (If you didn't find that, then you'd better go back and see where you went wrong.)

Do you agree with the following guess, or conjecture?

Conjecture 1. *Every amount from* 8 *upwards* can *be obtained.*

Of course, if you agree with the Conjecture, then you must justify your faith. If you don't agree with it, then you have to find a number above 8 that can't be made from 3 and 5.

Exercises

15. If you believe in Conjecture 1, then go on to steps (h) and (i). If you think Conjecture 1 is false, then you have to prove it's false and come up with a conjecture of your own. From there you go on to steps (h) and (i) and possibly back to (g) again.
16. Find an equivalent conjecture to Conjecture 1 with
    (a) 3¢ and 7¢ stamps;
    (b) 3¢ and 11¢ stamps;
    (c) 3¢ and 12¢ stamps.
    Generalise.
17. Repeat Exercise 16 with
    (a) 4¢ and 5¢ stamps;
    (b) 4¢ and 11¢ stamps;
    (c) 4¢ and 6¢ stamps.
    Generalise.
18. Repeat Exercise 16 with
    (a) 6¢ and 7¢ stamps;
    (b) 7¢ and 9¢ stamps;
    (c) 9¢ and 33¢ stamps.
    Generalise.

1.7. A Little Explanation

Conjecture 1 is certainly true. How did you prove it?

This is usually the hardest part of problem solving. The reason is not that it is difficult to write out a proof. Sometimes proofs are easy. No, the reason that proof writing is difficult is that it's not the fun part of problem solving. The fun part is solving the problem. Seeing what the right answer is and "knowing" how you could prove it, is somehow psychologically more interesting than writing out a careful answer.

But I say unto you, he that does not write out a proof has not necessarily solvethed the problem. You only really know you're right when you've safely passed into the haven of step (i).

We've procrastinated long enough. Let's get at it. Well we certainly can do 8, 9,10,11,12,…, 23, 24, 25. Can we do 26? You could work this out from scratch but I've just seen a quicker way. Think a minute. You can get 26 from something you've already produced.

Actually you can get 26 from either 21 or 23 by adding 5 or 3, respectively. And that just about solves the 3 and 5 problem. Because surely 27, 28, 29, 30 and all the rest can be got in exactly the same way from earlier amounts.

So in fact we only have to show that we can get 8 c, 9 c and 10 c After that all the rest follow just by adding enough 3stamps. Conjecture 1 must be true then.

Exercise

19. (*a*) Write out a formal proof of Conjecture 1.
    (b) Prove your corresponding conjectures for the amounts in Exercises 16, 17 and 18.

1.8. Tidying Up

Mathematicians like to produce their results with a flourish by calling them *theorems*. These are just statements that can be proved to be true. We'll now present Conjecture 1 as a theorem.

Theorem 2. *All numbers* $n \geq 8$ *can be written in the form 3a + 5b, where a and b are not negative.*

Proof. First note that 8 = 3 + 5, 9 = 3 x 3 + 0x 5 and 10 = 3 x 0 + 2x 5. If $n \geq 8$, then n is either 8 + $3k$, 9 + 3k or 10 + 3k for some value of k. Hence if $n > 8$ then either n = 3(k + 1) + 5, 3(k + 3) or 3k + 2 x 5. □

Well at this stage we are still not satisfied. A good mathematician would ask "is 8 best possible?" By that he would mean "is there a number *smaller* than 8 for which Theorem 2 is true?" In other words, is there some $c < 8$ such that for all $n \geq c$ we can express n in the form 3a + 5b, where a and b are not negative?

But in Table 1 that you completed in Section 1.6, the number 7 should have been given a cross. So clearly there is no number less than 8 which does the job and 8 *is* best possible.

Exercise

20. State and prove the corresponding theorems for your conjectures of Exercise 19(b). In each case show that your results are best possible.

Of course some of you will have realised that we haven't yet completely solved Problem 3, which, after all, asked us to find *all* amounts of postage that can be made with 3 and 5 stamps. We'd better answer that now.

We are going to answer it in the form of a *Corollary.* A corollary is something which follows directly from a result we have just proved. The result below is a simple

corollary of Theorem 2 because we can use Theorem 2 plus Table 1 to prove it.
Corollary 1. *If* $n = 0,3,5,6$ *or any number greater than or equal to* 8, *then* $n = 3a + 5b$, *where a and b are some non-negative numbers.*
Proof. By Theorem 2, the corollary is true for $n \geq 8$. By Table 1, the corollary is true for $n < 8$. □

It may worry some of you that we included 0 in the list of the Corollary. I have Machiavellian reasons for doing that. These will be revealed in due course.
Exercise
21. State and prove corollaries for all the theorems of Exercise 20.

Table 2.

stamps	c	
3¢,	5¢	8¢
3¢,	7¢	12¢
3¢,	11¢	20¢
3¢,	13¢	...
3¢,	14¢	...
3¢,	16¢	...

1.9. Generalise

We've now built up quite a bit of information about 3and *s*ccombinations (*a*mong other things). For instance we know part of Table 2, where c indicates the best possible value in the sense of Theorem 2. In other words, all $n \geq c$ can be obtained and $n = c - 1$ cannot be obtained.
Exercise
22. (*a*) Complete Table 2.
(b) Generalise. In other words, conjecture c if you only have 3 and sc stamps.

By now you will have realised that there are essentially two cases for the 3¢. and *s*¢. problem. In Exercise 16 you will have come across the problem of whether s is divisible by 3 or not. Clearly if s *is* divisible by 3, then you can only ever get amounts which are multiples of 3. Further you can get all multiples of 3. Let's consider what happens when $s = 12$.

Suppose $n = 3a + 12b$, where a and b are not negative. If $a = b = 0$, then $n = 0$. Otherwise $3a + 12b$ is divisible by 3 and so therefore is n. Further if $b = 0$, $n = 3a$. Hence every multiple of 3 can be obtained.

We have thus proved the following lemma.
Lemma 1. *If* $n = 3a + 12b$, *where a and b are not negative, then n must be a multiple of* 3 *and can be* any *multiple of* 3.

The word “lemma” means “little result”. When it grows up it could become a theorem. We usually call results lemmas if they are of no intrinsic value but together with other results they do fit together to help prove a theorem.

Usually theorems are results which are important in themselves, like Pythagoras’ Theorem, for instance.
Exercise 23. Prove the following lemma.

Lemma 2. *Let s be any multiple of* 3. *If n = 3a + sb, where a and b are not negative, then n* must *be a multiple of* 3 *and can be* any *multiple of* 3.

But we've strayed from Table 2. Suppose *s* is not a multiple of 3, what is c? In other words what did you get as your answer to Exercise 22(b)? Can you prove it?

Conjecture 2. $c = 2(s - 1)$.

For $s = 5$ we proved $c = 8$ (Theorem 2) by first showing we could get 8, 9 and 10. After that we just added 3's. The same strategy will work for $s = 7$, 11 and so on (provided *s* is not a multiple of 3). Can we do the same for *s* in general? If we can show that we can get $2s - 2$, $2s - 1$ and $2s$ using 3's and s's, then we can add on enough 3's and we can get any n.

Well one of this triumvirate of numbers is easy. Surely you don't want me to prove that I can get 2s! So how do you get $2s - 1$ and $2s - 2$?

Think about *s* for a minute. When you divide *s* by 3 you either get a remainder of 1 or a remainder of 2. This means that you can write *s* either as $3t + 1$ or as $3t + 2$ where $t > 0$. Let's have a look at the case $s = 3t + 1$. Now

$$2s - 2 = 6t + 2 - 2 = 6t = 3(2t).$$

Certainly then we can get $2s - 2$ in this case because $2s - 2$ is just a multiple of 3. So what about $2s - 1$? After a bit of thought I'm sure you would have realised that

$$2s - 1 = s + (s - 1) = s + [(3t + 1) - 1] = s + 3t.$$

We can surely get $s + 3t$ using just s's and 3's.

This only leaves the case $s = 3t + 2$.

Exercises

24. Prove the following theorem.

Theorem 3. *Let s be any number not divisible by* 3. *All numbers* $n \geq 2(s - 1)$ *can be written in the form n = 3a + sb, where a and b are not negative.*

25. Is $2s - 2$ best possible in Theorem 3?
26. Put Exercises 24 and 25 together along with the situation where s is a multiple of 3 to form a Theorem 4. Prove the theorem.
27. Repeat Exercise 26 for
    (a) 2¢ and *s*¢ stamps;
    (b) 5¢ and *s*¢ stamps;
    (c) 4¢ and *s*¢ stamps;
    (d) 6¢ and *s*¢ stamps.
28. All numbers $n > c$ can be written in the form $n = ra + sb$, where *r* and s have no factors in common and *a* and *b* are not negative.

What is the best possible value for *c* in terms of *r* and s? Prove it.

While we've been concentrating on the upper end of things, the "all $n > c$" part, something interesting has slipped past us at the lower end. Have a look at Table 3.

Table 3.

r	s	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	5	√	x	x	√	x	√	√	x										
3	7	√	x	x	√	x	x	√	√	x	√	√	x						
3	8	√	x	x	√	x	x	√	x	√	√	x	√	√	x				
3	10	√	x	x	√	x	x	√	x	x	√	√	x	√	√	x	√	√	x

Is there any pattern in all this? Are we able to say anything about those $n < c$ for which $n = 3a + sb$?

Exercises

29. (*a*) Conjecture some pattern in Table 3.

(b) Extend Table 3 by considering s = 11, 13, 14.

(c) Go back to (*a*). If your original conjecture looks good prove it. If your original conjecture turned out to be wrong, try another guess.

30. Repeat Exercise 29 with *r* =4.

Does the same conjecture hold for *r* =4 as for *r* =3? Try other values of *r.*

1.10. In Conclusion

I thought it might be useful to give a complete proof of the stamp problem. So here it is. I expect that many of you will find this extremely tough. Have a look at it and then forget it, but come back in a year's time and have another go.

You might like to try a “complete proof” of the consecutive number problem for yourself. It's not quite as tough as the stamp problem.

Theorem A. *Let r, s be positive integers with* (r, s) = 1. *Then there exist non-negative integers a, b such that ar* + *bs* = *c for all c* > *(r* — 1)(s — 1).

Proof. From Theorem 1 we know that *a, (* exist such that *ar* + *(3s* = 1. Further, for any integer n, r(*a* — *ns)* + *s((* + *nr)* = 1. Hence we can choose *n* so that either one of the brackets is positive and the other negative. So we may assume that *α* is positive and *β* is negative.

Assume *c* ≥ *(r* — 1*)(s* — 1). Returning to *αr* + *βs* = 1 it is clear that *acr* + *(cs* = *c* and, for any integer n, r(ca — *ns)* + *s((c* + *nr)* = c. We now choose *n* = *n'* so that *(c* + *n'r* is the smallest positive (or zero) value to satisfy this last equation. Clearly 0 ≤ *(c* + *n'r* ≤ *r* — 1 because if *(c* + *n'r* ≥ *r* then *(c* + *(n'* — 1)r is positive (or zero) and is less than *(c* + *n'r* which was assumed smallest.

Hence r(ca — n's) = *c* — *s((3c* + *n'r)* ≥ *c* — *s(r* — 1) > (r — 1)(s — 1) — *s(r* — 1). In other words, r(ca — *n's)* > —*(r* — 1). So *ca* — *n's* > 1/r — 1. But since *ca* — *n's* is an integer, this last inequality proves that *ca* — *n's* > 0.

We thus take *a* = *ca* — *n's* and *b* = *(c* + *n'r* and the conclusion of the theorem follows. ?

Remark B. In Theorem A, (r — 1)(s — 1) is best possible.

Suppose there exist *a*, *b* > 0 such that *ar* + *bs* = (r — 1)(s — 1) — 1. Then *ar* + *bs* = *rs* — *r* — *s*. This implies (b + 1)s = *r(s* — *a* — 1).

But (r, s) = 1. Hence *r* is a factor of *b* +1 and *s* is a factor of *s* — *a* — 1. Since 0 ≤ *s* — a —1 < s, then *s* — *a*— 1 = 0. However this means that b+1 = 0, so *b* = —1. But we assumed that *b* ≥ 0. So we have a contradiction.

Theorem C. *Let r and s be positive integers with* (r, s) = 1. *If x and y are both non-negative integers less than (r* — 1)(s — 1) *whose sum is (r* — 1) (s — 1) — 1, *then precisely one of x and y is expressible in the form ar* + *bs where a and b are both non-negative.*

Proof. From Theorem 1 and the argument in Theorem A we can find x1, x2, y1, y2 such that *x* = xir + X2*s* and *y* = yir + *ys* where 0 ≤ x2, y2 ≤ *r* — 1.

Now

$$(r-1)(s-1)-1 = x+y$$
$$= (x_1+y_1)r + (x_2+y_2)s.$$

Hence

$$rs - s - r = (x_1+y_1)r + (x_2+y_2)s.$$

So

$$(x_1+y_1+1)r = s(r-1-x_2-y_2).$$

Since (r, s) = 1, r must divide $r - 1 - x_2 - y_2$. But $r - 1 - x_2 - y_2 \geq 1 - r$ since $r - 1 > x_2, y_2 \geq 0$. Hence $r - 1 - x_2 - y_2 = 0$. This means $x_i + y_i + 1 = 0$. Hence one of x_i and y_i is not negative. So one of x and y is expressible in the form $ar + bs$ where a and b are not both negative.

Suppose x and y are *both* expressible in this form. Then $x = x_i r + x_2 s$ and $y = y_i r + y_2 s$ where $x_i, x_2, y_i, y_2 \geq 0$. This implies that

$$(r-1)(s-1)-1 = x+y = ar+bs$$

with $a = x_i + y_i \geq 0$ and $b = x_2 + y_2 \geq 0$. But from the Remark B above we know that $(r - 1)(s - 1) - 1$ is never expressible in the form $ar + bs$ with $a, b \geq 0$. Hence not both x and y are expressible in this form. □

Corollary 2. *Of the integers between* 0 *and* $(r - 1)(s - 1) - 1$ *inclusive, half are expressible in the form* $ar + bs$ *with* $a, b \geq 0$ *and half are not.*

Proof. This is an immediate consequence of Theorem C. □

1.11. Epilogue

This may have been your first foray into problem solving. If you worked hard and have not looked at the solutions till you've had answers, then it will also probably have been your first foray into mathematics.

The way we've meandered through this chapter is very roughly the way a mathematician would tackle a research problem. As I said in (*a*) in Section 1.3 the only difference between problem solving and research is that someone knows before you start the precise question to ask and also knows the answer.

Actually though, if we had tried the stamp problem just over a hundred years ago we would really have been doing research. One of the main theorems of this chapter was proved by the mathematician Sylvester in the 19th Century.

1.12. Solutions

DON'T EVEN DARE PEEK AT THE SOLUTIONS TO AN EXERCISE UNTIL YOU'VE GENUINELY TRIED TO SOLVE THE EXERCISE.

1. Glug, glug.
2. Fill up the 5 litre jug and pour 3 litres into the 3 litre jug. Drink this 3 litres and then transfer the remaining 2 litres to the 3 litre jug. Fill the 5 litre jug. In the two jugs you now have 7 litres.

   Is this more efficient? Why?
3. All except (j). But that pleasure is to come.

4. (*a*) $a = -2$, $b = 1$; $a = 5$, $b = -2$, etc.

In general $a = 7s — 2$ and $b = 1 — 3s$ for every integer s. Can you prove this?

(b) $a = —1$, $b = 1$; $a = 24$, $b = —19$, etc.

In general $a = 5s — 1$ and $b = 1 — 4s$ for every integer s. Can you prove this?

5. Let a and (3 be such that ac + (3d =1. Then $a = ds + a$ and b = (3 — cs. Now try to prove it.

6. $73 = (5 + 3a) \times 5 + (16 — 5a) \times 3$.

So we can measure out 73 litres if we use the 5 litre jug (5 +3a) times and the 3 litre jug (16 — 5a) times. (Here “use” = “fill” if the number is positive and “empty” if it's negative.)

(a) First let's minimise water wastage. This will be done if no water is wasted which requires $5 + 3a \geq 0$ and $16 — 5a \geq 0$.

Hence we need $a \geq — 1$ and $a \leq 3$. We then have the following table.

a	−1	0	1	2	3
uses of 3 litre jug	21	16	11	6	1
uses of 5 litre jug	2	5	8	11	14

So if we fill the 3 litre jug (16 — 5a) times and the 5 litre jug (5 + 3a) times for a = —1, 0,1, 2, 3 and dump the contents in the container, we will produce 73 litres of water.

Here then there are 5 best possible ways because we waste no water with any of them.

(b) Minimising jug use is a little harder. There are three cases to consider.

(1) 5 + 3a and 16 — 5a are both positive;

(2) $5 + 3a > 0$ and $16 — 5a < 0$;

(3) $5 + 3a < 0$ and $16 — 5a > 0$.

(Clearly $5 + 3a$ and $16 — 5a$ cannot both be negative since 73 is positive. Also note that neither 5 + 3a nor 16 — 5a can be zero since 73 is not divisible by 3 or 5.)

Case 1. $5 + 3a > 0, 16 — 5a > 0$.

We have seen that this means a = —1, 0, 1, 2, 3. For minimum jug usage $a = 3$. (Here the jugs are used 15 times.)

Case 2. $5 + 3a > 0$, $16 — 5a < 0$.

Hence $a \geq — 1$ *and* $a \geq 4$. So $a \geq 4$. We use the 5 litre jug at least 17 times and the 3 litre jug at least 4 times (emptying it). So here we need to use the jugs 21 times at least.

Case 3. $5 + 3a < 0$, $16 — 5a > 0$.

Hence $a \leq —2$ *and* $a \leq 3$. So $a \leq —2$. We use the 5 litre jug at least once (emptying it) and the 3 litre jug at least 26 times. This means at least 27 handlings of jugs.

By considering all three cases we see that 15 is our best answer.

$$73 = 14 \times 5 + 3.$$

7. $11 = (6 + 7a) \times 3 + (— 1 — 3a) \times 7$.

Minimum waste: Since not both 6 + 7a and —1 — 3a can be positive there must be wastage. It occurs when $a = —1$.

Minimum use: Treating the two cases (6 + 7a positive, —1 — 3a negative; 6 + 7a negative, —1 — 3a positive) gives $a = —1$ again.

8. $1 = 7 - 2 \times 3$,

$m = m \times 7 - 2m \times 3$.

We can then fill the 7 litre jug *m* times and from this water fill and discard the jug 2m times. The residue is *m* litres.

9. (*a*) $11 = (6 + 13a) \times 4 + (-1 - 4a) \times 13$.

The best solution seems to come when $a = 0$.

(b) $1 = 13 - 3 \times 4$,

$\therefore\ m = m \times 13 - 3m \times 4$.

10. If you think the answer is yes, try r = 2, s = 4 and m = 7.

However, let (r, s) = t. (This means that *t* is the highest factor common to both *r* and *s*.)

Result 1. *Let* (r, s) = 1. *Then m litres can be obtained from r and s.*

Proof. Since (r, s) = 1, then by Theorem 1, there exist integers *a* and *b* such that $ar + bs = 1$. Hence $m = mar + mbs$. ?

Result 2. *Let* (r, s) = *t. Then there exist integers a and b such that* $t = ar + bs$.

Proof. If (r, s) = *t* then there exist *r'* and *s'* such that $tr' = r$ and $ts' = s$. Now since (r, s) = *t* it follows that $(r', s') = 1$. Hence by Theorem 1 there exist integers *a, b* such that

$$ar' + bs' = 1,$$
$$\therefore atr' + bts' = t,$$
$$\therefore ar + bs = t.$$

□

Result 3. *Let* (r, s) = *t. Then m litres can be obtained from jugs of size r and s litres if and only if m is a multiple of t.*

Proof. Let $m = m't$. By Result 2

$$t = ar + bs,$$
$$m = m't = (m'a)r + (m'b)s.$$

Suppose *m* does not have a factor of *t* and $m = cr+ds$. Since (r, s) = *t* then *t* divides $(cr + ds)$. Hence *t* divides m. But this contradicts the assumption that *m* does not have a factor of t. So $m = cr + ds$. ?

As for a best possible solution, let's concentrate on minimum wastage. Now suppose that $m = ar + bs$. Hence $m = (a + ns)r + (b - nr)s$.

If $(a + ns)$ and $(b - nr)$ can both be positive we waste no water. Otherwise we keep adding (if *a* is negative) or subtracting (if *a* is positive) multiples of *s* until $0 > a + ns > -s$. So choose $a = a + ns$ where $0 > a > -s$.

Similarly we can operate on $b - nr$ and choose (3 such that 0 > (3 = b — n'r > —r.

Hence m = (a + fs)r + ((+ gr)s for some *f* and *g*. The minimum wastage of water is then the minimum of *ar* or (3s.

11. (*a*) 2, 3, 4, 5, 6 (unique);

(b) 9, 10, 11 and 6, 7, 8, 9 and 4, 5, 6, 7, 8 (not unique);

(c) 6, 7, 8, 9, 10 (unique);

(d) 9, 10, 11, 12, 13, 14, 15, 16 and 18, 19, 20, 21, 22.

12. A table of values might help.

(a)

$k+1$	4	5	10	20	25	50	100
k	3	4	9	19	24	49	99
$2a+k$	$2a+3$ $=25$	$2a+4$ $=20$	$2a+9$ $=10$	$2a+19$ $=5$	$2a+24$ $=4$	$2a+49$ $=2$	$2a+99$ $=1$
a	11	8	$1/2$	-7	-10	$-17/2$	-49

So 11, 12, 13, 14 and 8, 9, 10, 11, 12 are the only ones possible.

(b)

$k+1$	4	5	8	10	16	...
k	3	4	7	9	15	
$2a+k$	$2a+3$ $=40$	$2a+4$ $=32$	$2a+7$ $=20$	$2a+9$ $=16$	$2a+15$ $=10$	
a	$37/2$	14	$13/2$	$7/2$	$-5/2$	

So we have the unique 14, 15, 16, 17, 18.

(c) See 11(d).

(d) 38, 39, 40, 41, 42.

13. {198,199,…, 202}; {55, 56, 57,…, 70}; {28, 29, 30,…, 52}.
14. "Ay, there's the rub!" (Hamlet, Act III, Scene I).
15. I believe in Conjecture 1. (You will too by the time you've slaved your way through this chapter.)
16. Every amount onward from (*a*) 12? (b) 20?

For (c) I've pulled a fast one. Clearly you can only get multiples of 3. For the generalisation we'll talk about 3 and *s* c stamps. Well, if *s* isn't a multiple of 3, then pretty clearly…

17. Every amount onwards for (*a*) 12? (b) 30?

For (c) we once again see that 4 and 6 have a common factor of 2. This probably means you can get every even number from 4 upwards.

For the generalisation we'll talk about and *s*/ stamps. Well, if (4, s) = 1 (4 and *s* have no factors in common), then pretty clearly…

18. (*a*) 30? (b) 48? (c) It's that old problem again (9, 33) = 3. Probably we can only get multiples of 3 from 60 on. Did I say 60? Why?
19. (*a*) I'll do my bit in a minute. What does your proof look like? Did it convince any of your friends or your teacher or your mum?

(b) For 16(*a*) and 16(b) what's the first run of three consecutive obtainable numbers. It's all up the number line from there. For 16(c) wait till you see Lemma 1. For 17(*a*) and 17(b) look for the first string of four consecutive obtainable numbers. For 17(c) notice that you can get 4 and 6. Now if you add 4 to each of these you get 8 and 10. Hence you can get all even numbers from 4. (Obviously you can't get any *odd* numbers.)

For 18(*a*) look for the first run of 6 and for 18(b), the first run of 7. For 18(c),…9 and 33 are cute, huh? Is it any help that 60, 63, 66 are the first run of multiples of 3? (What has this problem got to do with 3c and 11 c?)

20. I'll just do 6 and 7c. The others follow a similar pattern.

Theorem. *All numbers* $n \geq 30$ *can be written in the form* $6a + 7b$, *where a and b are not negative. The number* 30 *is best possible.*

Proof. 30 = 5 x 6, 31 = 4 x 6 + 7, 32 = 3 x 6 + 2 x 7, 33 = 2 x 6 + 3 x 7, 34 = 6 + 4 *x* 7, 35 = 5 *x* 7.

Any $n \geq 30$ can be written as $30 + 6k$, $31 + 6k$, $32 + 6k$, $33 + 6k$, $34 + 6k$ or $35 + 6k$ for some value of *k*. Hence any number greater than or equal to 30 can be written in the required form.

Suppose 30 is not best possible. Then 29 = 6a + 7b, where *a* and *b* are not both negative.

Hence 7b = 29 — 6a. But 7b ≥ 0, so 0 ≤ *a* ≤ 4. No matter which of these values of *a* we take, 29 — 6*a* is not a multiple of 7. This means that 29 = 6a + 7b, where *a* and *b* are not both negative. □

21. I'll just do 6c and 7c again.

Corollary (to the theorem in the solution to Exercise 20). *If n* = 0,6,7,12,13,14,18,19,20,21,24,25,26,27,28 *or any number greater than or equal to* 30, *then n* = 6a + 7b, *where a and b are some non-negative numbers.*

Proof. By the Theorem (of the solution to Exercise 20) the corollary is true for *n* ≥ 30.

An exhaustive check shows that the other values listed are the only ones possible. □

22. (*a*) 13 gives 24; 14 gives 26; 16 gives 30.

(b) Ah now. It's on the tip of my tongue….

23. **Proof.** Clearly, by putting *b* = 0 we can get any multiple of 3 we want.

If *n* = 3a + *sb*, where *s* = 3t, then *n* = 3(*a* + *tb).* Hence *n* must be a multiple of 3. □

24. **Proof.** We note that if 2s — 2, 2s — 1 and 2s can be written in the form 3a + *sb*, then all numbers *n > 2s* — 2 can be obtained from these by adding an appropriate multiple of 3.

Clearly 2s = 3 x 0 + s x 2. We now consider 2s — 2 and 2s — 1.

Case 1. s = 3t +1.

Now 2s — 2 = 3 x 2t and 2s — 1 = *s* + 3t. Both of these values can be obtained using 3 and *s*¢· stamps.

Case 2. s = 3t + 2.

Now 2s — 2 = s + s — 2 = s + 3t and 2s — 1 = (6t + 4) — 1 = 3(2t +1). Again both of these values can be obtained using 3 and *s*c stamps. Hence we can write all numbers *n > 2s* — 2 in the required form. ?

25. Yes. We have already shown that for *s* = 5, 2s — 3 = 7 is not possible. However we can show that 2s — 3 is *never* possible, no matter what the value of s. The following is a corollary to Theorem 3.

Corollary. *If s is not divisible by* 3, *then any number n > 2s* — 2 *can be written in the form n* = 3a + *sb, where a and b are not negative. 2s* — 2 *is best possible.*

Proof. The first part of the proof is precisely that of Theorem 3. Suppose 2s — 3 = 3a + *sb*, where *a* and *b* are not negative. Then 2s = 3(*a* +1) + *sb.*

Therefore (2 — *b)s* = 3(*a* + 1).

But the left-hand side of the equation is divisible by 3 since 3 divides 3(*a* + 1), the right-hand side of the equation. Since *s* is not a multiple of 3, then 2 — *b* is divisible by 3. But *b* ≥ 0 as is 3(*a* + 1). Hence 2 — *b* must be zero (there is no other number between 2 and zero which is divisible by 3).

If 2 — *b* = 0 then *a* +1 = 0. Hence *a* = —1. But this is a contradiction since *a* ≥ 0.

We cannot therefore obtain 2s — 3 in the form 3a + *sb*, where *a* and *b* are not negative. Thus 2s — 2 is best possible. □

26. **Theorem 4. (*a*)** *Let n be any number which can be written in the form* 3a + *sb where a and b are not negative.*

(i) *If s is a 'multiple of* 3, *then n is any multiple of* 3.

(ii) *If s is not a multiple of* 3, *then n is any number greater than or equal to* 2*s* — 2 *Further* 2*s* — 2 *is best possible.*

Proof. This follows immediately from Lemma 2 and the Corollary to Theorem 3. □

27. (*a*) If *s* is even then we can only get even numbers. Clearly we get all even numbers.

If *s* is odd we get all numbers from *s* — 1 on. This is best possible.

(b) If *s* = 5t, then *n* is any multiple of 5. Otherwise we can get any $n \geq 4s - 4$. This is best possible. To prove this we (i) consider the cases *s* = 5t +1, 5t + 2, 5t + 3, 5t + 4 and (ii) note that if 4s — 5 = 5a + *sb*, then we get a contradiction.

(c) There are actually three things to consider here. If (4, s) = 1, then we get everything from 3s — 3 on. This result is best possible.

If (4, *s)* = 4, then we get only multiples of 4.

However if (4, *s)* = 2, then we get all even numbers from *s* — 2 on. Perhaps this is a little unexpected.

Look at it this way. Let *s* = 2r. Then we're searching for *n* of the form 4a + 2rb = 2(2a + rb). Now we know from (*a*) that *2a* + *rb* gives us all the numbers from *r* — 1 on. Hence 2(2a + *rb)* must give all the even numbers from 2(r — 1) on. (Note that (r, 2) = 1 since *s* is not divisible by 4.) Finally notice that 2r — 2 = *s* — 2.

(d) Here (s, 6) = 1, 2, 3 or 6. For (s, 6) = 1 we get 5s — 5 as best possible and for *(s*, 6) = 6 we get only multiples of 6.

Using the argument of (c) we see that if (s, 6) = 2 we get everything even from *s* — 2 onwards (put *s* = 2r and use Theorem 4) and if (s, 6) = 3 we get every multiple of 3 from *s* — 3 on (put *s* = 3r and use (*a*)).

28. So what did you guess? How about *(r* — 1*)(s* — 1)?

If this is correct, the proof is not going to be as easy as it is for the various particular values of *r* that we've considered so far. With a particular value of *r* we were able to break the proof up into a number of cases. Then we dealt with each case separately. The problem with dealing with *r* is that it is not fixed and we will have a lot of cases to handle. After all *r* could be 10^{14}!

One way of tackling this problem is to use Theorem 1. From that we know that *a* and *(* exist such that 1 = *ra* + *s(*. (I'm only considering the case where (r, s) = 1 here.) Obviously one of a, *(* is negative. Now *n* = *rna* + *sn(*. We haven't quite got *a* = *na, b* = *n(* with neither negative yet but try a little fiddling. If *n* = *r(na)* + *s(n()* then so does *r(na* — s) + *s(n(* + r). So if *(* was negative and *a* positive, we might be able to make *n(* "less negative" by adding r. This will be at the expense of making *na* into *na* — *s* which is "more negative".

Of course this procedure can be continued.

n = *r(na* — s) + *s(n(3* + r) = *r(na* — 2s) + *s(n(3* + 2s).

It is possible that for *n* > *(r* — 1)(s — 1), that we can add enough r's to *n(* to make it positive (or zero) and that subtracting s's from *na* doesn't change the multiple of *r* to a negative number. If so you have your proof.

This idea, of course, is not new. We used it in the jug problem. (See the solution to Exercise 10, for instance.)

When you've done that you only have to show that *(r* — 1*)(s* — 1) is best possible.

This particular exercise is not easy.

29. (*a*) Well, er…

(b) I'm sure you can do this.

(c) Are there as many ticks as crosses? Is there any elegant way of pairing ticked numbers with crossed ones?

30. See Exercise 29.

Oh I suppose I should come clean. Take x and y so that $x + y = (r — 1)(s — 1) — 1$. The hard part now is to show that precisely one of x and y is a tick. Try it first for $r = 3$ and then for some other particular values of r before trying it for general r.

---

[a]This is the sum of an arithmetic progression. It's easy enough to deduce this simplification when you notice that the average of the sum $s = a + (a +1) + ∠∠∠ + (a + k)$ is both $s/(k + 1)$ and $1/2[a + (a + k)]$.

[b]Actually, Eröds is a sufficiently interesting person that you might like to look him up on the web or read about him in "The Man Who Loved Only Numbers" by Paul Hoffman. Why do some people have an Erdos number and why can't *you* have an Erdos number of 1 but I do?

Chapter 2
Combinatorics I

2.1. Introduction

In this second chapter I want to look at some combinatorial problems. Along the way I hope you'll be stunned and stimulated into mathematical activity and come to realise that mathematics is not a complete body of knowledge sitting in a box somewhere, all sewn up and tied with a neat bow. Rather I hope you will see it as an area that is growing exponentially, daily; that it is something which is being created by humans. I also hope that you will get some idea of the way that it is growing.

Don't worry if at first you can't do all of a group of exercises here. Try the earlier questions. When you feel more confident with the various techniques, go on to the later questions.

You should also not feel that you need to go through all of Section 2.3 before Section 2.4. If counting appeals to you more than pigeons, then do Section 2.4 first (or even half of Section 2.4 followed by some of Section 2.3 and back to Section 2.4 and so on).

Good luck.

2.2. What is Combinatorics?

A good question. Well if you look in a dictionary you'll see it's…OK so maybe my pocket dictionary is a little small. And maybe too if your dictionary at home is a few years old you may not be able to find "combinatorics" there either.

Now of course it may just be that combinatorics is one of those words that isn't fit for polite society. But hang on. Nobody seems to be shocked when *I* say it in public. That can only mean that "combinatorics" is one of those secret words that can only be spoken in the inner mathematical holy of holies. (Wherever that is.) So it must be in a Mathematical dictionary somewhere. Surely it's on the web!

Let's see then. "Combinatorics investigates the different possibilities for the arrangement of objects." "Combinatorics is a branch of mathematics that studies discrete objects."

Well I'm not really sure that that helped any. So let me go to my own experience. Combinatorics is the mathematics of counting,…without counting. Er, combinatorics is playing with sets of objects,…when you're not doing set theory. Er, well, er, combinatorics is the mathematics of structure,…when you're not doing geometry or algebra or whatever that's not combinatorics.

I guess combinatorics is hard to define. Possibly this is because combinatorics is a relatively new and growing area of mathematics. Although you can probably find glimpses of it earlier, it's really only been around a couple of hundred years. Indeed the bulk of what we know on the subject has only been known since the last half of the 20th Century.

Mathematical Reviews is a journal that tries to publish abstracts of all the latest mathematical results. The combinatorics' (or combinatorial theory) section of *Maths Review* is one of the largest. There seems to be more research going on in this area than in almost any other field of mathematics.

Now I must admit that this is partly because combinatorics is the waste paper basket of mathematics. What I mean by that is that if its mathematics and you don't know what to call it, then call it combinatorics. So here are some things that are combinatorics.

Latin squares are square arrays of numbers that have the property that no number occurs more than once in any row or column. The arrays below are Latin squares.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
3	1	4	5	2
5	3	1	2	4
4	5	2	3	1
2	4	5	1	3

Finding Latin squares and how they relate to one another is part of combinatorics. They have important applications in designing experiments. And they are now extremely popular in the form of the Sudoku puzzles. Here we have some very special, partially filled 9×9 Latin squares, and the problem is to complete the Latin square by putting the remaining entries in.

You might like to think about the conditions that are needed on a set of entries of a Latin square so that the Latin square can be filled uniquely.

0-1 sequence (or *binary sequences).* These are just strings of zeros and ones. When we require the sequences to have special properties relative to each other they give binary codes. For instance, 1111, 1010, 1100, 1001 is a binary code. Using strings like this with specified properties, we can measure the distance from here to the moon with extreme accuracy and we can also protect international banking transactions. Obviously (?!) 0-1 sequences are part of combinatorics.

Matchings. Suppose I have a list of jobs at a given factory and a list of people with the jobs they can do. Then matching theory will tell me whether or not I can assign a job to each person so that no two people do the same job and all jobs are taken. Naturally the organisation of the sets involved in this task is part of combinatorics.

In this chapter I want to concentrate on two areas of combinatorics — basic counting and the pigeonhole principle. I will only be able to scratch the surface of these two areas of combinatorial theory so undoubtedly there will be another chapter on the topic later, maybe in another book. There are a lot of books available on combinatorics these days because many universities now give courses on this topic. If you would like to get hold of more material we suggest, as a first look at the subject, that you consider R. Brualdi *"Introductory Combinatorics*", Second Edition, North Holland, New York, 1992. But any book recommended for a first undergraduate course will be fine. You can also look around the web for specific topics.

2.3. The Pigeonhole Principle

This is all very simple and obvious if you think about it. The famous principle simply states "if there are n pigeonholes and n +1 pigeons to go into them, then at least one pigeonhole must get 2 or more pigeons". What could be simpler or more obvious?

Problem 1. *You can use the pigeonhole principle to come up with some startlingly trivial facts. For instance, to the nearest dollar, there are at least two wage earners in your country who earn precisely the same amount.*

Discussion. The easy way to see that is to observe there aren't too many people in the country earning more than $200,000 a year. (If there are forget about them.) But there must be more than 200,001 wage earners, earning less than \$200,000. With the dollar amounts as pigeonholes and the wage earners as pigeons, the pigeonhole principle tells us that there are 2 wage earners at least, who earn the same amount of money in a year.

By the way, in Europe, the Pigeonhole Principle is often referred to as Dirichlet's (box) Principle.

Exercises

1. Prove that in a group of 13 people at least two have their birthday in the same month.
2. Prove that in a group of 32 people there are at least two whose birthdays are on the same date in some month.
3. I know that among p people at least two were divorced on the same day of the week. What is the smallest value of p that will guarantee this?
4. In Swooziland, bank notes each have a single digit preceded by three letters.
   (i) How many notes do I need before I can be sure there are two of them whose identification starts with the same letter?
   (ii) Repeat (i) for notes whose identification starts with the same *two* letters.
5. Suppose, car registration plates have two letters and three numbers. Is it true that in the car park on the opening day of the Olympics there were two cars with the same three digit numbers on their plates?
6. Prove that any 5 points chosen within a square of side length 2, there are two whose distance apart is at most $\sqrt{2}$. (Is this true for 4 points?)
7. (a) Prove that of any 5 points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{2}$.
   (b) Prove that of any 10 points chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{3}$.
   (c) Determine an integer m_n such that if m_n points are chosen within an equilateral triangle of side length 1, there are two whose distance apart is at most $\frac{1}{n}$.

Now we can develop the idea of the pigeonhole principle further. If we have five pigeons and two pigeonholes it should be clear that no matter how the pigeons go to roost (or whatever pigeons do), then there must be one hole which has to hold at least three pigeons. In more general terms:

Given n pigeonholes and $mn + 1$ pigeons there is one pigeonhole which contains at least $m + 1$ pigeons.

This version of the pigeonhole principle contains the first version as a special case. As such we say it is a *generalisation* of the first. Mathematicians are always trying to generalise results. I'll point out generalisations of other results as they arise. We have already thought about this idea in Chapter 1.

Problem 2. *Students in a university lecture have black, brown, red, green, or blue and white hair. There are* 101 *students in the lecture. Show there are at least* 21 *students who have the same colour hair.*

Discussion. The pigeonholes here are the hair colours. There are 5 of these. The pigeons are the 101 students.

In this question then, $n = 5$ and $mn + 1 = 101$. So $m + 1 = 21$. By the more general pigeonhole principle, there must be at least 21 students in the lecture who have the same colour hair.

Exercises

8. Some 31 diplomats from Finland, Greece, Italy, Romania, New Zealand and Singapore went out to dinner together after an afternoon session at the United Nations. Prove that there was one country that was represented by at least 6 diplomats.
9. The heights of 27 students in a Geography class were measured to the nearest 5 cm. There was a range of heights from 150cm to 180cm. There were at least *t* students with one of these heights. What is the largest value of *t* you can guarantee?
10. Thirteen schools took part in an athletics competition at Murrayfield. There were 1514 student spectators. Show that there was one school that was cheered on by at least 117 students.

One of the classic problems to use the pigeonhole principle is the party problem.

Problem 3. *Prove that in a group of six people at a party there are at least three people who mutually know each other or there are three who are mutual strangers.*

Discussion. To start this off a diagram is useful. Let the six people be represented by dots and draw a line between two people who know each other; draw a broken line between people who don't know each other. So in Figure 2.1, *a* and *b* know each other, *a* and *d* know each other, *b* and *c* know each other and so do *c* and d. Any other pair are strangers. We assume too that if *x* knows *y*, then *y* knows *x*.

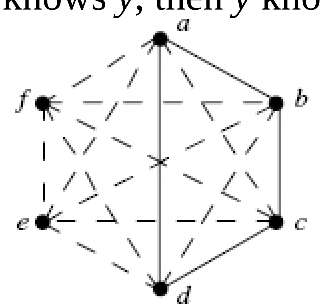


Figure 2.1.

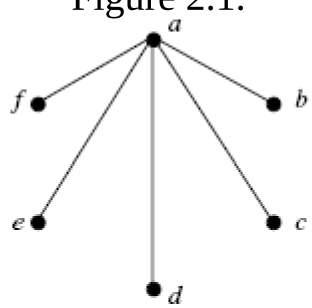


Figure 2.2.

(In Figure 2.1 there are no three who are mutual acquaintances but *a*, *c*, *e*, among others, are three mutual strangers.)

How do we show there are at least three mutual acquaintances or at least three mutual strangers? Well, we have to show that in our dot, line and broken line diagram, there is either a solid line triangle or a broken line triangle.

Consider person *a* in Figure 2.2. Potentially there are 5 lines that can be drawn from *a* to the other dots. We apply the pigeonhole principle by taking two pigeonholes — one hole for lines and one hole for broken lines. So one pigeonhole must contain at least three pigeons. In other words, there must be at least three lines or at least three broken lines coming from *a*.

Let us suppose without loss of generality that there are at least three solid lines out of *a*. Further, without prejudicing our argument, we may as well suppose that we have the situation of Figure 2.3. Here *a* is joined to (knows) *b*, *c* and *d*.

What can we say about *b*, *c* and *d*? If one pair from these three are friends, then join them by a line. Say *b* and *c* know each other. From Figure 2.4 we see we've got our solid triangle. And we have our solid triangle if any pair of *b*, *c*, *d* are friends.

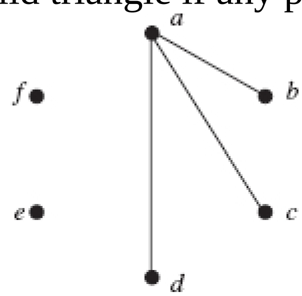


Figure 2.3.

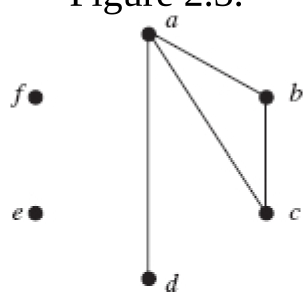


Figure 2.4.

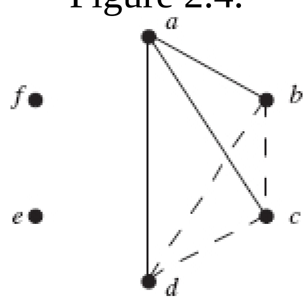


Figure 2.5.

But what if none of *b*, *c* and *d* know each other? Then join them all by broken lines and we have a broken triangle (see Figure 2.5).

So whatever happens we have a triangle of some kind. We have therefore proved that of the six people at the party, there are either at least three mutual acquaintances or at least three who have never met each other.

Exercises

11. Seventeen people correspond by mail with one another — each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic. (International Mathematical Olympiad 1964.) (Is this still true if there are only 16 people corresponding?)
12. Show that the 6 (as in 6 people) in Problem 3 is best possible. In other words, show that the property of three people knowing or not knowing each other, does not hold for 5 people.

    (See Chapter 1 for problems involving the idea of “best possible”.)
13. Show that among our six friendly party people there are (i) two groups of three who mutually know each other, (ii) two groups of three who mutually don't know each

other or (iii) a group of three who do and a group of three who don't.

In the answer to Problem 3 we used the phrase “without loss of generality”. This is one of the stock phrases of mathematical proofs. It means that when a certain symmetry exists (as here between solid lines and broken lines) we can argue on the assumption that one of them happens. This assumption does not alter the validity of the argument.

Why not? In the present case suppose we dropped the “without loss of generality”. We could argue as we did, first assuming that there were at least three lines and we would get the result we wanted. However, to complete the argument we would need to consider the case when at least three broken lines came out of *a*. But the argument for the broken line case is exactly the same as for the line case, except that we replace “line” everywhere by “broken line” and “broken line” everywhere by “line”. (Check it out to make sure.) To avoid this tedious repetition we use the phrase “without loss of generality”.

One other thing, you should by now be realising that the hint to deciding that the pigeonhole principle can be used is the words “show there are at least…” or “show there exists some number among other numbers”.

Now try the following set of problems which all use a version of the pigeonhole principle. The clue to these solutions is to decide how to put the problems together so that you can sort out pigeons from pigeonholes. We are now into the harder type of problem so the pigeonholes are not always going to be obvious.

Exercises

14. Show that given any 52 integers, there exist two of them whose sum, or else whose difference, is divisible by 100. (Does this result hold for 51 integers?) If 100 is replaced by 10, what should 52 be replaced by? Generalise the result as far as you can.

    (Hint: For the “52” problem first reduce the numbers to the set {0,1,2…,99}. Then take your pigeonholes as 0, 50 and the pairs (1, 99), (2, 98) (49, 51). What good does it do to know that at least two numbers are in one of these pigeonholes?)

15. (a) Prove that in any set of 27 different odd numbers all less than 100, there is a pair of numbers whose sum is 102.

    (b) How many sets of 26 such numbers can we choose such that no pair in any of these sets gives a sum of 102? (American Mathematical Olympiad 1981.)

16. Show that given any 17 numbers it is possible to choose 5 whose sum is divisible by 5. Generalise this result.

17. Inside a cube of side 15 units there are 11,000 given points. Prove that there is a sphere of unit radius within which there are at least 6 of the given points. (Unit radius = radius one.) (British Mathematical Olympiad 1978.)

18. A chessmaster who has 11 weeks to prepare for a tournament, decides to play at least one game every day, but in order not to tire himself he decides not to play more than 12 games in any 7 day period. Show that there exists a succession of days during which he plays *exactly* 21 games. (Is there a sequence of days when he plays exactly 22 games?)

19. A student has 37 days to prepare for an exam. From past experience she knows that she will require no more than 60 hours of study. She also wishes to study at least 1 hour per day. Show that no matter how she organises her study, there is a succession

of days during which she studies exactly 13 hours. (Assume she works for a whole number of hours per day.) Can this problem be generalised?

We've now discovered that there are a few types of pigeonhole principle problems. There are the easy, almost trivial examples such as Problem 1 and Exercise 1. Then come the geometrical types of Exercises 6 and 7. “Sequence of days problems” like Exercises 18 and 19 are another variant.

The “people” problems of Problem 3 and Exercise 11 are related to a variant of the pigeonhole principle known as Ramsey Theory. Books have been written on this subject although it only originated in 1930, when F.P. Ramsey proved a theorem that was important for the foundation of logic.

Problems in this area usually deal with a number of people and “coloured links” between them. For instance, in Problem 3 we could have linked two people with a red line if they knew each other and a blue one if they didn't. Similarly in Exercise 11 we could have linked two people in red if they corresponded on topic 1, blue if they corresponded on topic 2 and white if they corresponded on topic 3. In each example we want to know if there is a triangle in just one of the colours.

Looking at these problems in this way it's easily seen that Exercise 11 is an extension of Problem 3. Clearly we can extend the problem to links in four colours.

Now n people stand in a field and hold ribbons coloured red, white, blue and green. Each pair of people share precisely one ribbon between them. How big does n have to be to ensure that there are three people linked by ribbons of only one colour?

Is it obvious that there is such an n? Before you panic, Ramsey's Theorem tells us there is. Unfortunately it doesn't tell us how big n is.

If this topic appeals to you, you might be interested in reading Martin Gardner's Mathematical Games section of the *Scientific American*, Volume 237, No. 5, November 1977. You might also look up “Ramsey Theory” on the web and see how complicated and difficult the whole thing is.

The following two problems are quite difficult.

Exercises

20. A *4-clique* is a set of four people who are all linked in the same colour. In an office two people are either friendly or they hate each other. How big must the staff of the office be in order for there to be either a friendly 4-clique or a hateful one?
21. Find the smallest n in the four colour ribbon problem.

Well, that was combinatorics. At least, it was one of the concepts of combinatorics. And now here's another.

2.4. Counting without Counting

This section is a basic introduction to systematic counting. Before we know it we'll have a link with the expansion of algebraic expressions.

The easiest way to learn to swim is to jump in the deep end.

Problem 4. *How many positive integers with* 5 *digits can be made up using the digits* 1, 2 *and* 3.

Discussion. Suppose we look for all the 2 digit numbers first then work up to 5. We can make a list: 11, 12, 13, 21, 22, 23, 31, 32, 33. So there are 9.

The reason for this seems to be that there are three numbers that can go in the first place. For every number in the first place there are three numbers that can go in the second place. 3×3 = 9.

Right then, let's tackle 5 digits. There are 3 choices for the first place, 3 for the second, 3 for the third, 3 for the fourth and 3 for the fifth. Altogether we've got 3 × 3 ×3 × 3 × 3 = 3^5 = 729.

Problem 5. *How many positive integers with n digits can be made up using just the digits 1, 2 and 3.*

Discussion. 3×3 ×…×3, *n* times. So the answer is 3^n.

Exercises

22. How many 10 digit numbers can be made using the digits 2, 3, 4, 5 and 6?
23. How many 6 digit numbers are there whose digits are all non-zero even numbers?
24. How many 7 digit numbers can be made up using just odd digits?
25. How many numbers between 1000 and 9999 have only even digits (including zero)?
26. The Morse code uses dots and dashes. Each letter of the alphabet is made up of at most 4 of these signals (dots and/or dashes). How many different letters are possible in Morse code?
27. (a) In the plane, coordinates are of the form (x, *y*). How many different points in the plane can be found whose x- and y-coordinates come from the set {0, 1}.
    (b) Repeat (a) for three dimensions where coordinates are of the form (*x*, *y*, *z*).
28. (a) Show there are four sets which can be made from the two elements *a* and *b*.
    (b) Show that eight sets can be made from the three elements a, *b* and *c*.
    (c) Why are the numerical answers to Exercise 27(a) and Exercise 28(a) the same? Why is it likewise for Exercise 27(b) and Exercise 28(b)?
    (d) Show that 2^n sets can be made with *n* elements.

But what if we are restricted in the number of times we can use a number or letter? Consider the following problem.

Problem 6. *How many "words" (strings of letters, most of them not words in the dictionary) can be made from the letters A, C, T if we use each letter only once?*

Discussion. If you haven't met this type of problem before and have no strategy, then it is best, first of all, to use trial and error. So, writing down all possible words *systematically* gives

ACT; ATC; CAT; CTA; TAC; TCA.

There are thus 6 words.

Alternatively we see that there are 3 possible choices for the first letter. Once the first letter is chosen we have 2 choices for the second letter. Finally, there is only 1 possible choice for the last letter. We can therefore produce 3× 2×1 = 6 "words" from the letters A, C, T.

Exercises

29. Using each letter only once, how many "words" can be made from the letters in the word (i) BEAT; (ii) SLATE?
30. In how many ways can the letters in the word FLIGHT be arranged?
31. How many 6-letter words in which at least one letter appears more than once, can be made from the letters in the word F, L, I, G, H, T? (You may use any letter as often as you like.)

In general then, we can see that if we have *n* distinct letters, each used once, we can produce

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

words.

For convenience we write *n*! (pronounced "*n* factorial") for the expression

$$n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

So we can rearrange the letters in the word FLIGHT in 6! (=720) ways.

Problem 7. *In how many ways can the letters in the word DID be arranged?*

Discussion. The two D's are a problem. Let's suppose for a start that they were different. Call them D_1 and D_2. Then we'd have the 3! words

$$D_1D_2I; \quad D_2\,D_1I; \quad D_1ID_2; \quad D_2ID_1; \quad ID_1D_2; \quad ID_2D_1$$

But since D_1 and D_2 are the same, $D_1D_2I = D_2D_1I = DDI$. The other words occur in pairs too. Hence $D_1ID_2 = D_2ID_1 = DID$ and $ID_1D_2 = ID_2D_1 = IDD$.

So the number of different words here is 3! ÷ 2 = 3. These are obviously, DDI, DID and IDD.

Exercises

32. How many "words" can be made from the following words, where all the letters are used?
    (i) BOOT; (ii) TOOT; (iii) LULL; (iv) MISSSISSSIPPPI.
33. How many 7-digit numbers can be made using two 1's, three 2's and two 3's?
34. There are 12 runners in a cross-country race. There are 3 runners each from the Hasty Harriers Club, the Runaway Racers Club, the Country Cross Club, and the Achilles Athletic Club. In how many ways can the teams cross the finish line (assuming no ties)?

Problem 8. *How many "words" can be made up from r A's, s B' s and t C's?*

Discussion. Let $n = r + s + t$. If we assume all the A's are different, and all the B's are different and all the C's are different, then there are *n*! words. But the A's are *not* distinct. So the *n*! words occur in groups of *r*! words which are in fact the same. There are thus $\frac{n!}{r!}$ words where the A's are not distinct.

Then again the B's are all the same. So the $\frac{n!}{r!}$ words occur in groups of *s*! which are the same. So there are $\frac{n!}{r!s!}$ words.

Finally, the C's are not distinct so we just have $\frac{n!}{r!s!t!} = \frac{(r+s+t)!}{r!s!t!}$ different words.

Exercises

35. How many rearrangements are there of the letters in the words
    (i) ENGINEERING; (ii) MATHEMATICAL?
36. How many words can be formed from the letters
    (i) AABBB; (ii) AAABBBB?
37. How many binary sequences (strings of 0's and 1's) of length 10, can be made using four 0's and six 1's? (A binary sequence *can* start with zero.)
38. How many *n*-digit numbers can be made up using r_1 1's, r_2 2's, r_3 3's and r_4 4's, where $n = r_1 + r_2 + r_3 + r_4$.

Problem 9. *How many subsets of size* 3 *can be chosen from a set of size* 6?

Discussion. When in doubt, write them out. Let the elements of the set be a, b, c, d, e, f. Working systematically starting with the a's we get

$$\begin{array}{ccccc} abc, & abd, & abe, & abf, & acd, \\ ace, & acf, & ade, & adf, & aef, \\ bcd, & bce, & bcf, & bde, & bdf, \\ bef, & cde, & cdf, & cef, & def. \end{array}$$

The answer, assuming we haven't missed one, is 20. I think I did it correctly here but what if I want the subsets of size 3 in a set of size 106? How can I be sure that I won't miss any subsets then?

We've clearly got to find a systematic way to do the counting. There's a clue back at Exercise 29 where we counted sets using 0's and 1's. In that example a 0 in the 6th position say, indicated that that sixth element wasn't in the set. On the other hand a 1 in the 3rd position showed that the 3rd element *was* in the set. So we can represent the subsets of size 3 above using 0's and 1's as follows:

$$\begin{array}{ccccc} 111000, & 110100, & 110010, & 110001, & 101100, \\ 101010, & 101001, & 100110, & 100101, & 100011, \\ 011100, & 011010, & 011001, & 010110, & 010101, \\ 010011, & 001110, & 001101, & 001011, & 000111. \end{array}$$

I've dropped the commas and brackets in the binary sets to make life easier but you should realise that 101001 is the same as acf and 001101 is the same as cdf.

But we know how to count binary sequences of length 6 with three 0's and three 1's. The answer is $\frac{6!}{3!3!}$ = 20. Just what we got by trial and error.

Exercises

39. How many subsets of size 3 can be chosen from a set of size 7?
40. How many subsets of size 5 can be chosen from a set of size 9?
41. How many subsets of size 4 can be chosen from a set of size 10?
42. How many subsets of size r can be chosen from a set of size 8? Check your answer for the specific values 1, 2, 5 for r.

Problem 10. *How many subsets of size r can be chosen from a set of size n?*

Discussion. This is just $\frac{n!}{r!(n-r)!}$. Check it out using the values for n and r in Exercises 39-41. This turns out to be a useful number so we will write it as ${}^nC_r$. (You will see it written as $\binom{n}{r}$ and in even some other ways.) The C comes from the fact that ${}^nC_r$ is sometimes called the number of *combinations* of n things taken r at a time. This just means the number of ways of choosing a subset of r things from a set of size n.

Exercises

43. Calculate (i) ${}^5C_3$; (ii) ${}^{16}C_3$; (iii) ${}^{999}C_{998}$.
44. Show that ${}^nC_r = {}^nC_{n-r}$.
45. In how many ways can three different letters be chosen from the full alphabet?
46. In a particular trotting event, five horses line up at the barrier and four are in a line behind them. In how many ways can the five front horses be chosen?

One more little wrinkle is needed. The question of 0!. Do we need it? Well, suppose we want to calculate ${}^5C_5$. Clearly the number of ways of choosing 5 objects from 5 objects is just 1. You just do it, you can only do it, in one way. So

$$1 = {}^5C_5 = \frac{5!}{5!0!} = \frac{1}{0!}.$$

For this equation to make sense we must have 0! = 1. So we make a special case for 0. By convention we agree that 0! = 1.

Exercises

47. Calculate

(i) ${}^{5}C_{0}$; (ii) ${}^{16}C_{0}$; (iii) ${}^{k}C_{0}$; (iv) ${}^{k}C_{k}$.

48. Simplify

(i) ${}^{n}C_{0}$; (ii) ${}^{n}C_{1}$; (iii) ${}^{n}C_{2}$.

49. Prove by direct calculation that ${}^{3}C_{0} + {}^{3}C_{1} + {}^{3}C_{2} + {}^{3}C_{3}$ If we change all the 3's to 4's *does* equality still hold? What expression with C's in, adds up to 2^{4} then?

50. Prove the following by direct calculation:

(i) ${}^{4}C_{3} = {}^{3}C_{3} + {}^{3}C_{2}$; (ii) ${}^{10}C_{7} = {}^{9}C_{7} + {}^{9}C_{6}$.

Generalise the previous two results.

Make a concerted attempt at the above problems before continuing.

Surprisingly the last few problems were set with more than practice in mind. I am leading you inexorably on to, fanfare stage left, *Pascal's Triangle.* Isn't that beautiful?

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

. . . .

Pascal's Triangle

First, in case you are meeting this for the first time and can't see the pattern, to get a new number simply add together the two numbers in the row directly above. For instance,

3 3 10 5

6 15

Thus the next row of the triangle will be

1 7 21 35 35 21 7 1

Oh. I forgot to tell you to put 1's on the ends of each row before you start.

What has all this got to do with combinations? Go back to Exercise 50. When you calculated ${}^{3}C_{0} + {}^{3}C_{1} + {}^{3}C_{2} + {}^{3}C_{3}$ you should have got 1 + 3 + 3 + 1. These are exactly the numbers, in order, of the 3rd row of Pascal's Triangle. (I'm cheating a little. The row with just 1 in it I'm going to consider to be the zeroth row.)

Check out ${}^{5}C_{0}$, ${}^{5}C_{1}$, ${}^{5}C_{2}$, ${}^{5}C_{3}$, ${}^{5}C_{4}$, ${}^{5}C_{5}$ and you'll see that you get the numbers in the 5th row.

In general, the nth row is formed by the integers ${}^{n}C_{0}$, ${}^{n}C_{1}$, ${}^{n}C_{2}$,…, ${}^{n}C_{n-2}$, ${}^{n}C_{n-1}$, ${}^{n}C_{n}$, in that order.

How can that be?

Lemma.[a] ${}^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{r-1}$

Proof.

$$\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!} \\
&= \frac{n!}{(r-1)!(n-r)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\} \\
&= \frac{n!}{(r-1)!(n-r)!}\left\{\frac{n-r+1+r}{r(n-r+1)}\right\} \\
&= \frac{n!(n+1)}{(r-1)!r(n-r)!\,(n-r+1)} \\
&= \frac{(n+1)!}{r!(n+1-r)!} \\
&= {}^{n+1}C_r. \qquad \square
\end{aligned}$$

Maybe that Lemma didn't help either. It was the generalisation I was looking for in Exercise 51 though. Perhaps a diagram will cause the penny to drop.

$$\begin{array}{ccccccc}
{}^nC_{r-2} & & {}^nC_{r-1} & & {}^nC_r & & {}^nC_{r+1}\cdots \\
& \cdots\ {}^{n+1}C_{r-1} & & {}^{n+1}C_r & & {}^nC_{r+1}\cdots &
\end{array}$$

This is just how Pascal's Triangle is constructed. The rth term in row $n + 1$ is the sum of the two terms immediately above it. These are just ${}^nC_{r-1}$ and ${}^nC_r$.

Once we have 1, 1 (think of these as ${}^1C_0$, ${}^1C_1$) from row 1, and 1's on the left and right of each row (think of these as ${}^nC_0$, ${}^nC_n$), the lemma tells us that all other entries in the triangle are ${}^nC_r$'s. The triangle could easily have been called the Combinations Triangle.

Even this though would only make the triangle an interesting oddity if it were not for the following.

Exercise

51. Expand the following in increasing powers of x.
 (i)$(1+ x)^3$; (ii)$(1+ x)^5$; (iii)$(1+ x)^6$.

Assuming you've done the problem you should now see that the coefficients of these expansions are precisely the numbers in the corresponding row of Pascal's Triangle.

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4.$$

The 1, 4, 6, 4, 1 are just the entries in order, of the fourth row of the Triangle. Hence

$$(1+x)^4 = {}^4C_0 + {}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4.$$

(If you put $x = 1$ in this expression you should see why Exercise 49 works.)

This then should give us a quick way of expanding $(1 + x)^{12}$. There's no need for us to calculate Pascal's Triangle down to the 12th row (thank goodness!). By what we've said

$$\begin{aligned}
(1+x)^{12} &= {}^{12}C_0 + {}^{12}C_1x + {}^{12}C_2x^2 + {}^{12}C_3x^3 + {}^{12}C_4x^4 \\
&\quad + {}^{12}C_5x^5 + {}^{12}C_6x^6 + {}^{12}C_7x^7 + {}^{12}C_8x^8 + {}^{12}C_9x^9 \\
&\quad + {}^{12}C_{10}x^{10} + {}^{12}C_{11}x^{11} + {}^{12}C_{12}x^{12}.
\end{aligned}$$

To finish this off, all we need to do is to calculate all the ${}^{12}C_r$ terms.

Exercises

52. Using combination notation, then simplifying, expand
 (i)$(1+ x)^6$; (ii) $(1 + x)^{10}$.

53. Find the coefficient of x^{15} in
(i)$(1+ x)^{17}$; (ii) $(1 + x)^{22}$.
54. What is the sum of the coefficients in the expansion of $(1 + x)^6$?
55. Simplify ${}^nC_0 + {}^nC_1 + {}^nC_2 +\ldots+ {}^nC_n$.
What has this got to do with the fact that there are 2^n subsets of a set of size *n?*

An expression of the form $(1 + x)^n$ is called a binomial expression (bi = two, nom… = numbers and 1 and *x* are two numbers). Thus the various coefficients of the powers of *x* are called *binomial coefficients.* So the terms ${}^nC_r$ are given the collective name, *binomial coefficients.*

It should be no surprise therefore that the next result is the *Binomial Theorem.* It generalises what we have been saying about the expansions of binomial expressions.

Theorem (Binomial Theorem).

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n.$$

Proof.

$$(1+x)^n = (1+x)(1+x)\cdots(1+x).$$

$$\leftarrow \quad n \text{ terms} \quad \rightarrow$$

If we can prove that the coefficient of x^r is ${}^nC_r$ for $r = 0, 1,\ldots, n$ we must be finished.

Now we get an x^r term by taking *x* from *r* of the *n* brackets (1 + x). Further, this is the only way to get an x^r term. So there are as many x^r terms as there are ways of choosing *r* of the *n* brackets. This is simply ${}^nC_r$ by the definition of ${}^nC_r$ and our earlier counting. Hence the coefficient of x^r is ${}^nCr$. □

Exercises

56. By replacing x by a suitable value, use the Binomial Theorem to expand the following
(i) $(1 + 2a)^3$; (ii)$(1 - 3b)^4$; (iii) $(1 + 4c)^5$.
57. Expand the following
(i) $(x + y)^3$; (ii) $(x + y)^4$; (iii) $(x - y)^5$.

By generalising the last exercise we obtain an extension of the Binomial Theorem.

Theorem (Binomial Theorem Plus).

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_r x^{n-r} y^r + \cdots + {}^nC_n y^n.$$

This can be proved in the same way as the Binomial Theorem was proved. It allows us to expand any binomial expression to any positive integer power.

Exercise

58. Expand the following:
(i) $(x+2y)^4$; (ii) $(a+3b)^3$; (iii) $(2a+3b)^4$;
(iv) $(c-d)^3$; (v) $(2c-3d)^6$; (vi) $(3a+2/a)^4$.

Armed with binomial coefficients we can launch into more serious counting. See how we use binary sequences in another way.

Problem 11. *The Origami Motor Company has just released two new model cars — the Ki and the Wi. I want to buy* 12 *of the Origami vehicles for my sales people. How many different choices do I have?*

Discussion. Let's change this problem into a string of 0's and 1's. Here the zeros are just place markers to keep the Kis and Wis apart. So we need just one 0. The 1's represent cars. Each 1 before the 0 represents a Ki; each 1 after the 0 represents a Wi.

For instance 1111101111111 represents a purchase of 5 Kis and 7 Wis. Indeed every string of twelve 1's and one 0 represents a possible purchase. On the other hand every possible purchase can be represented by a string of twelve 1's and one 0. From what we have seen earlier there are ${}^{13}C_{12}$ possible binary sequences of this form.

So there are 13 possible choices of cars. (They are 0111111111111, 1011111111111, 1101111111111, 1110111111111, 1111011111111, 1111101111111, 1111110111111, 1111111011111, 1111111101111, 1111111110111, 1111111111011, 1111111111101, 1111111111110.)

If the Origami Motor Company had produced Kis, Wis and Wikis we would still have had twelve 1's because that is the number of cars I'm going to buy. However we would now need two 0's. In this case the 1's before the first 0 would count Kis, the 1's between the two 0's would count Wis and the 1's after the second 0 would count Wikis. There'd be ${}^{14}C_{12}$ choices then.

Exercises

59. On Sunday, my local shop sells freshly baked white rolls, brown rolls, sesame seed rolls and poppy seed rolls. In how many ways can I buy a dozen fresh rolls?
First express your answer as a single binomial coefficient.
60. Last week my wife won second prize in the lottery. She immediately ran downtown to a dress shop that sold red dresses, white dresses, blue dresses, green dresses and pink dresses. She bought twelve dresses. In how many ways could she have done this?
First express your answer as a single binomial coefficient.
61. I have c colours of paint and g golf balls. How many ways can I colour the golf balls? (Only one colour per ball please.)
62. How many solutions are there, in non-negative integers, of
(i) $x + y + z = 8$; (ii) $x + y + z + w = 18$.
(Use 0,1 sequences.)

We conclude this section with a set of problems of a combinatorial nature that are based on the ideas in this booklet. Some of them are very hard.

Exercises

63. How many distinct positive divisors does the number $73{,}950{,}800 = 2^4 \cdot 5^1 \cdot 7^5 \cdot 11$ have?
64. A fast food shop sells five different types of hamburgers. How many different combinations of nineteen hamburgers can one buy from this shop?
65. How many selections of three numbers each can be made from the set {1,2…,99, 100} if no two consecutive numbers can be included?
66. Prove that ${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \ldots + {}^{r-1}C_{r-1}$.
67. Find the number of solutions satisfying the inequality
$$0 \leq x_1 + x_2 + x_3 \leq 30,$$
if x_1, x_2 and x_3 are non-negative integers. For example, $x_1 = 5$, $x_2 = 0$ and $x_3 = 18$ is a solution.

68. By determining the constants a, b and c such that $k^3 = a\binom{k}{3} + b\binom{k}{2} + c\binom{k}{1}$ for all positive integers $k \geq 3$, find an explicit formula for the sum of the series $1^3 + 2^3 + 3^3 + + n^3$.

(Recall that $\binom{n}{r} = {}^nC_r$.)

69. (a) Express each of the following sums as a single binomial coefficient.

(i) $\binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \cdots + \binom{r+n}{n}$;

(ii) $\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \cdots + \binom{n}{k}\binom{m}{0}$ where $k \leq \min(m, n)$.

(b) Evaluate the sums

(i) $\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2$;

(ii) $\binom{2n}{1} + \binom{2n}{3} + \binom{2n}{5} + \cdots + \binom{2n}{2n-1}$.

70. Is it possible to choose 1983 distinct positive integers, all less than or equal to 100,000, no three of which are consecutive terms of an arithmetic progression? Justify your answer. (IMO 1983 No. 5.)

2.5. A Sigma Aside

Many of the expressions that have been written in the last section can be considerably shortened by the use of sigma notation. It's a way of cutting out those three little dots that have appeared from time to time in various expressions.

First of all Σ is the Greek upper case sigma (*σ* is the lower case sigma). Since *s* is for sum and Σ is the Greek *s*, mathematicians use Σ as part of the notation for **S**umming things.

Consider the expression 1 + 2 + 3 + 4. This can be written as $\sum_{i=1}^{4} i$.

What the Σ notation means is, start with *i* =1, then add what you get with *i* = 2, then add what you get with *i* = 3, then add what you get with *i* = 4. You stop at 4 since that is the largest value of *i* on the Σ.

In this way you should see that $\sum_{i=1}^{5} i = 1+2+3+4+5$ and $\sum_{i=3}^{6} i$ = 3 + 4 + 5 + 6. On the other hand, something like $\sum_{i=1}^{n} i$ avoids the three little dots, for $\sum_{i=1}^{n} i = 1+2+3+\cdots+n$.

What is $\sum_{i=1}^{4} i^2$ then? Simply $1^2 + 2^2 + 3^2 + 4^2$. The point is that you substitute each *i* value from 1 (at the bottom of the Σ) up to 4 (at the top of the Σ) in the expression i^2 and add them all together.

Exercises

71. Write the following sums out in full. Well, include three little dots (ellipses) if you have to!

(i) $\sum_{i=1}^{5} i^2$; (ii) $\sum_{i=1}^{n} i^2$; (iii) $\sum_{i=2}^{5} i^3$;
(iv) $\sum_{i=3}^{6} i^4$; (v) $\sum_{i=1}^{3} \frac{1}{i}$; (vi) $\sum_{i=1}^{6} i!$.

72. Write the following sums using Σ notation.

(i) $1+2+3+4+5+6+7$;
(ii) $2+4+6+\cdots+12$;
(iii) $2+4+6+\cdots+2n$;
(iv) $3+9+27+81$;
(v) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$;
(vi) $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}$;
(vii) ${}^3C_0+{}^3C_1+{}^3C_2+{}^3C_3$;
(viii) ${}^4C_1+{}^4C_2+{}^4C_3+{}^4C_4$;
(ix) ${}^3C_0+{}^3C_1x+{}^3C_2x^2+{}^3C_3x^3$;
(x) ${}^5C_0+{}^5C_1x+{}^5C_2x^2+{}^5C_3x^3+{}^5C_4x^4+{}^5C_5x^5$;
(xi) ${}^nC_0+{}^nC_1+\cdots+{}^nC_n$;
(xii) ${}^nC_0+{}^nC_1x+{}^nC_2x^2+\cdots+{}^nC_rx^r+\cdots+{}^nC_nx^n$.

73. Express each of the following as sums.

(i) $\sum_{k=0}^{n}\binom{r+k}{k}$; (ii) $\sum_{s=0}^{k}\binom{n}{s}\binom{m}{k-s}$;

(iii) $\sum_{i=0}^{n}(-1)^i\binom{n}{i}^2$; (iv) $\sum_{r=0}^{n-1}\binom{2n}{2r+1}$;

(v) $\sum_{k=0}^{n-r}\binom{n-1-k}{r-1}$; (vi) $\sum_{i=0}^{3}\binom{3}{i}a^{3-i}b^i$.

74. State the Binomial Theorem (p. 45) using sigma (summation) notation.

The sigma notation will be useful on many occasions in the future. Practice it and use it when you can.

2.6. Solutions

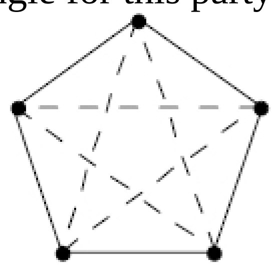
1. The pigeonholes are the 12 months of the year. The pigeons are the 13 birthdays. By the pigeonhole principle there must be at least 2 birthdays in the same month.
2. The pigeonholes are the dates 1,2,…,31. The pigeons are the 32 people. There must be one pigeonhole that gets at least 2 people.
3. We need $p \geq 8$, otherwise we could get at most one pigeon (a divorcee) assigned to each pigeonhole (day of the week).
4. (i) 27 (because there are 26 letters in the alphabet);
   (ii) 26 x 26 + 1 = 677.
5. Yes, if there were at least 1001 cars in the car park. (I have assumed XY 000 is a legal registration.)
6. Divide the 2 by 2 square into four unit (sidelength 1) squares. The squares are the pigeonholes; the points are the pigeons. Hence by the pigeonhole principle there is one unit square which contains at least 2 points. In a square, the maximum distance apart that 2 points can be, occurs when they are on opposite corners. So in a unit square any 2 points are at most $\sqrt{2}$ apart. Hence of the 5 points there are 2 whose distance apart is at most $\sqrt{2}$.

   Given only 4 points, they can be at the corners of the large square and so any pair are at least a distance 2 apart. Hence the "$\sqrt{2}$" statement does not hold for 4 points.
7. (a) Divide the equilateral triangle up into 4 equal equilateral triangles of sidelength 1/2. By the pigeonhole principle there are at least 2 of the chosen 5 points in one of the smaller equilateral triangles. Two such points are at most 1/2 apart.
   (b) Divide the large triangle into 9 smaller equilateral triangles of sidelength 1/3. Of the 10 points at least 2 are in a smaller triangle and are thus at most 1/3 apart.

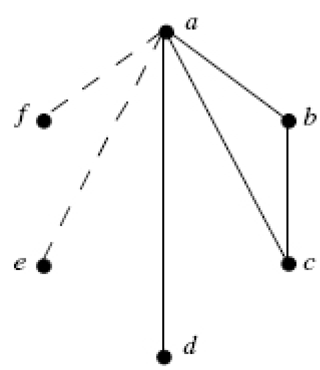
(c) There are n^2 equilateral triangles of sidelength $1/n$ that can be placed in the larger triangle. Hence $m_n = n^2 + 1$.

(The n^2 can be calculated by area considerations or even by adding 1 + 3 + 5+… + 2n - 1.)

8. If the countries represent pigeonholes, n = 6. Since there are 31 = 6 x 5+1 pigeons, one pigeonhole contains at least 5 + 1 = 6 pigeons.
9. The possible heights are 150, 155, 160, 165, 170, 175, 180. Hence there are 7 pigeonholes. Since 27 = 3 x 7 + 6, there were at least 3 + 1 = 4 pigeons in one of the pigeonholes. Hence the maximum value of t is 4. (Five of the remaining pigeonholes could be occupied by 4 and the other one by 3. Clearly we can't force a 5.)
10. 1514 = 116 x 13 + 6. One school had at least 116 + 1 = 117 supporters.
11. Take one of the 17 people at random. Colour the edges joining 2 people by red, blue or green depending on which topic they are corresponding. Now there are 16 edges from the chosen person to the others. By the pigeonhole principle at least one of the colours is used 6 times on these edges. Suppose, without loss of generality, this colour is red. Let these 6 red edges be joined to *a, b, c, d, e, f*. If any pair of *a, b, c, d, e, f* is joined in red we are done. Hence only 2 colours are used between *a, b, c, d, e, f*. We are thus in the party problem situation where we know that there is at least one monochromatic triangle.
12. There is no monochromatic triangle for this party of 5.



13. We know by the Discussion of Problem 3 that we have at least one triangle. Without loss of generality suppose it is solid and joins *a*, *b*, *c*. By the argument of Figures 2.3–2.5, if two of *ad*, *ae* and *af* are solid or *ad*, *ae*, *af* are all broken, then another triangle is forced. Call this argument A. Without loss of generality, this leaves *ad* solid and *ae*, *af* broken.



We get another triangle involving *d* unless *bd* and *cd* are both broken. But then one of *de*, *df* is broken and we apply argument A to get a solid triangle, or both *de* and *df* are solid and argument A again gives another triangle.

14. Since we are dealing with divisibility by 100, we can, without loss of generality, assume our 52 numbers are chosen from 0,1, 2,…, 99. Any extra multiplies of 100 can be discarded. How can the sums of pairs add to 100? We could have

$$\begin{array}{ccccccc} 1 & 2 & 3 & & 48 & & 49 \\ + & + & + & \ldots & + & \text{or} & + \\ 99 & 98 & 97 & & 52 & & 51 \end{array}$$

(The numbers are listed this way so that you can see the pattern.) Take as our 51 pigeonholes the numbers 0, 50 and the pairs (1, 99), (2, 98),…, (49, 51). So by the pigeonhole principle if we choose 52 *distinct* numbers we are forced to choose 2 from some pair (i, 100 — i) or two from 0 or two from 50. These latter two pairs obviously add to a multiple of 100. If we have *i* and 100 — i, then the same thing happens. The only possibilities remaining are that we chose *i* and *i* or 100 — *i* and 100 — i. In both these cases the difference is divisible by 100. **The case of 51.** Now 51 is an extremal case. If we choose the 51 numbers 0,1, 2,…, 49, 50, there is no pair whose sum or difference is 100. **100 replaced by 10.** The question now is, for what (smallest) *n* integers is it true that any pair have sum or difference divisible by 10?

Looks like 7. Is 6 extremal? (The proof is along the same lines as for 100.)

For 10” it is i x 10” + 2. Can you see where this came from?

15. (a) Line up the odd numbers to give sums of 102 where possible. So we get 1, (3, 99), (5, 97),…, (49, 53), 51. There are 24 pairs and the two numbers 1 and 51. By the pigeonhole principle if we choose 27 different odd numbers we are forced to pick a pair (i, 102 - i) for some odd i. (Why don't we have two of the form *i* or 100 - *i* here?)

(b) To pick 26 so that no pair adds to 102 we must choose 1, 51 and one number from each pair (i, 102 - i) for *i* = 3, 5, 7,…, 49. There are 2 choices for each of 24 pairs so there are 2^{24} choices.

16. Any number has a remainder of 0, 1, 2, 3, 4 when divided by 5. If among the 17 chosen numbers there are 5 whose remainders are 0, 1, 2, 3 and 4, then their sum has remainder 0 + 1 + 2 + 3 + 4. Hence their sum is divisible by 5.

Suppose then that among the 17 numbers only 4 of the remainders are possible. By the pigeonhole principle one of these remainders must occur at least 5 times. Choose 5 numbers with the same remainder and their sum is divisible by 5.

Is there more than one generalization here?

17. First find the dimensions of the largest cube that will fit inside a sphere of radius one. Such a cube, of side *a*, will have its main diagonal of length 2, since this is a diameter of the sphere. Hence $3a^2 = 4$ by Pythagoras' Theorem (applied twice). The volume of such a cube is $a^3 = \frac{8}{3\sqrt{3}}$ and there are $\frac{15\times15\times15}{a^3}$ such cubes in the larger cube. On average $\frac{11000a^3}{15\times15\times15}$ points lie in each small cube. But $\frac{11000a^3}{15\times15\times15} = \frac{704\sqrt{3}}{243} > 5$. Hence there is a small cube which contains at least 6 points. It follows that there is a sphere of unit radius which contains at least 6 points.

18. If a_i is the number of games played up to and including the ith day, then $a_1 < a_2 < a_3 < \ldots < a_{77}$. (We have 77 distinct numbers here.) Now consider $a_1, a_2, a_3, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. This is a total of 154 numbers, the largest of which is $a_{77} + 21$.

Now return to the chessmaster. In any 77 days he plays at most 12 x 11 = 132 games. Hence $a_{77} \leq 132$ and so $a_{77} + 21 \leq 153$.

By the pigeonhole principle, with 154 numbers between 1 and 153 at least two must be the same. Hence for some *i* and some *j* we must have $a_i = aj + 21$. So $a_i - aj = 21$. There is therefore a string of days from day $j + 1$ to day *i* when 21 games are played.

The case for 22. Repeating the argument we have ai, $a_2, \ldots, a_{77}$, ai + 22, $a_2 + 22, \ldots, a_{77} + 22$. Further $a_{77} \le 132$, so $a_{77} + 22 \le 154$. We have 154 numbers confined between 1 and 154. If two are equal we are done. Otherwise every number between 1 and 154 occurs.

Hence $a_1 = 1$. But then $a_1 + 22 = 23$. Because of the ordering of the numbers, $a_2 = 2$, $a_3 = 3$, $a_4 = 4, \ldots, a_{22} = 22$. So he plays 22 games in the first 22 days.

Generalise. For what *m* is it true that there is a sequence of days in which he plays precisely *m* games?

There must be a limit to *m* surely? Can *m* be as high as 77?

Try replacing 11 weeks by *w* weeks. Then try replacing 12 games per 7 day period by *g* games.

(This most general form has been worked out by R. Hemminger and B.D. McKay, Integer sequences with proscribed differences and bounded growth, *Discrete Mathematics*, 55, 1985, 255-265.)

19. Repeat the argument of Exercise 18. Suppose she studies *a*i hours up to and including the ith day. Then $a_1 < a_2 < \ldots < a_{37}$. We are also told that $a_{37} \le 60$, so $a_{37} + 13 \le 73$.

By the pigeonhole principle two of the 74 numbers $a_1, a_2, \ldots, a_{37}, a_1 + 13, a_2 + 13, \ldots, a_{37} + 13$ are equal. Hence the result follows.

20. There must be at least 18 people, however, this is far from being easy.

21. Suppose there were only three coloured ribbons. How large would *n* have to be to ensure a triangle in one of the colours?

One way to do this would be to force a situation where six people were holding two colours, because we know this forces a monochromatic triangle. This could be done if one person was forced to be joined to six people by one colour, say red. You see in that case, if a pair of the six were joined by a red ribbon, then we'd have a red triangle. If not, the six people shared white and blue ribbons which forces a red or a white triangle.

The pigeonhole principle then tells us that we need 3 x 5 + 1 = 16 ribbons coming from one person. Hence *n* would need to be 17.

A graph on p. 45 of Capobianco and Molluzzo shows that 16 isn't quite big enough to have monochromatic triangles. So 17 is the smallest number here.

Now go back to the original problem. We have four colours. How big is *n* in order to guarantee a monochromatic triangle? If we use the same approach as in the three colour case we get $n \le 66$. The difficulty is showing that 65 people can't necessarily force a triangle in one colour. Can you do it?

22. 5^{10}.

23. There are four non-zero even digits. Hence we can produce 4^6 of the required numbers.

24. 5^7.

25. 4×5^3 (the first digit can be chosen from 2, 4, 6, 8, the rest from 0, 2, 4, 6, 8).

26. $2^4 + 2^3 + 2^2 + 2 = 30$. (…--,.-…-,---…,---- are not used. Digits and punctuation use five or six signals.)

27. (a) $2^2 = 4$; (b) 8.

28. (a) Ø, {*a*}, {*b*}, {*ab*};

(b) Ø, {*a*}, {*b*}, {*c*}, {*a, b*}, {*a, c*}, {*b, c*}, {*a, b, c*};

(c) 00 → Ø (neither *a* nor *b* used), 10 → {a} (*a* used, *b* not used, 01 → {b} (*a* not used, *b* used), 11 → {a, *b}* (both used) (b) 000 → Ø, 100 → {a}, 010 → {b}, etc.

(d) There are 2^n (binary) sequences of 0's and 1's — a 0 in the *ith* place means element *i* is not in the subset corresponding to that sequence, a 1 in the ith place means element *i is* in.

29. (i) $4 \times 3 \times 2 \times 1 = 24$; (ii) 120.

30. 720.

31. There are 6^6 possible words using the letters F, L, I, G, H, T. Of these 720 use each letter only once. Hence $6^6 - 720 = 45{,}936$ have some letter appearing more than once. (If this worries you try F, L then F, L, I, etc. until you see the pattern.)

32. (i) $\frac{4!}{2} = 12$; (ii) $\frac{4!}{2!2!} = 6$; (iii) $\frac{4!}{3!} = 4$; (iv) $\frac{14!}{4!6!3!} = 840840$.

33. $\frac{7!}{2!3!2!} = 210$.

34. $\frac{12!}{3!3!3!3!} = 369600$.

35. (i) $\frac{11!}{3!3!2!2!} = 277200$; (ii) $\frac{12!}{2!3!2!} = \text{lots}$.

36. (i) $\frac{5!}{2!3!} = 10$; (ii) $\frac{7!}{3!4!} = 35$.

37. $\frac{10!}{4!6!} = 210$.

38. $\frac{n!}{r_1!r_2!r_3!r_4!}$.

39. $\frac{7!}{3!4!} = 35$.

40. $\frac{9!}{5!4!} = 126$.

41. $\frac{10!}{4!6!} = 210$.

42. $\frac{8!}{r!(8-r)!}$. (For $r = 1$: $\frac{8!}{7!} = 8$, $r = 2$: $\frac{8!}{2!6!} = 28$, $r = 5$: $\frac{8!}{5!3!} = 56$.)

43. (i) $\frac{5!}{3!2!} = 10$; (ii) 560; (iii) 999.

44. ${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = {}^nC_{n-r}$.

45. ${}^{26}C_3 = 2600$.

46. ${}^9C_5 = 126$.

47. (i) 1; (ii) 1; (iii) 1; (iv) 1.

48. (i) 1; (ii) n; (iii) $\frac{n(n-1)}{2}$.

49. LHS = 1 + 3 + 3 + 1 = 8 = 2^3 = RHS.

No, because ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 = 15 \neq 2^4$.

However, ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$ *does* equal 2^4. (What's going on here?)

50. (i) LHS = 4 = 3 + 1 = RHS; (ii) LHS = 120 = 84 + 36 = RHS. Generalisation: ${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$.

(For a proof see the Lemma on p. 43.)

51. (i) $1 + 3x + 3x^2 + x^3$;

(ii) $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$;

(iii) $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

52.

(i) $(1+x)^6 = {}^6C_0 + {}^6C_1x + {}^6C_2x^2 + {}^6C_3x^3 + {}^6C_4x^4 + {}^6C_5x^5 + {}^6C_6x^6$;

(ii) $(1+x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + {}^{10}C_3x^3 + {}^{10}C_4x^4 + {}^{10}C_5x^5 + {}^{10}C_6x^6 + {}^{10}C_7x^7 + {}^{10}C_8x^8 + {}^{10}C_9x^9 + {}^{10}C_{10}x^{10} = 1 + 10x + 45x^2 + 120x^3 + 210x^4 + 252x^5 + 210x^6 + 120x^7 + 45x^8 + 10x^9 + x^{10}$.

53. (i) ${}^{17}C_{15} = 136$; (ii) ${}^{22}C_{15} = 170544$.

54. 2^6. (Just let x = 1.)

55. 2^n. This is precisely the number of subsets of a set of size n. You can count these sets other than by using binary sequences. After all, there are ${}^nC_0$ subsets with 0 elements, ${}^nC_1$ subsets with 1 element,…, and ${}^nC_n$ subsets with *n* elements.

56.

(i) $1 + 6a + 12a^2 + 8a^3$;

(ii) $1 - 12b + 54b^2 - 108b^3 + 81b^4$;

(iii) $1 + 20c + 160c^2 + 640c^3 + 1280c^4 + 1024c^5$.

57.

(i) $x^3 + 3x^2y + 3xy^2 + y^3$;

(ii) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$;

(iii) $x^5 - 5x^4y + 10x^3y - 10x^2y^3 + 5xy^4 - y^5$.

58.

(i) $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$;

(ii) $a^3 + 9a^2b + 27ab^2 + 27b^3$;

(iii) $16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$;

(iv) $c^3 - 3c^2d + 3cd^2 - d^3$;

(v) $64c^6 - 576c^5d + 2160c^4d^2 - 4320c^3d^3 + 4860c^2d^4 - 2916cd^5 + 729d^6$;

(vi) $81a^4 + 216a^2 + 216 + \frac{96}{a^2} + \frac{16}{a^4}$.

59. ${}^{15}C_3 = 455$.

60. ${}^{16}C_4 = 1820$.

61. ${}^{c+g-1}C_{c-1} = \frac{(c+g-1)!}{(c-1)!g!}$.

62. (i) Here 1011101111 means *x* =1, *y* = 3 and z = 4. If we consider a sequence of eight 1's and two 0's, each sequence corresponds to a solution of *x* + *y* + z = 8 (and vice-versa). The number of solutions is ${}^{10}C_2 = 45$;

(ii) ${}^{21}C_3 = 1330$.

63. The general strategy with all problems is this "if you can't do them, try doing an easier one". So, what about the divisors of $72 = 2^3 \cdot 3^2$? List them. We get $1(= 2^0 3^0)$, $2, 2^2, 2^3, 3, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, 3^2, 2 \cdot 3^2, 2^2 \cdot 3^2, 2^3 \cdot 3^2$. That makes 12. They are all of the form $2^a 3^b$ where a = 0,1, 2, 3 and b = 0,1, 2. Four choices for *a* and 3 for *b* gives 4 x 3 = 12. So for $2^4 \cdot 5^2 \cdot 7^5 \cdot 11$ we should have 5 x 3 x 6 x 2 = 180 divisors.

64. This is a binary sequence problem like Exercise 59. So we get ${}^{23}C_4 = 8855$.

65. Suppose we just count all the ways of taking three numbers from the set. This is simply A = ${}^{100}C_3$.

But then we've counted all the choices 1,2, *n* for *n* = 3, 4,…, 100 and 2, 3, *m* for *m* = 4, 5,…, 100 and so on. Combine *i* and *i* +1 into one. There are 99 ways of choosing two numbers from the set {1,2,…, *i* - 1, a_i, *i* + 2,…, 99,100} where a_i is the combined element i, i + 1. So there are 99 ways of choosing three numbers, two of which are i and i + 1. Now there are 99 possible values of i, so there are B = 99 x 99 ways of choosing three numbers which contain i and i + 1.

Now A — B *almost* counts the selections of three numbers with no two consecutive numbers included. Unfortunately the B count discards 2, 3, 4 twice — once with i = 2 (**2, 3**, 4) and once with i = 3 (2, **3, 4).** In fact, all triples of consecutive numbers are counted twice except 1, 2, 3 and 98, 99, 100. So let C = 96, the number of triples 2, 3, 4; 3, 4, 5;…; 97, 98, 99. Then A - B + C counts what we're after.

The required number of selections is 161700 - 9801 + 96 = 151995.

66. By the Lemma on p. 43,

$$\begin{aligned}{}^{n}C_{r} &= {}^{n-1}C_{r-1} + {}^{n-1}C_{r}\\ &= {}^{n-1}C_{r-1} + ({}^{n-2}C_{r-1} + {}^{n-2}C_{r}) \text{ again by the Lemma}\\ &= {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + ({}^{n-3}C_{r-1} + {}^{n-3}C_{r}) \text{ by the Lemma.}\end{aligned}$$

Eventually this gives

$${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \cdots + {}^{r}C_{r-1} + {}^{r}C_{r}.$$

But ${}^{r}C_{r} = 1 = {}^{r-1}C_{r-1}$ and the result follows.

(This result is best proved by the principle of mathematical induction — see Chapter 6.)

67. We get ${}^{2}C_{2} + {}^{3}C_{2} + {}^{4}C_{2} + \ldots + {}^{30}C_{2}$ by looking at each equation $x_1 + x_2 + x_3 = i$ for i = 0,1, 2,…, 30. By Exercise 66, this is ${}^{31}C_{3}$ = 4495.

68.

$$k^3 = \frac{ak(k-1)(k-2)}{6} + \frac{bk(k-1)}{2} + ck, \quad \text{for } k \geq 3.$$

$$\therefore k^3 = \frac{a}{6}k^3 + \left(-\frac{a}{2} + \frac{b}{2}\right)k^2 + \left(\frac{a}{3} - \frac{b}{2} + c\right)k.$$

Clearly $a = 6$. Hence $b = 6$ and $c = 1$.

Now $1^3 + 2^3 + 3^3 + \cdots + n^3 = 1^3 + 2^3 + 6({}^{3}C_{3} + {}^{4}C_{3} + \cdots + {}^{n}C_{3}) + 6({}^{3}C_{2} + {}^{4}C_{2} + \cdots + {}^{n}C_{2}) + ({}^{3}C_{1} + {}^{4}C_{1} + \cdots + {}^{n}C_{1})$.

By Exercise 66, ${}^{3}C_{3} + {}^{4}C_{3} + \cdots + {}^{n}C_{3} = {}^{n+1}C_{4}$, ${}^{2}C_{2} + {}^{3}C_{2} + \cdots + {}^{n}C_{2} = {}^{n+1}C_{3}$ and ${}^{1}C_{1} + {}^{2}C_{1} + \cdots + {}^{n}C_{1} = {}^{n+1}C_{2}$.

Hence

$$\begin{aligned}&1^3 + 2^3 + \cdots + n^3\\ &= 1^3 + 2^3 + 6\,{}^{n+1}C_{4} + 6({}^{n+1}C_{3} - {}^{2}C_{2}) + ({}^{n+1}C_{2} - {}^{1}C_{1} - {}^{2}C_{1})\\ &= 1 + 8 + 6\frac{(n+1)(n)(n-1)(n-2)}{4!} + 6\frac{(n+1)n(n-1)}{6}\\ &\quad - 6 + \frac{(n+1)n}{2} - 1 - 2\\ &= \frac{(n+1)n(n-1)(n-2)}{4} + (n+1)n(n-1) + \frac{(n+1)n}{2}\\ &= \frac{(n+1)n}{4} \times \{(n^2 - 3n + 2) + 4(n-1) + 2\}\\ &= \frac{(n+1)^2 n^2}{4}.\end{aligned}$$

See Chapter 6 for an alternative proof.

69. (a) (i) Using the lemma that came before the Binomial Theorem we get

$$\binom{r+n}{n}+\binom{r+n-1}{n-1}+\cdots+\binom{r+1}{1}+\binom{r}{0}$$
$$=\left[\binom{r+n+1}{n}-\binom{r+n}{n-1}\right]$$
$$+\left[\binom{r+n}{n-1}-\binom{r+n-1}{n-2}\right]$$
$$+\cdots+\left[\binom{r+2}{1}-\binom{r+1}{0}\right]+\binom{r}{0}.$$

After cancelling and noting that $\binom{r+1}{0}=\binom{r}{0}$ we get $\binom{r+n+1}{n}$.

(ii) The coefficient of x^k in $(1+x)^{n+m}$ is $\binom{n+m}{k}$.

Now $(1 + x)^{n+m} = (1 + x)^n (1 + x)^m$. To get an x^k term from the right-hand side of this last equation, take x^s in $(1 + x)^n$ and x^{k-s} in $(1 + x)^m$.

The two respective coefficients are $\binom{n}{s}$ and $\binom{m}{k-s}$. Multiplying gives $\binom{n}{s}\binom{m}{k-s}$. As we vary s from 0 to k we pick up all s^k terms on the right-hand side. Hence

$$\binom{n}{0}\binom{m}{k}+\binom{n}{1}\binom{m}{k-1}+\cdots+\binom{n}{k}\binom{m}{0}=\binom{n+m}{k}.$$

(b) (i) Use the technique of (a)(ii). Consider $(1-x^2)^n = (1-x)^n(1+x)^n$. Now the coefficient of x^n on the left is 0 if n is odd and $\binom{n}{n/2}$ if n is even. On the right we get

$$\binom{n}{0}\binom{n}{n}-\binom{n}{1}\binom{n}{n-1}+\cdots+(-1)^n\binom{n}{n}\binom{n}{0}.$$

But $\binom{n}{r}=\binom{n}{n-r}$, so the coefficient is

$$\binom{n}{0}\binom{n}{0}-\binom{n}{1}\binom{n}{1}+\cdots+(-1)^n\binom{n}{n}\binom{n}{n}.$$

Hence the expression of the exercise is 0 if n is odd and $\binom{n}{n/2}$ if n is even.

(ii) The trick here is to notice that

$$(1+x)^{2n}=\binom{2n}{0}+\binom{2n}{1}x+\binom{2n}{2}x^2$$
$$+\cdots+\binom{2n}{2n-1}x^{2n-1}+\binom{2n}{2n}x^{2n}.$$

Putting x =1 then x = -1 into this equation gives

$$2^{2n}=\binom{2n}{0}+\binom{2n}{1}+\binom{2n}{2}+\cdots+\binom{2n}{2n-1}+\binom{2n}{2n}.$$
$$0=\binom{2n}{0}-\binom{2n}{1}+\binom{2n}{2}-\cdots+\binom{2n}{2n-1}-\binom{2n}{2n}.$$

Subtracting we find that

$$\binom{2n}{1}+\binom{2n}{3}+\cdots+\binom{2n}{2n-1}=2^{2n}\div 2=2^{2n-1}.$$

70. We construct a set T containing even more than 1983 integers, all less than 10^5 such that no three are in arithmetic progression, that is, no three satisfy $x + z = 2y$.

The set T consists of all positive integers whose base 3 representations have at most 11 digits, each of which is either 0 or 1 (i.e., no 2's). There are $2^{11} - 1 > 1983$ of them, and the largest is

$$1+3^2+3^3+\cdots+3^{10}=88573<10^5.$$

Now suppose $x + z = 2y$ for some x, y, z $\in T$. The number 2y, for any $y \in T$, consists only of the digits 0 and 2. Hence x and z must match digit for digit, and it follows that $x = z = y$. Hence T contains no arithmetic progression of length 3, and the desired selection is possible.

71.

(i) $1^2+2^2+3^2+4^2+5^2$; (ii) $1^2+2^2+\cdots+n^2$;
(iii) $2^3+3^3+4^3+5^3$; (iv) $3^4+4^4+5^4+6^4$;
(v) $1+\frac{1}{2}+\frac{1}{3}$; (vi) $1!+2!+3!+4!+5!+6!$.

72.

(i) $\sum_{i=1}^{7} i$; (ii) $\sum_{i=1}^{6} 2$
(iii) $\sum_{i=1}^{n} 2i$; (iv) $\sum_{i=1}^{4} 3$
(v) $\sum_{i=2}^{6} \frac{1}{i}$; (vi) $\sum_{i=2}^{7}$
(vii) $\sum_{i=0}^{3} {}^3C_i$; (viii) $\sum_{i=1}^{4}$ 4
(ix) $\sum_{i=0}^{3} {}^3C_i x^i = (1+x)^3$; (x) $\sum_{i=0}^{5}$ 5

73. (i) See Exercise 69(a)(i). (Note that anything can be used to sum with. Here we've chosen *k;* before we've used i.);
(ii) See Exercise 69(a)(ii); (iii) See Exercise 69(b)(i);
(iv) See Exercise 69(b)(ii); (v) See Exercise 66;
((vi) $\binom{3}{0}a^3+\binom{3}{1}a^2b+\binom{3}{2}ab^2+\binom{3}{3}b^3=(a+b)^3$.

74. For all natural numbers n, $(1+x)^n=\sum_{r=0}^{n}\binom{n}{r}x^r$.

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[a]A lemma is a baby theorem. It's a little result not big enough to be a theorem. When it grows to adolescence it might be called a proposition.

Chapter 3
Graph Theory

3.1. Introduction

This excursion into the realms of dot-to-dots aims to give you an introduction to the fast growing world of graph theory. Although Euler kicked things off in 1736 when he tackled the Königsberg Bridge problem, the bulk of research work has been done in the last 50 years or so.

Perhaps the reason for the growth in graph theory is the fact that dots and lines provide simple models for a variety of situations. It is also of some interest to computer scientists. These two facts alone would have got graph theoretical research moving. However, it turns out that there are a host of interesting pure mathematical problems hidden among the dots and lines. Consequently pure mathematicians have taken to graphs like ducks to water.

3.2. Königsberg

In the pleasant summer days of the early 1730's, it was the fashion among the gentry of Königsberg to take Sunday strolls around the bridges of their fair city. (See Figure 3.1; this can be found at http://www-groups.dcs.st-and.ac.uk/~history/Miscellaneous/Konigsberg.html.) They observed after much trial and error that it did not seem to be possible to start at any point in the city and promenade in such a way that they crossed every bridge once and only once.

It took a killjoy mathematician to spoil their fun and tell them that, try how they may, there was no way. Königsberg's bridges were such that it was just not possible to walk across each one once and only once. The mathematician was Leonard Euler (pronounced Oiler) and he published his results in 1736.

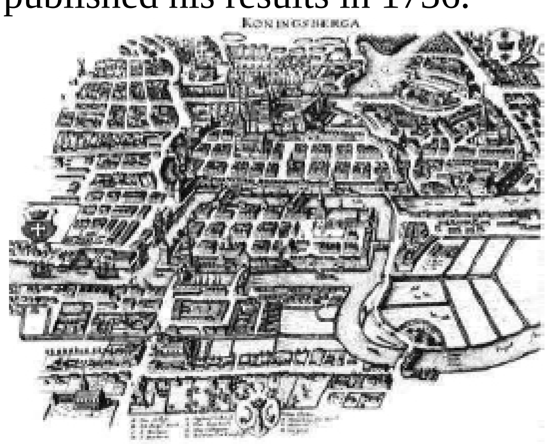


Figure 3.1. The bridges of Konigsberg as they were in 1736.

Euler accomplished his coup by a basic piece of mathematical modelling. What is the essence of the problem? Did the street layout of old Konigsberg matter? Was it important that the bridges were bridges or could they have been planks in rice paddies in China?

If you haven't seen the problem before have a go at it now.

Exercises

1. Show that Euler was right.

2. For the city with bridges as shown in Figure 3.2 show that it is possible to walk around so that each bridge is crossed once and only once.

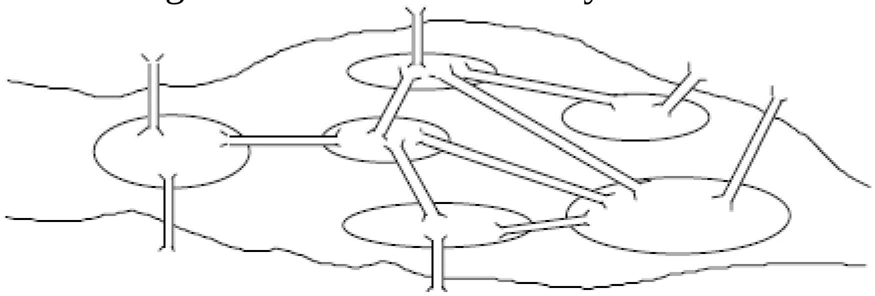


Figure 3.2

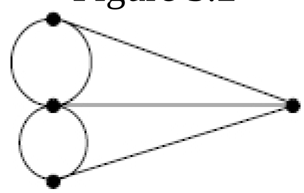


Figure 3.3

Euler reasoned that it didn't matter very much what shape the various land forms were. The only things that were important were (i) that there was land and (ii) that the various bits of land were joined. So he represented the land by dots and joined two dots by a line for every bridge between two corresponding pieces of land. This led him to the picture in Figure 3.3.

Now Euler had to decide if he could pass around his dot and line model of Königsberg, using each line once and only once. So at least the problem had been whittled down to size. If by no other means, he could now go ahead by trial and error. Provided he covered all cases he would come to the conclusion that either there was a suitable route across all the bridges or there wasn't.

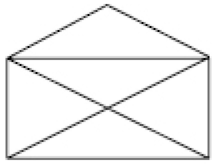
Exercises

3. By systematic use of the lines in Figure 3.3, exhaust all possibilities and show that there is no suitable path.
4. Convert Figure 3.2 to a dot and line model. Is there a suitable path round this model? If so, how many ways are there of getting round using each bridge once and only once?

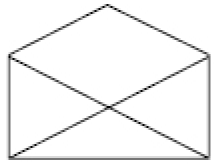
But Euler was made of sterner stuff. Possibly he did use some system originally to prove that there was no route which used each bridge precisely once. However, he soon discovered that he could generalise the Königsberg situation so that no matter how the dots were joined by lines he could tell you whether or not an “each bridge precisely once” tour of the city was on. Let's call such a tour an *Euler tour.*

Exercises

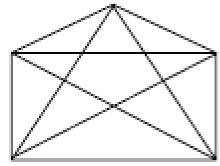
5. Using the models from Figures 3.1 and 3.2, try to come up with some condition on the number of lines at each dot that will tell you when an Euler tour is possible. (Remember that you have to almost always go in and then out of a dot during an Euler tour.)
6. Show that drawing figures (such as those below) so that the pen does not have to leave the paper is equivalent to finding an Euler tour. Hence decide which figures can be drawn without the pen leaving the paper and draw those for which this is possible.



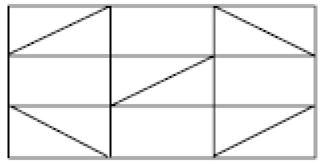
(1)



(2)



(3)



(4)

Have a look at Euler's original paper. (It is translated for you into English in the July 1953 edition of the Scientific American. This is reproduced in "*Mathematics in the Modern World*", Readings from the Scientific American, Freeman and Co., San Francisco, 1968 and in "*Graph Theory 1736-1936*", by N. Biggs, E.K. Lloyd and R.J. Wilson, Oxford University Press, Oxford, 1976.) It's at this stage that he seems to wander a bit before he comes up with the following conclusion.

Theorem 1 (Euler). *A dot and line model contains an Euler tour if and only if*

(1) *all the dots have an even number of lines attached to them, or*

(2) *all but two of the dots have an even number of lines and the Euler tour starts and finishes at the dots attached to an odd number of lines.*

If you think about it, it soon becomes clear that if an Euler tour exists then any dot except possibly the first and the last, has to have an even number of lines associated with it. Clearly you have to go into the dot and out again. That uses up two bridges at that dot. Keep going back to that dot from time to time and each time you'll use two more. If you don't have an even number of bridges you eventually get stuck at a dot.

If the dot you start from is "even", then you first go out and use one line. Every time after that you use two lines. That's an odd number used, so you have to end up at the original dot to complete the Euler tour and every dot is "even". So we've explained the reason for condition (1) of the theorem.

Exercises

7. Argue that if the first dot of an Euler tour is odd, then all the others are even, except the very last dot. This gives condition (2) of the theorem.
8. But Euler's Theorem says "If and only if". In other words, if conditions (1) or (2) hold, then the model has an Euler tour. Have a go at proving this.

Euler's Theorem was published in 1736 in *Commentarii Academiae Scientiarum Imperialis Petropolitanae.* This is a mathematical journal. Mathematicians send their new results to journals. If the editor of the journal thinks the result is of sufficient merit, then it is published. This is how new theorems and techniques are made available to all mathematicians. In 1736, with the solution of the Königsberg bridge problem, Euler published the very first result in Graph Theory.

At this point you might like to go onto the web and do some searching about the city of Königsberg. It has a fascinating history especially as far as bridges are concerned. Many of the original bridges were destroyed in World War II or demolished later (look it up on the web) so at various times Euler would have found a tour for the gentry of the fair city of Königsberg.

3.3. So What is a Graph?

A graph is just a thing with dots and lines. But let's be a bit more formal about this. Let *G* be a *graph*. Then *G* consists of a set of *vertices* VG and a set of *edges* EG, which join vertices of VG. Unlike Euler, we will insist that any two vertices in a graph have at most one edge between them.

Exercises

9. Draw up the table shown below.

VG	number of possible graphs
1	
2	
3	
4	

Complete the table.

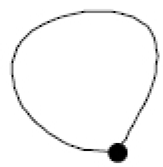


Figure 3.4.

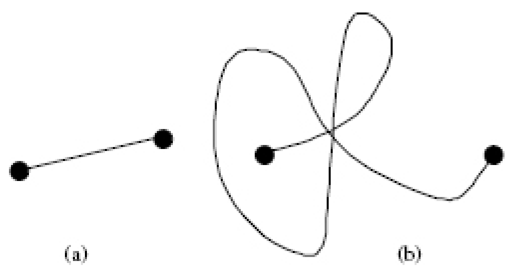


Figure 3.5.

So how many graphs are there with one vertex? The only difficulty here is in deciding whether a single vertex can have an edge drawn from itself to itself as in Figure 3.4.

Such an edge is called a *loop*. In the graphs we are talking about at the moment we will not allow loops. You should therefore have found that there is only one graph with one vertex. He's a very lonely fellow.

So how did you go with two vertices? How many graphs have two vertices? Well there must be at least one — take two lonely fellows and put them together. Are there any more? If you look at Figure 3.5 you will see two candidates. But are they the same? One has a plain, straight old edge. The other has a fairly fancy, up market, curly edge. Now if we are going to take looks into account in this game we're going to find ourselves with an infinite collection of graphs on two vertices — there'll be one for every fancy edge you can dream up.

Let us then decide that the graphs of Figure 3.5 are the same. They consist of two vertices and one edge between the vertices. Such a pair of graphs that are essentially the same, we shall call *isomorphic* (of the same form).

Now there is no reason why there should not be two or more edges joining a pair of vertices. When that happens we say that there is a *multiple edge* between the vertices. We have seen multiple edges already. Euler used them. However we will not let our graphs have multiple edges. Having saidthat, occasionally it is useful to include multiple edges and loops too. The only place this is done in this chapter is in Exercise 72, p. 92. There are therefore only 2 possible graphs, in our sense, with 2 vertices.

Have a go at the three vertex graphs. How many non-isomorphic (different) graphs are there on three vertices?

Clearly there is one graph consisting of three lonely vertices. The next decision to be made is, are the graphs of Figure 3.6 isomorphic or not? When you've made up your mind there, move on to the graphs with 2 edges and then 3 edges.

After all that hard work you should have just 4 graphs. All the graphs of Figure 3.6 are isomorphic. By suitable movements in the plane, you can put the vertices of (a) on top of those of (b). In so doing, the one edge of (a) can be made to sit on the single edge of (b). And, of course, you can do the same for (c).

By now some of you will have seen a pattern. The number of graphs is 1 (for 1 vertex), 2 (for 2) and 4 (for 3). It's obvious that we'll get 8 (for 4). Or is it?

Now it may have occurred to some of the more precocious amongst you that Figure 3.7 actually contains *11* non-isomorphic graphs. This is indeed so. So unfortunately the pattern has broken down.

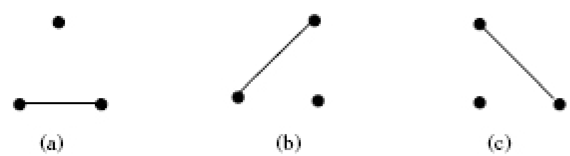


Figure 3.6.

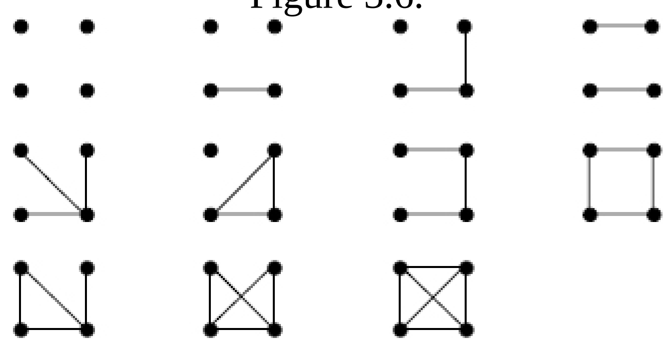


Figure 3.7.

Figure 3.8.

But I can see that many of you have found *more* than 11 non-isomorphic graphs. I'm sorry to say you only think you have. If you check things out carefully you will find some of your extra graphs are isomorphic to some of those in Figure 3.7. For instance the graphs of Figure 3.8 are isomorphic.

And the number of different graphs on *n* vertices is not 2^{n-1}. The actual count of the number of graphs on *n* vertices is, in fact, quite difficult. It relies on an advanced method of counting called *Pólya enumeration.* We won't bother with it here.

Exercises

10. How many graphs are there on 5 vertices?
11. How many of the graphs on 4 or fewer vertices have Euler tours?

Let's have a look at another idea now. The *degree* of a vertex *v*, written as deg *v*, is simply the number of edges the vertex is incident with; the number of lines going into the dot, if you like. The degrees of the vertices of the graph of Figure 3.9 are shown in circles.

This definition opens up a number of possibilities. Explore the ideas of the following exercises.

Exercises

12. For each graph you have drawn on up to 5 vertices add the degrees of all the vertices. What do you notice about the number you get for each graph? In what way is it associated with the graph? Can you formulate a general result?

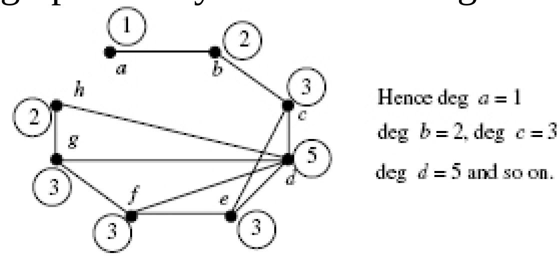


Figure 3.9

13. (a) Are there any graphs with 5 vertices which have vertices of degrees 1, 2, 3, 4 and 5?
    (b) Are there any graphs with 6 vertices which have vertices of degree 0, 1, 2, 3, 4 and 5?
    (c) Are there graphs, all of whose vertices have different degrees?
14. We say that a graph is regular if every vertex has the same degree. It is *regular of degree r* if every vertex has degree *r*.
    (a) Find all the regular graphs on up to 5 vertices.
    (b) How many regular graphs of degree 0 are there on *n* vertices?
    (c) How many regular graphs of degree 1 are there on *n* vertices?
    (d) How many regular graphs of degree 2 are there on *n* vertices?
    (e) Do there exist graphs which are regular of degree 3 on *n* vertices for all values of *n?*
    (f) Do there exist graphs which are regular of degree 4 on *n* vertices for all values of *n?*
    (g) Show that there are graphs which are regular of degree *r* for all positive integers *r.*

If we go back to the ideas of Exercise 12 we find the following result.

Theorem 2. $\Sigma_{v\in VG} \deg v = 2|EG|$.

But first we had better explain the notation. "deg *v*" is easy, we know that is short for the degree of the vertex *v*. And |EG| just means the size of the set EG, that is the number of edges of *G*. So what is Σ ?

In Section 2.5 we introduced the sigma or summation notation. Here we're using Σ to sum again. This time, however, we're summing over a set, rather than over consecutive numbers.

Recall from Chapter 2 that $\Sigma^4_{i=1}\, i = 1 + 2 + 3 + 4$. Suppose now we put $A = \{1, 2, 3, 4\}$. Then $\Sigma_{i\in A}\, i$ is equivalent to $\Sigma^4_{i=1}\, i$. In the former case we sum over all members of

A. That's obviously the same as summing from 1 to 4. So if VG = $\{v_1, v_2, \ldots, v_n\}$, $\Sigma_{v \in VG}$ deg v means deg v_1 + deg v_2 +…+ deg v_n.

Now let's go back to where we were. I wanted to prove a theorem.

Theorem 2. *In any graph G*, $\Sigma_{v \in VG}$ deg v = 2|EG|.

Proof. deg v counts the number of edges incident with the vertex v. As we go round all the vertices of VG adding up the degrees, we count all the edges of G. However we count them each twice, for if $e = uv \in$ EG then we count e once in deg u and once in deg v. Hence

$$\sum_{v \in VG} \deg v = 2|EG|.$$

□

This simple result has a surprising number of uses. For a start we have this corollary. (A corollary to a theorem is a result which follows as a direct result of the theorem.)

Corollary. *In any graph G, there are an even number of vertices of odd degree.*

Proof. Let's divide VG into two sets — the vertices of odd degree, *X*, and the vertices of even degree *Y*. Then

$$2|EG| = \sum_{v \in VG} \deg v = \sum_{v \in X} \deg v + \sum_{v \in Y} \deg v.$$

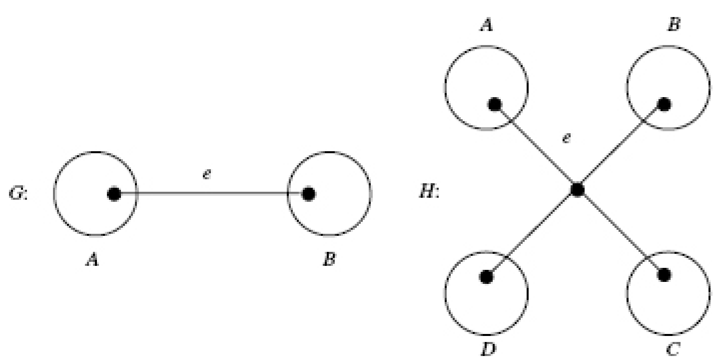
Since 2|EG| and $\Sigma_{v \in Y}$ deg v are both even, then so is $\Sigma_{v \in X}$ deg v. In this last sum, however, each term deg v, is odd. The only way the sum of odd numbers can be even, is if there are an even number of them.

Hence the corollary follows.

□

Exercises

15. (a) Show that in a cubic graph (a graph which is regular of degree 3), the number of vertices is even and the number of edges is divisible by 3.
 (b) Generalise this result to all graphs which are regular of odd degree, *r*.
 (c) If *G* is a regular graph of degree *r* and |EG| is even, what can be said about *r* or *G* or both?



16. (a) The graph *G* above is cubic and |*A*| = |*B*| Is |A| even, odd or can it be either? (The blob for *A* and *B* represents an arbitrary collection of vertices and edges.)
 (b) The graph *H* is regular of degree 4. Describe *H* completely. (If you are finding this difficult, first find the smallest graph which looks like *H*.)
17. (a) What is the smallest graph (i.e., has the fewest vertices) which is regular of degree 2?
 (b) What is the smallest cubic graph?
 (c) What is the smallest graph which is regular of degree 4?

(d) What is the smallest graph which is regular of degree 6?

18. The smallest graph which is regular of degree $n - 1$ has n vertices. In this graph every vertex is joined to every other vertex. This graph is known as the *complete graph* on n vertices and is denoted by K_n. Find $|EK_n|$.

Now find $|EK_n|$ using another approach in which your answer is expressed as a Binomial Coefficient (see Chapter 2).

19. A *bipartite* graph $G = (X, Y)$ is one in which $VG = X \cup Y$, where X and Y are disjoint (have no elements in common), and every edge of G has one end in X and the other in Y.
    (a) Find all the bipartite graphs on 4 and fewer vertices.
    (b) Find all the regular bipartite graphs on 6 and fewer vertices.
    (c) If G is a regular bipartite graph of degree $r > 1$, what can be said about $|X|$ and $|Y|$?
    (d) What is the smallest regular bipartite graph of degree 2?
    (e) What is the smallest regular bipartite graph of degree 3?
    (f) What does the smallest regular bipartite graph of degree r look like?

20. A bipartite graph $G = (X, Y)$ is called a *complete bipartite graph* if every vertex of X is joined to every vertex of Y. If $|X| = m$ and $|Y| = n$, we denote G by $K_{m, n}$.
    (a) Show that in $K_{m, n}$, every vertex of Y is joined to every vertex of X.
    (b) Use the notation $K_{m, n}$ to describe the graphs of Exercise 19(d), (e), (f).
    (c) Find $|EK_{m, n}|$.
    (d) Find $\{\deg v : v \in VK_{m, n}\}$.
    (e) For what values of m, n and t are $K_{m, n}$ and K_t isomorphic?

3.4. Ramsey[a]

Remember the problem in Chapter 2 that went, “Show that at a party of 6 people, there are 3 who are mutual acquaintances or that there are 3 who have never met each other”? That problem is exactly the same as Exercise 21(a).

Exercises

21. (a) Colour all the edges of K_6 either red or blue. Show that there must be a red triangle or a blue triangle.
    (b) Show that the edges of K_5 can be coloured red or blue so that there is no monochromatic triangle.
    (c) Colour the edges of K_{17} either red or white or blue. Show that there must be a monochromatic triangle.
    (d) Is (c) possible if we replace K_{17} by K_{16}

22. Colour the edges of $K_{m, n}$ either red or blue. For what values of m and n do there exist monochromatic triangles?

23. We can think of $K_{2,2}$ as being a “square”.
    (a) Arbitrarily colour the edges of $K_{3,3}$ red or blue. Must $K_{3,3}$ contain a monochromatic square?
    (b) Arbitrarily colour the edges of $K_{n, n}$ red or blue. Find the smallest value of n for which $K_{n, n}$ contains a monochromatic square.

Does this bring back fond memories of Chapter 2? One way of expressing what Ramsey did is the following.

Theorem 3 (Ramsey). *Arbitrarily colour the edges of K_n with any one of r different colours. Let m be some fixed integer. Then for n sufficiently large, K_n contains a monochromatic K_m.*

In the case $r = 2$ and $m = 3$ we know by the 6 people party problem that "n sufficiently large" means just "n ≥ 6". Every party with at least 6 people contains 3 who know each other or 3 who don't.

In the case of $r = 3$ and $m = 3$ we know that n has to be at least 17. So here "n sufficiently large" means "n ≥ 17".

However, in general, Ramsey gave us no clue as to how big "*n* sufficiently large" is. Indeed Ramsey Theory is a very difficult area of graph theory to work in because it is very difficult to find precise values of *n* for even small values of *r* and *m*.

Paul Erdös (who I have talked about before) and George Szekeres have proved the following result. The upper bound here though seems to be gross. For most known values of "n sufficiently large" the Erdös–Szekeres bound is a long way away from the actual value.

Theorem 4 (Erdös-Szekeres). *Arbitrarily colour the edges of K_n, red or blue. If K_n contains a monochromatic K_m then $n \leq {}^{2m-2}C_{m-1}$.*

To finish this section have a go at the following problems. They do not necessarily have anything to do with Ramsey Theory.

Exercises

24. At a party people shake hands as they are introduced. Not everybody necessarily shakes hands with everyone else, of course.
    (a) Show that there have to be two people who shake hands the same number of times.
    (b) Show that the number of people who have shaken hands an odd number of times is even.
25. "There should be three roads on this map", the traveller complained. "I know there's one road from Ashville to Blogsville, another from Blogsville to Crudville and another from Crudville to Ashville."

    "Well they're not all marked in", his wife replied.

    Draw a sketch of each of the possible maps that could have been printed of the three towns. How many such maps are there?

    If Dampville is a fourth town and there is still at most one road between each pair of towns, what is the maximum number of possible roads and how many possible maps could the inefficient publishers make (assuming they were still in business)?

    Suppose now there are *n* towns and at most one road between any pair of them. What is the maximum number of possible roads? How many possible maps could the printers make? How many possible maps are there with *r* roads printed in?
26. My wife and I recently attended a party at which there were four other married couples. Various handshakes took place. No one shook hands with himself (or

herself) or with his (or her) spouse and no one shook hands with the same person more than once.

After all the handshakes were over I asked each person, including my wife, how many hands he (or she) had shaken. To my surprise each gave a different answer. How many hands did my wife shake?

3.5. Euler Tours (Revisited)

Euler started all this off in 1736 by solving the question of when can you go round a graph and use every edge once and only once. The result is surprisingly easy to state.

Theorem 5 (Euler). *A graph G has an Euler tour if and only if* (1) *every vertex has even degree or* (2) *precisely two vertices have odd degree.*

Exercise

27. Perhaps Euler's Theorem as stated above is surprisingly easy to state because it is wrong. What is wrong with it?
 Give an example of a graph that satisfies (1) but does not have an Euler tour.
 Give an example of a graph that satisfies (2) but does not have an Euler tour.
 Actually Euler's Theorem is “almost” right. How can it be fixed?

The problem, of course, is with graphs which have two or more “bits”. There's no way we can find an Euler tour, which after all is a walk around a graph, if we have to be air-lifted from one part of the graph to another. That's tantamount to taking our pencil off the paper. So when does a graph have two or more “bits”?

We'll say a graph G is *connected* if it is possible to get from any vertex of G to any other vertex of G, simply by using edges of G.

In this way the graph in Figure 3.10(a) is connected but that in Figure 3.10(b) isn't. In the latter graph, for instance, there in no way of getting from u to v using only edges of the graph.

We can fix up our problem with Euler's Theorem by inserting the word “connected”. We'll also distinguish between the two types of “tour” by calling the one that ends in a different place from where it started, a “trail”.

Theorem 6 (Euler). (1) *A connected graph has an Euler tour if and only if every vertex is of even degree.*
(2) *A connected graph has an Euler trail if and only if precisely two vertices are of odd degree.*

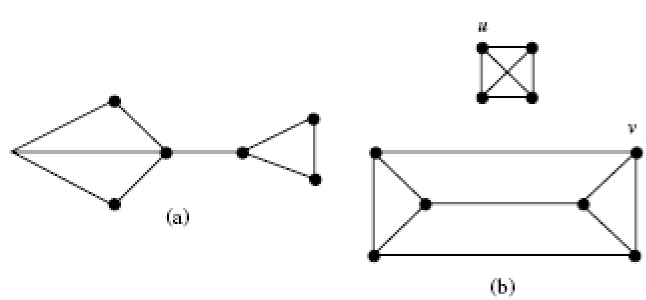


Figure 3.10.

Exercises

28. Find all connected graphs on 5 or fewer vertices. Which of them have Euler tours and which have Euler trails?

29. Show that conditions (1) and (2) of Euler's Theorem can be replaced by "at most two vertices of G have odd degree".

3.6. Knight's Tours

Is it possible to move a knight around a chessboard so that it lands on every square once and only once? Do chessboards have *knight's tours* or *trails*?

First let's recall what a knight is and how it moves. As you can see in Figure 3.11 a knight moves two squares in a straight line and then one square at right angles to this line. For a knight in the middle of a board there are 8 possible moves.

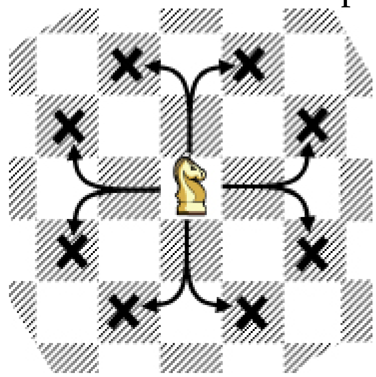


Figure 3.11.

There seems at first sight to be some sort of link between knight's tours/trails and Euler tours/trails. The same sorts of ideas seem to be involved. So let's try to make a graph out of the chessboard. Suppose the squares are vertices. Let's join two vertices if a knight can move from one to the other. Call this the *knight's graph* of the board. But knights and Euler are actually a little different. For the knight we don't have to use *every* possible edge of the knight's graph. We only have to be able to get him to every *vertex.*

This is almost totally unintelligible so let's do an example. Take the 3×3 chessboard. What is the knight's graph of the 3×3 board? We've shown it in Figure 3.12(b). Figure 3.12(a) shows how we've numbered the squares to produce the knight's graph of Figure 3.12(c).

So the knight's graph of a 3×3 board is the union of a cycle on 8 vertices and an isolated vertex.

Exercises

30. Does a knight have a tour on a 3×3 board?
31. Draw the knight's graph of a 4×4 and 5×5 board.
    Do either of these boards have a knight's tour? (One does, one doesn't.)
32. Can you find knight's tours on 6×6, 7×7 and 8×8 boards?
33. Try writing a computer program which will test knight's tours for any $n \times n$ board.
    Does every $n \times n$ board have a knight's tour for $n \geq 5$?
34. If all that is getting too hard, then try rectangular boards instead of square ones.
    (a) Show that a 3×4 board has a knight's tour while a 3×5 board doesn't.
    (b) For what n does a $3 \times n$ board have a knight's tour?

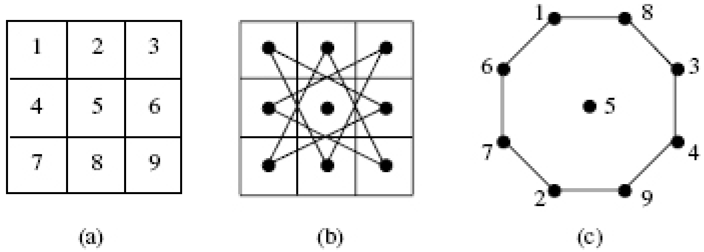


Figure 3.12.

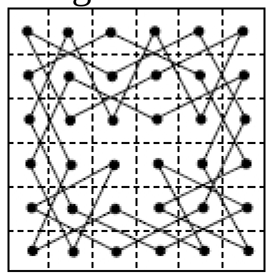


Figure 3.13.

(c) Show that a 4 × 5 board has a knight's tour.

(d) Does every 4 × n board, for $n \geq 5$, have a knight's tour?

We can impose one further restriction on our knight's tour. Make it start and stop at the same place. (Just like the Euler tour.)

Say that a knight's graph of an $m \times n$ board has a *knight's cycle* if it has a knight's tour where the two ends are a knight's move apart or, equivalently, where you can begin and end on the same vertex. In Figure 3.13, a knight's cycle is shown on a 6 × 6 board.

Exercises

35. Investigate the possibility of a knight's cycle on 4 × 4, 5 × 5, 6 × 6, 7 × 7 and 8 × 8 boards.
For which n do $n \times n$ boards have knight's cycles?

36. (a) Which 3 × n boards have knight's cycles?
(b) Which 4 × n boards have knight's cycles?

You should have found that so far, no board with an odd number of squares has a knight's cycle. Why is this so? Or rather, is this always so?

There's one thing about knight's moves that we haven't exploited yet. Have another look at Figure 3.11. There the knight is sitting on a *white* square. All its moves land it on *black* squares. The reverse is also true. A knight on a black square can only move to a *white* square.

So actually knight's graphs are *bipartite* graphs (see Exercise 19). We can divide their vertex set into two sets — one, W say, corresponding to the white vertices and the other, B say, corresponding to the black vertices. In a knight's graph, there are no edges joining any two vertices of W or *any* two vertices of B. There are only edges which join some vertex of W to some vertex of B.

Exercise

37. (a) Suppose mn is odd. Show that an $m \times n$ board does *not* have a knight's cycle.
(b) Suppose mn is even. Is it true that an $m \times n$ board has a knight's cycle?

Clearly there is a lot more that we could do on knight's tours and cycles. Do it. What results do you get? Can you find all of those results somewhere on the web?

3.7. Hamilton

A biography of Sir William Rowan Hamilton appears on the MacTutor site (http://www-history.mcs.st-and.ac.uk/). He was born in Dublin in 1805 and made significant contributions to applied mathematics and noncommutative algebra. Most of his life was spent as Astronomer Royal of Ireland.

When it comes to graph theory he is best known for Hamiltonian paths and cycles. However, if you look into the literature carefully, you will see that we should probably be talking about "Kirkman cycles" after the Rev T. Kirkman who seems to have played with these objects first. You will also find a lot that is of historical interest in the MacTutor site, so I recommend that you log in to it.

Now we've been talking about knight's graphs. But we don't have to stop there with just graphs based on chessboards. Take any old graph and ask if it is possible to move around it so that each vertex is used once and only once, not getting back where you started. If it is, we say that the graph has a *Hamiltonian path.* Such a graph is shown in Figure 3.14.

The sequence of consecutively adjacent vertices 1, 2, 3, 4, 5, 6, 7, 8, 9 gives us the Hamiltonian path.

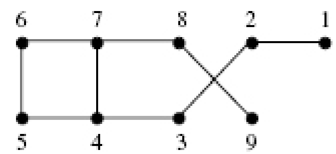


Figure 3.14.

Graphs with a Hamiltonian path whose ends are adjacent have *Hamiltonian cycles*. The graph of Figure 3.14 does not have a Hamiltonian cycle but it would have if there was an edge between 1 and 9.

Exercises

38. Which of the graphs of Exercise 28 have Hamiltonian paths and which have Hamiltonian cycles?

39. (a) For what n does K_n have a Hamiltonian cycle?

(b) For what values of m and n does $K_{m,\, n}$ have a Hamiltonian cycle? (Remember knights.)

40. Prove that a Hamiltonian graph is connected. (A graph with a Hamiltonian cycle is said to be *Hamiltonian.)*

41. What is the smallest connected graphs on 10 vertices which is not Hamiltonian? (Here "smallest" is in terms of edges.)

42. Are all connected regular graphs of degree 2 Hamiltonian?

43. Are all connected regular graphs of degree 3 Hamiltonian?

44. So who was Kirkman anyway?

It's pretty obvious (see Exercise 39) that complete graphs have Hamiltonian cycles. Let's have a look at K_5 for a minute and see what happens to it (Figure 3.15) when we remove a Hamiltonian cycle.

After removing the cycle 1, 2, 3, 4, 5, 1 from K_5 we are left with the cycle 1, 3, 5, 2, 4, 1. Does this work for *every* complete graph?

Exercises

45. (a) Remove a Hamiltonian cycle from each of K_4, K_6, K_7, K_8, K_9. Are the resulting graphs Hamiltonian?

(b) For which n is it true that K_n with a Hamiltonian cycle removed is Hamiltonian?

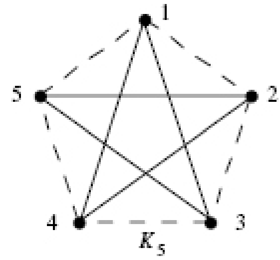


Figure 3.15.

(c) Let $G_m = (K_m - H_m)$. By this I mean K_m with a Hamiltonian cycle H_m removed. From (b) we know that for some m at least, G_m is itself Hamiltonian. Now form $G'_m = G_m - H'_m$. In other words subtract a Hamiltonian cycle from G.

For what m is G'_m Hamiltonian?

46. For what n is it true that K_n is an edge disjoint union of Hamiltonian cycles? In other words, for what n can you start with K_n and consecutively subtract Hamiltonian cycles till you have no edges left?

(As always in this sort of problem you have to first conjecture what the right answer is and *then* you have to prove that your conjecture is true.)

47. So what about the other complete graphs, the ones which *aren't* a union of their Hamiltonian cycles. What do you get left with when you remove the Hamiltonian cycles from the last graph in the sequence G_m, G'_m, etc.?

(a) What do you have left when you remove as many disjoint Hamiltonian cycles as you can from K_4 and K_6?

(b) Conjecture a result for the values of n for which K_n is *not* the union of disjoint Hamiltonian cycles. (What do you have left when you remove as many disjoint Hamiltonian cycles as you can?)

(c) Prove your conjecture. (Or disprove it and then make a better conjecture.)

48. Are there similar results for Hamiltonian paths?

(a) Are there two Hamiltonian paths in K_4 which have no edges in common? Is K_4 the disjoint union of two Hamiltonian paths?

(b) Repeat (a) using K_5, K_6, K_7, K_8, K_9.

49. Repeat Exercise 46 with the word "cycle" replaced by "path".

50. Repeat Exercise 47 with the word "cycle" replaced by "path".

3.8. Trees

At one end of the graph extremes on n vertices are the complete graphs. These have the maximum number of edges possible for a graph on n vertices. At the other end of the scale, are graphs with no edges. These are just collections of vertices — the graphs which are regular of degree zero.

Exercise

51. Somewhere between K_n and graphs of degree 0 are the connected graphs on n vertices.

(a) What are the smallest connected graphs (with fewest edges) on 3 vertices?

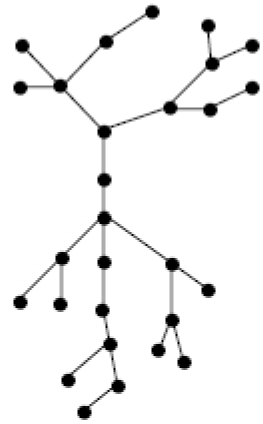


Figure 3.16.

(b) What are the smallest connected graphs on 4, 5 and 6 vertices?

(c) How many edges are there in the graphs you found in (a) and (b)?

(d) Conjecture a relation between the number of vertices and edges in smallest connected graphs.

(e) Prove your conjecture or go back to (d).

Connected graphs on *n* vertices which have the smallest possible number of edges are called *trees*. This is because they look like trees (see Figure 3.16). Admittedly pretty bare trees but with a bit of imagination you can see branches and roots.

3.9. Planarity

Printed circuits are fundamental to today's electronics industry. In simplified form, printed circuits can be thought of as graphs. The points of a printed circuit are the vertices of the graph and two points with current carrying copper between them are joined by an edge in the graph.

Now printed circuits have a very important property — no two of the joins cross. If they did, then current wouldn't flow as it was supposed to. The printed circuit would fail.

The graphs we get from printed circuits therefore, also have the property that no two edges cross. Such graphs are called *planar* graphs because they can be drawn in the *plane* so that no two edges cross.

Exercises

52. Show that all graphs on four or fewer vertices are planar.
53. Which graphs on 5 and 6 vertices are *not* planar?
54. An artist is having trouble constructing a wall hanging. The concept is to use six different pieces of material of varying lengths that are to be sewn to backing material at each end. The artist wants to limit the number of places where the material is sewn to the backing material to four. No two ends of each piece of material are to be sewn to the same place. Can she do this without any of the pieces of material overlapping?
55. The artist's next project is to use ten pieces of material and five sewing points. Can she do this without any of the pieces of material overlapping?
56. Now the artist wants to suspend nine pieces of cloth between two rods. Three pieces of cloth must meet at three different points on each rod. Must two pieces of cloth overlap?
    (Ask an older person what this has to do with gas, electricity and water.)
57. (a) Which complete bipartite graphs are planar?
    (b) Which complete graphs are planar?

(c) Which trees are planar?

There's a sense in which there are only two non-planar graphs, despite the fact that you should have discovered an infinite collection of non-planar graphs in Exercise 57.

We'll call a graph H, a *homomorphism* (or *homomorphic form)* of another graph G if we get H from G by adding vertices of degree 2 arbitrarily on various edges of G. From the example of Figure 3.17 it looks as if H has caught the measles.

Obviously there are an infinite number of homomorphic forms of any graph because we can add as many spots (vertices of degree 2) as we like.

One other idea is needed before we can reveal all about non-planar graphs. A graph S is a *subgraph* of G if we get the S from G by removing some edges and some vertices. Figure 3.18 gives subgraphs S_1, S_2, S_3 of the graph G.

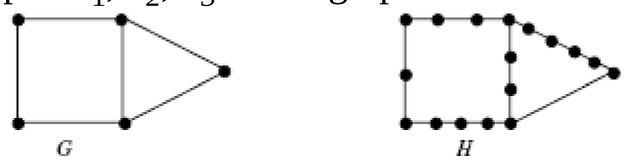


Figure 3.17.

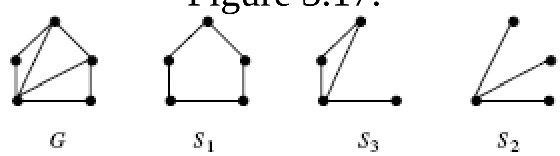


Figure 3.18.

Exercises

58. Draw five homomorphic forms of K_4.
59. Draw all subgraphs of K_4.
60. Show that if the number of vertices of G is less than or equal to n, then G is a subgraph of K_n.

    Which graphs are subgraphs of $K_{m,\,n}$?

The following theorem due to *Kasimir Kuratowski*, a Polish mathematician, says that K_5 and $K_{3,3}$ are the only (in some sense) non-planar graphs.

Theorem 7 (Kuratowski). *G is non-planar if and only if it contains a subgraph which is a homomorphic form of* K_5 *or* $K_{3,3}$.

So if we want to check to see whether a graph is planar or not, all we have to do is to check to see whether it has a measly form of K_5 or $K_{3,3.}$

Is the graph J of Figure 3.19, planar or not?

If you delete the edges joining 1 and 2, 2 and 3, 4 and 5 and 5 and 6, you get $K_{3,3.}$ So J contains a subgraph which is a homomorphism of $K_{3,3}$ and hence is non-planar.

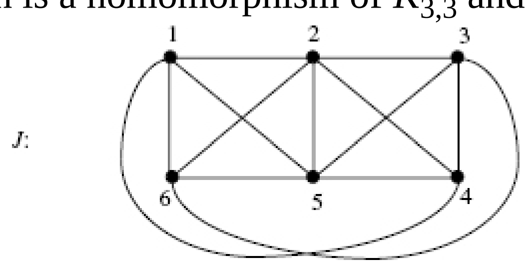


Figure 3.19.

But J also contains a subgraph which is a measly form of K_5. To see this delete the vertex 3 and the edges 23, 34, 35, 36. The graph we have left would be K_5 except that there is no edge between 4 and 6. So add back the edges 34 and 36. The graph we've got now is a homomorphism of K_5 with the vertex 3 being the only "measle".

Exercises

61. Use Kuratowski's Theorem to find all the connected graphs on 6, 7, 8 and 9 vertices which are non-planar.
62. Is the graph of Figure 3.20 planar or not?
63. Find at least six regular graphs on 10 vertices which are non-planar.
64. Who was Kuratowski and who named K_5 and $K_{3,3}$ after him?

As well as having vertices and edges, planar graphs have faces. Look at the planar graph in Figure 3.21(a). This graph has four faces. These are the regions enclosed by the edges of the graph. No face has an edge cutting across it.

Notice that we call the region which is "outside" all the edges of the graph a face too (this is F_1 in Figure 3.21(a)). This is because we can turn the graph inside out if we like and make F_1 an interior face of the same graph. We show this way of looking at things in Figure 3.21(b). In this drawing of the graph of Figure 3.21(a), F_3 has become the outside face.

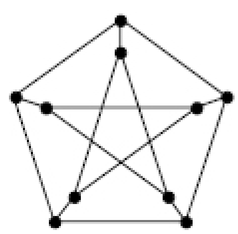


Figure 3.20.

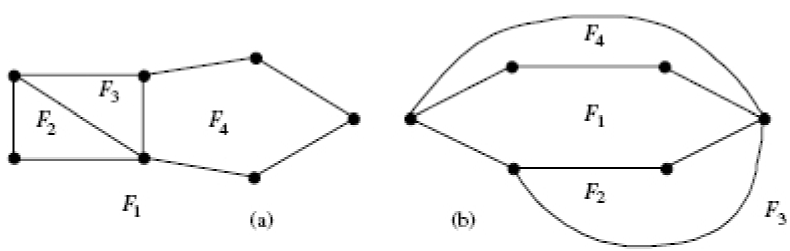


Figure 3.21.

Exercises

65. Show that the faces F_2 and F_4 can also be drawn as the outside faces of the graph of Figure 3.21(a).
66. Draw K_4 as a planar graph. How many faces does it have?
67. Do a thorough investigation of all the connected planar graphs on 5 or fewer vertices. For each graph find an equation linking v, e and f, where v is the number of vertices, e the number of edges and f the number of faces of each graph. Show that there is one such equation which holds for all these graphs.
(At this point it is well worth reading Imre Lakatos (1976). *Proofs and Refutations.* Cambridge: Cambridge University Press. You may sympathise with some of the discussion.)

3.10. The Four Colour Theorem

As part of everybody's mathematical culture they should know about the Four Colour Theorem. I've put together a quick run through the ideas here and show how to prove the Five Colour Theorem. Any of you who get keen on the topic should follow this up further by looking on the web or browsing in a library.

The proof of the Four Colour Theorem turned out in the end to be very similar in nature to that of the Five Colour Theorem — it took longer to prove because there were more difficult cases and a new idea was needed.

What are these theorems all about? Well in 1852, a student, Francis Guthrie, who should have been doing his Geography homework, started colouring in the counties of England. To his surprise he discovered that he only needed four colours to colour the counties so that no two counties with a common boundary had different colours.

It turns out that this result holds not just for the counties of England but for any collection of regions that anyone can ever dream up. So if you divide the plane up into any number of regions, if you then colour in all the regions so that no two regions with a common boundary have the same colour, then you only need *four* colours to complete the job. (If the regions only have a single point in common, then they don't have to have different colours.) This result is known as the Four Colour Theorem and it took over125 years from the time it was posed to its solution in 1976 by Appel and Haken, two mathematicians who were working at the University of Illinois in the USA.

I'll give a quick proof here of the Five Colour Theorem. As I said earlier this proof shows the main ideas used in the proof of the Four Colour Theorem. The main steps involved are as follows. First we simplify the sort of maps involved. Then we change the map/region colouring problem into a graph/vertex colouring problem. Thirdly we show that a few configurations must always occur in these graphs. Finally we work on these configurations to get the result.

Five Colour Theorem. *The regions of a planar map can always be coloured with five or fewer colours so that no two regions with a common boundary have the same colour.*

Proof. Step 1. We can first of all assume that only three regions meet at any one point. To see this look at the situation of Figure 3.22.

In *A* we have five regions meeting at the point *P*. Replace *P* by a region to give *B* of Figure 3.22. It is now true that if we can five-colour the regions of *B*, we can do the same for *A*.

(Check this out. When you've done that, you'll know that we have shown that we only have to prove the Five Colour Theorem for maps where precisely three regions meet at a point.)

Step 2. We now make a graph from the map as follows. Put one vertex in every region of the map. Join two vertices if the regions they are in have a common boundary. (See Figure 3.23.) Note that the outside region gets a vertex too.

The graph we've got is called the *dual graph* of the map.

We now note two things. First, colouring the faces of the map so that no two faces with a common boundary have the same colour, has a graph equivalent. That equivalent is colouring the vertices so that no two adjacent vertices have the same colour. Our aim then, will be to try to colour the vertices of the dual graph in five (or fewer) colours so that no two adjacent vertices have the same colour. (Check this out for some small maps.)

Figure 3.22.

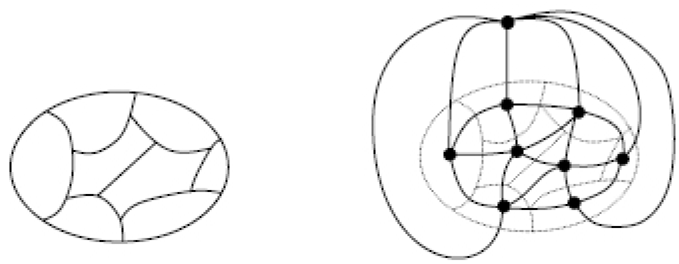


Figure 3.23.

The second thing we notice is that each face of the dual graph is a triangle. This is because each face of the dual graph encloses the point where three regions meet. (Satisfy yourself that this is OK.)

So each map produced after Step 1 gives a planar graph whose faces are triangles. Call such graphs *triangulations.*

If you think about it (and you should) the reverse is true. Every triangulation gives rise to a map of the type discussed in Step 1. (That is, exactly three regions come together at a point.)

As a result of the above discussion we now only need to prove that we can colour the vertices of triangulations with five or fewer colours so that no two adjacent vertices have the same colour.

Step 3. Here we are going to show that *any* triangulation has a vertex of degree 2, or degree 3, or degree 4, or degree 5.

To do this we work from Euler's formula $v — e + f = 2$. (Remember this came about as a result of the work in Exercise 67.) As a first step, let d_i be the number of vertices of degree i. Since every vertex is on at least one triangular face in a triangulation, then $d_1 = 0$. (Think about it. There are no vertices of degree 1.)

We now find expressions for v, e and f in terms of the d_i. Finding v is straightforward. We have $v = d_2 + d_3 +\ldots= \Sigma_{i\geq 2}\, d_i$. Now by Theorem 2, p. 69, $£_{v\in VG}$ deg $v = 2e$. But we can arrange degrees into groups with the same value. So

$$\begin{aligned} 2e &= \deg v_1 + \deg v_2 + \cdots + \deg v_n \\ &= (2 + \cdots + 2) + (3 + \cdots + 3) + \cdots + (s + \cdots + s). \end{aligned}$$

Now there are d_2 lots of 2 in the “2” bracket, since d_2 is the number of vertices of degree 2. Similarly there are d_3 lots of 3 in the “3” bracket, d_4 lots of 4 in the “4” bracket and so on. Hence

$$2e = 2d_2 + 3d_3 + \cdots + sd_s = \sum_{i\geq 2} id_i.$$

To pick up f we recall that all the faces of a triangulation are triangles. Count the edges around each triangle. This gives a tally of 3f because each of the f triangles has three edges. But in this count every edge has been counted twice — because every edge is on *two* triangles. This means that $3f = 2e$. Since we already knew that $2e = \Sigma_{i\geq 2}\ id_i$, we now have $3f = \Sigma_{i\geq 2}\ id_i\ id_i$. Putting all this into Euler's formula we get

$$\sum_{i\geq 2} d_i - \frac{1}{2}\sum_{i\geq 2} id_i + \frac{1}{3}\sum_{i\geq 2} id_i = 2.$$

$$\therefore \quad 6\sum_{i\geq 2} d_i - 3\sum_{i\geq 2} id_i + 2\sum_{i\geq 2} id_i = 12.$$

$$\therefore \quad \sum_{i\geq 2} (6-i)d_i = 12.$$

$$\therefore \quad 4d_2 + 3d_3 + 2d_4 + d_5 = 12 + \sum_{i\geq 6} (i-6)d_i.$$

Now we're in business. The right-hand side of this last equation is positive. In fact since $i \geq 6$, the right-hand side of this equation is at least 12. Hence $4d_2 + 3d_3 + 2d_4 + d_5 \geq 12$, which means that at least one of d_2, d_3, d_4, d_5 is strictly positive.

We conclude that every triangulation has either $d_2 > 0$, $d_3 > 0$, $d_4 > 0$ or $d_5 > 0$. This means that every triangulation contains a vertex of degree 2 or 3 or 4 or 5.

Step 4. The smallest triangulation is a triangle. That is a graph on three vertices each of whose vertices are of degree 2. The vertices of these are obviously colourable in five or fewer colours.

Now suppose we systematically have worked through all the triangulations on 3, 4, 5, and so on vertices and found them to be five-colourable. So now we've got to the graphs on *n* vertices and we're testing them.

From Step 3 we know that a triangulation *T* on *n* vertices has a vertex of degree 2, 3, 4, or 5.

If the triangulation *T* has more than three vertices and a vertex of degree 2, then part of it is as shown in Figure 3.24(a). Removing *v*, the edges incident with it and one of the edges u_1, u_2 we obtain the triangulation *T'*. As this has fewer vertices than *T*, *T'* can be coloured in five or fewer colours. Assign the colours that are assigned to the vertices of *T'* to the same vertices of *T*. Since u_i, u_2 take two colours, there is a colour that can be given to *v* to extend the five-colouring of *T'* to a five-colouring of *T*.

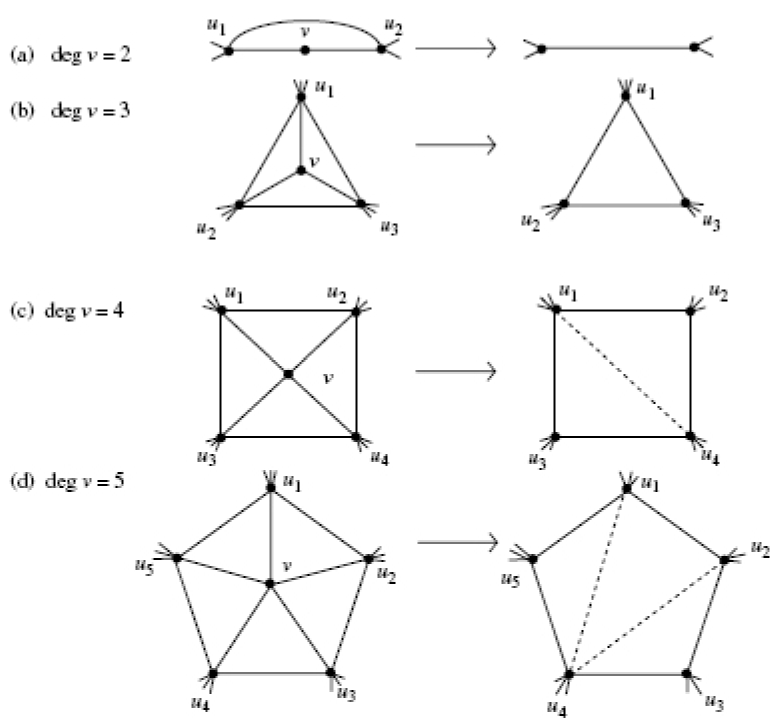


Figure 3.24.

Suppose then that T has a vertex of degree 3 but no vertex of degree 2. We show this situation in the first part of Figure 3.24(b). If we remove v and the edges joining it to u_1, u_2, u_3 in T we get a triangulation T'. This triangulation is on $n - 1$ vertices and we know that this is five-colourable. Colour the vertices of T as they were in T'. At worst u_1, u_2, u_3 take up 3 of these colours. So we can put a different fourth colour on the vertex v to give a five-colouring of T.

So now suppose T has no vertex of degree 2 or 3 but it does have a vertex v of degree 4. In Figure 3.24(c) we remove v from T and add the dotted edge u_1u_3 to make T' a triangulation. But T' is on $n - 1$ vertices and is five-colourable. If we colour T in the same way that T' is coloured, we find that u_1, u_2, u_3, u_4 take up at most four of the five colours. Hence there is a free fifth colour for v. We can therefore five-colour T.

At this stage we've coped with all triangulations on n vertices which have a vertex of degree 2, 3 or 4. The remaining triangulations must have a vertex v of degree 5 by Step 3. Remove v from T and add edges $u_1 u_4$, $u_2 u_4$ to give the triangulation T' indicated in Figure 3.24(d).

If we are lucky, when we colour T' in five colours and repeat this colouring on T, only four colours will be used on the vertices u_1, u_2, u_3, u_4, u_5. This leaves a fifth colour spare for v and gives a five-colouring of T.

But what happens if all *five* colours are used on u_1, u_2, u_3, u_4, u_5 First we cry a lot. Wait though. Suppose vertex u_i is coloured in colour c_i ($i = 1, 2, 3, 4, 5$). Just think about the bits of T' that are coloured in c_1 and c_3.

One of two things now happens, either u_1 is not connected to u_3 by a path alternatively coloured c_1 and c_3, or u_1 *is* connected to u_3 by a path alternatively coloured c_1 and c_3.

In the former case (Figure 3.25(a)) change c_1 for c_3 and c_3 for c_1 on the vertices connected to u_3 in the part of T' coloured c_1 and c_3. We've now neatly cut down the number of colours used on u_1, u_2, u_3, u_4 to four. This gives us the free fifth colour to use on v. We've five-coloured T!

But the bad news is that there might be a path from u_1 to u_3 alternatively coloured c_1 and c_3. In this case if we swap c_1 and c_3 we only swap c_1 for c_3 on u_1 and c_3 for c_1 on u_3. Thus there's been no gain.

In this case look at Figure 3.25(b) and especially at the part of T coloured with c_2 and c_4. There can't be a c_2 to c_4 path going from u_2 to u_4. If there were it would have to cut the $c_1 - c_3$ path going from u_1 to u_3. This cut couldn't be at a vertex (the vertex would have to be simultaneously coloured c_1 or c_3 and c_2 or c_4). This cut couldn't be an edge (T is *planar*). So there can't be a $c_2 - c_4$ path going from u_2 to u_4.

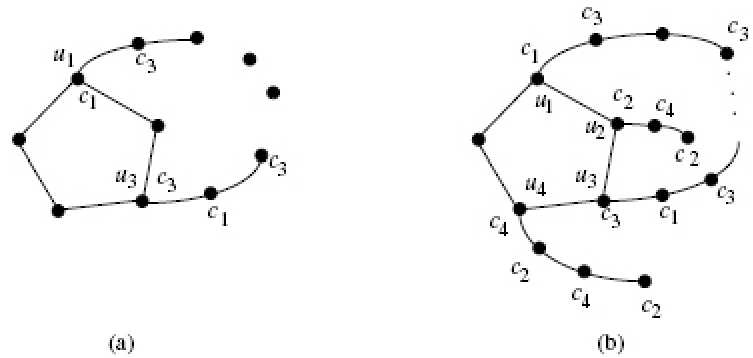


Figure 3.25.

Ah! Now we're in business. Since there is no $c_2 - c_4$ path from u_2 to u_4, interchange the colours c_2 and c_4 starting at vertex u_4. This has the effect of reducing the number of colours used on the vertices u_1, u_2, u_3, u_4 to four and we slap the fifth colour on v to complete the five-colouring of T.

Note that this argument for the degree 5 case first appeared in the false proof of the Four Colour Theorem by Kempe in 1879. As a result the argument is called the Kempe Chain argument. It can sometimes be used to help solve other colouring problems.

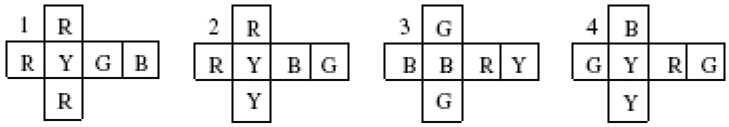
Exercises

68. Why can't exactly the same proof be used to prove the Four Colour Theorem?
69. Appel and Haken's proof of the Four Colour Theorem was essentially the same as that of the Five Colour Theorem. They first showed that some configuration had to be present in every triangulation (see Step 3 above). They then showed how to four-colour a triangulation on n vertices assuming it could be done for those on $n - 1$ (see Step 4) but they had to make heavy use of a computer at this stage.
    (a) How many configurations did Appel and Haken use in their proof? (We used four in the Five Colour Theorem proof.)
    (b) How many hours of computer time did Appel and Haken require?
    (c) Why have people been concerned about Appel and Haken's proof?
    (You will need to consult the web or a book to be able to answer these questions.)

3.11. Some Additional Problems

We pose the following harder graph problems with no hints or apologies. *Exercises*

70. In a group of nine people, one person knows two of the others, two people each know four others, four each know five others, and the remaining two each know six others. Show that there are three people who all know one another.
71. A certain bridge club has a special rule to the effect that four members may play together only if no two of them have previously partnered one another. At one meeting fourteen members, each of whom has previously partnered five others, turn up. Three games are played, and then proceedings come to a halt because of the club rule. Just as the members are preparing to leave, a new member, unknown to any of them, arrives. Show that at least one more game can now be played.
72. "Instant Insanity" is a game consisting of four cubes whose faces are coloured as shown below. (B is for Blue; G for Green; R for Red and Y for Yellow.)



The aim of the game is to build a tower by putting the cubes one on top of another so that the four resulting faces (each four times the side of one of the cubes) is a different colour. Solve this problem using graph theory. (You will need to use loops and multiple edges.)

73. Each of 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.

74. There are *n* couples at a party.

(a) In how many ways can they combine in pairs for dancing?

(b) In how many ways can they dance if no husband and wife dance together?

(c) What has this to do with derangements? (You may have to find out what derangements are.)

75. Those of you with an Australian bent might like to know that a squatter decided to leave his land to his five sons when he died. But since his sons had all become swagmen, he insisted that the land would go to four local troopers unless the sons were able to divide it into five regions in such a way that

(i) each pair of regions had a section of boundary fence in common, and

(ii) each region consisted of a simple, connected, piece of land.

Who got the squatter's land when he died?

The planetoid Doughnut is roughly Earthshape except that, being a doughnut, it has a large hole through its centre. The landlady of Doughnut decided to leave her land, which consisted of all the surface of the planetoid, to her five daughters after her death. She imposed the same conditions on her land ((i) and (ii) above) as had the squatter, except that if the conditions were not fulfilled, the land was to become the property of her four favourite Martian tenants.

Who inherited the planetoid when the landlady died?

(By way of explanation, a squatter is a farmer who may not necessarily have acquired his farm legally, a swagman is a tramp and a trooper is a policeman. The reference here is the song “Waltzing Matilda”.)

76. Let n be a positive integer and let $A_1, A_2, \dots, A_{2n+1}$ be subsets of set B. Suppose that

(a) each A_i has exactly $2n$ elements,

(b) each $A_i \cap A_j$ $(1 \le i \le j \le 2n + 1)$ contains exactly one element, and

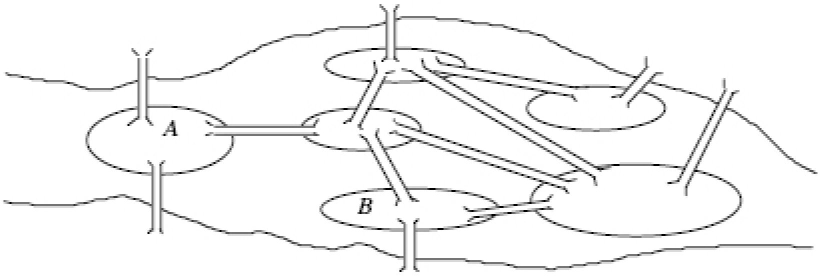
(c) every element of B belongs to at least two of the A_i.

For which values of n can one assign to every element of B one of the numbers 0 and 1 in such a way that each A_j has 0 assigned to exactly n of its elements? (IMO, 1988.)

3.12. Solutions

1. This can be done by systematic trial and error.

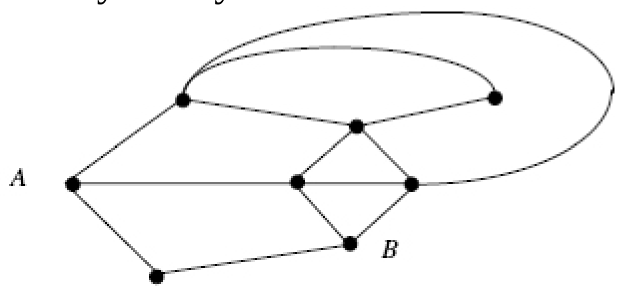
2.



This can be done in more than one way. However you will always have to start and finish on islands *A* and *B*. Why?

3. This is really what you did in Exercise 1.

4.



I think there might be as many as 486 different ways of doing this. Can anybody prove me right (or wrong)?

5. Is it possible to find an Euler tour if there are more than 2 dots with an odd number of lines?

   Is it possible to find a dot and line model with only one dot with an odd number of lines?

6. If dots are placed where more than 2 lines meet then we're back to the dot and line model. If this has an Euler tour then the figures can be drawn without taking the pen off the paper, and vice-versa.

   (1), (3) and (4) can be done. (2) has three places where an odd number of lines meet.

7. If the first dot is odd, then after you have left it the first time, every time you come back to it you must go out again. Hence you always use an odd number of its lines. As you go through other dots you must always go in and out except the last dot. Hence all dots, except the first and last are even.

8. We've proved so far that if an Euler tour exists, then (1) or (2) holds. Now we have to prove that if (1) or (2) hold, then there is an Euler tour. A proof can be found in most basic graph theory books.

9. Read the discussion after Exercise 9 in the text. This should confirm your results.

10. 34.

11. 7.

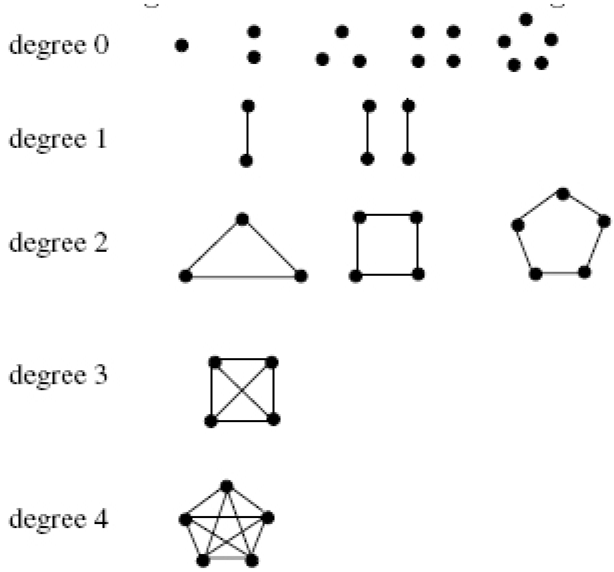
12. The number is even. So it is always 2t. What does *t* equal in each case?

13. (a) No. If a graph has *n* vertices it can have no vertex of degree bigger than *n* - 1.

    (b) No. Suppose a graph has *n* vertices and one vertex *v* has degree *n* — 1. Then every other vertex is joined to *v*. So every other vertex has degree at least one.

    (c) No. Suppose *G* has *n* vertices. If they all have different degrees, then by (a) the degrees are 0,1,…, *n* - 1. But by (b), *G* cannot have a vertex of degree *n* - 1 as well as a vertex of degree 0.

14. (a)



(b) 1.

(c) 1 if n is even, 0 otherwise.

(d) Now that's a tough one! It's not just one. (On eight vertices there are 2 graphs which are regular of degree 2.) Try to develop a formula.

If you know anything about partitions of numbers, the answer is the number of partitions of a number in which every part is at least 3. (You can find out above these by looking on the web.)

(e) No. It's certainly not possible for $n < 4$. But *even* for $n \geq 4$, there is a restriction on n. What is it?

(f) Yes, provided $n \geq 5$. To prove this, construct one on 5, then 6, then 7, then 8 and 9 vertices. Now use the fact that any number greater than 9 can be written as $5t + u$, where $u = 5, 6, 7, 8$ or 9.

(g) Take $r + 1$ vertices and join them all up.

15. (a) By the Corollary there must be an *even* number of vertices of degree 3. Since there are *only* degree 3 vertices, |VG| must be even.

Now $2|EG| = \Sigma_{v \in VG} \deg v = 3|VG|$. Since the right-hand side is divisible by 3, then so is the left-hand side. Hence |EG| is divisible by 3.

(b) Once again |VG| is even, by the Corollary. By the Theorem $2|EG| = r|VG|$. Since r is odd, r divides |EG|.

(c) r is odd and |VG| is divisible by 4, or r is even.

Note. You can now answer the second question I asked in the solution of Exercise 5. In 14(e), it is now clear that n is even.

16. (a) Remove e from G. This leaves one vertex in A (and B) of degree 2 while the rest are of degree 3. By the Corollary there must be an even number of vertices of degree 3. Hence $|A|$ is odd. (Similarly $|B|$ is odd.) It is easy to construct graphs like this with $|A|$ and $|B|$ odd.

(b) Remove the edge e. Then A is a graph with some vertices of degree 4 and one vertex of degree 3. But one is odd, so A, and hence H, does not exist.

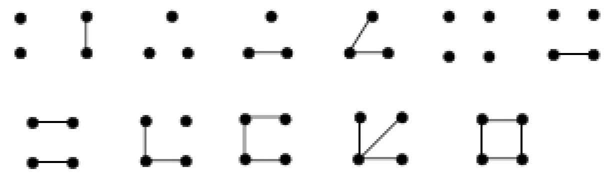
17. By Exercise 18 you will see that the answers are (a) K_3; (b) K_4; (c) K_5; (d) K_7.

18. By Theorem 2, $2|EK_n| = (n-1)|VK_n| = (n-1)n$. Hence $|EK_n| = \frac{n(n-1)}{2}$.

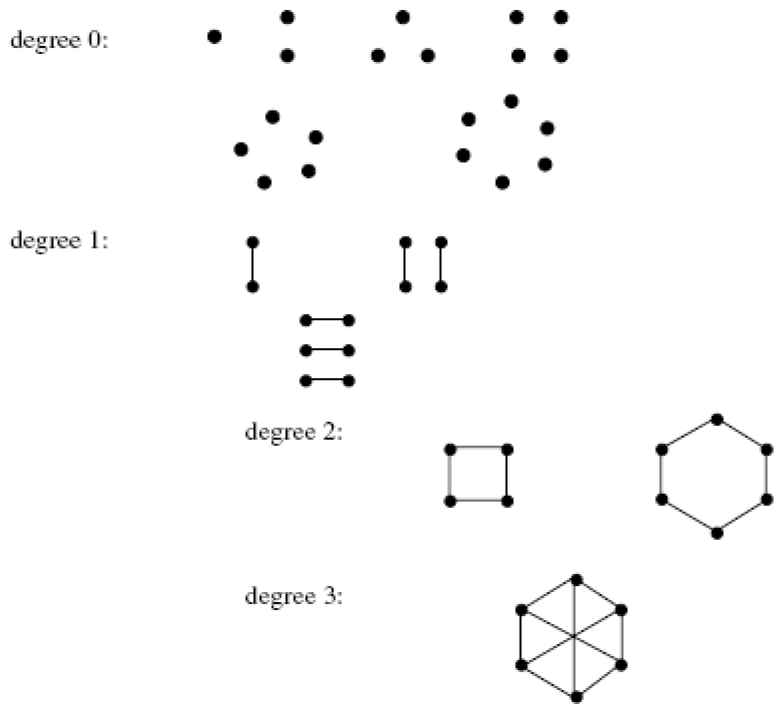
Alternatively, for each pair of vertices there is one and only one edge. Hence $|EK_n|$

is the number of ways of choosing 2 vertices from n. This is just ${}^nC_2$ (by Chapter 2). (Note that ${}^nC_2 = \frac{n(n-1)}{2}$.)

19. (a)



(b)



There are no others.

(Which of the following are true?

(i) All graphs which are regular of degree 0 are bipartite.

(ii) All regular graphs of degree 1 are bipartite.

(iii) All regular graphs of degree 2 are bipartite.

(iv) There are no bipartite graphs which are regular of degree 4.)

(c) If G is regular of degree r and bipartite, then there must be $r|X|$ edges from X to Y and $r|Y|$ edges from Y to X. These are the same edges so $|X| = |Y|$.

(d) The "square" on 4 vertices.

(e) The graph on 6 vertices in (b).

(f) By (c), $|X| = |Y|$. For a vertex in X to have degree r, $|Y| \geq$ r. If $|X| = |Y| = r$ and every vertex of X is joined to every vertex of Y, then we have the required graph, $K_{r,\,r}$.

20. (a) This is probably obvious given that every vertex of X is joined to every vertex of Y.

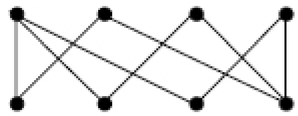
(b) Exercise 19(d) $K_{2,2}$ (e) $K_{3,3}$ (f) $K_{\mathrm{r},\,\mathrm{r}}$.

(c) For all $x \in X$, deg $x = |Y|$. Hence $|EK_{m,\,n}| = |X||Y|$.

(d) For $m \neq n$ the set is $\{|X|, |Y|\} = \{m, n\}$. For $m = n$, the set is $\{|X|\} = \{m\}$.

(e) Let $G = K_t = K_{m,\,n}$. Since $G = K_t$ any two vertices in G are adjacent. Since $G = K_{m,\,n}$ then these two vertices are in different sets X, Y. Hence $|X| = |Y| = 1$ and $t = 2$, $m = n = 1$.

21. (a) The argument is in Chapter 2.
  (b) See Chapter 2, p. 50, or use Figure 3.14 with the solid lines red and the dotted ones blue.
  (c) See Chapter 2, p. 50, Solution to Exercise 11.
  (d) No. Guess where you'll find the answer?
22. $K_{m,\ n}$ has no triangles monochromatic or otherwise. In fact $K_{m,\ n}$ contains no pentagon either. Why not? Generalise.
23. (a) No. It's not too hard to colour $K_{3,3}$ so that there are no monochromatic squares.
  (b) $K_{4,4}$ may not have a monochromatic square. (See the graph below.)



But it can be shown that $K_{5,5}$ does, no matter how the edges are coloured.
24. (a) See Exercise 13(c).
  (b) By Corollary p. 70.
25. There are 8 possible maps.
  With 4 towns there are 64 possible maps.
  With n towns there are $\frac{1}{2}n(n-1)$ possible roads, this is the number of ways of choosing 2 towns from the n towns on the map.
  Any particular road is either marked or not marked. There are therefore 2 possibilities for each road. The total number of possible maps is therefore 2 × 2 ×…× 2, where there are $\frac{1}{2}n(n-1)$ twos. Hence we have $2^{\frac{1}{2}n(n-1)}$. possible maps.
  We have to choose r roads from the $\frac{1}{2}n(n-1)$ roads available. Hence there are ${}^{\frac{1}{2}n(n-1)}C_r$ such maps. This is exactly $\frac{m(m-1)\cdots(m-r+1)}{r(r-1)\cdots 3\cdot 2\cdot 1}$, where $m = \frac{1}{2}n(n-1)$.
26. Since there are 10 people and no person shakes hands with their spouse or themselves, the maximum number of shakes for any person is 8.
  I observed that everyone else had shaken hands a different number of times. Therefore the number of handshakes is

8,7,6,5,4,3,2,1,0, h,

where h is the number of times I shook hands.
  Spouses don't shake hands. So the 8 shakes and 0 shakes must belong to spouses. If we could remove this couple and their handshakes from the party we'd have a party with

6,5,4,3,2,1,0, h – 1

number of handshakes.
  Spouses don't shake hands. So the 6 shakes and 0 shakes must belong to spouses. If we could remove this couple and their handshakes from the party we'd have a party with

4,3,2,1,0, h - 2

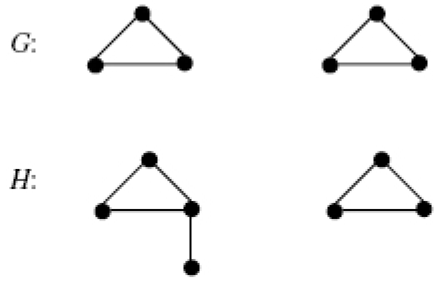
number of handshakes.
  Repeating the argument reduces the party to

2,1,0, h-3

number of handshakes. But here the “2” must have shaken with the “1” and the “h - 3”. Hence h - 3 = 1 or h = 4.

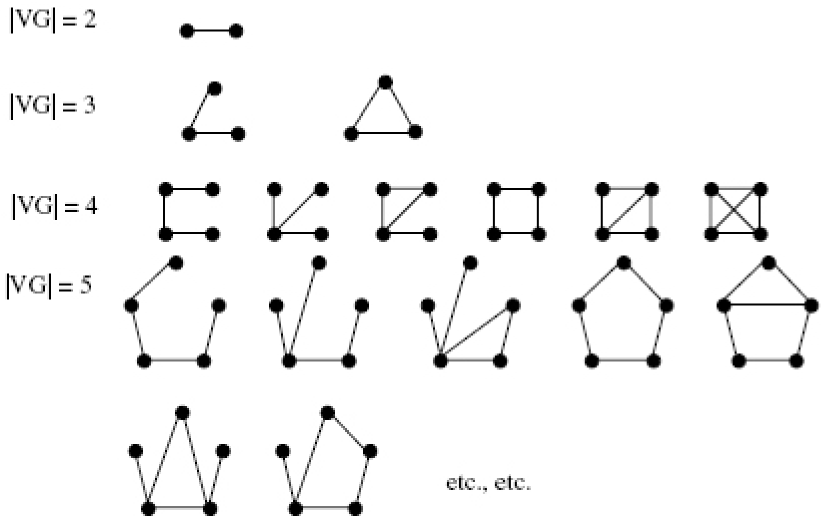
But the 2 and 0 must be spouses so my wife is the “1” here. Following this back to the original party shows that my wife shook 4 hands (as did I).

27.



For the graph *G*, there is no Euler tour. Likewise for *H*. We clearly must have a graph which is all connected together.

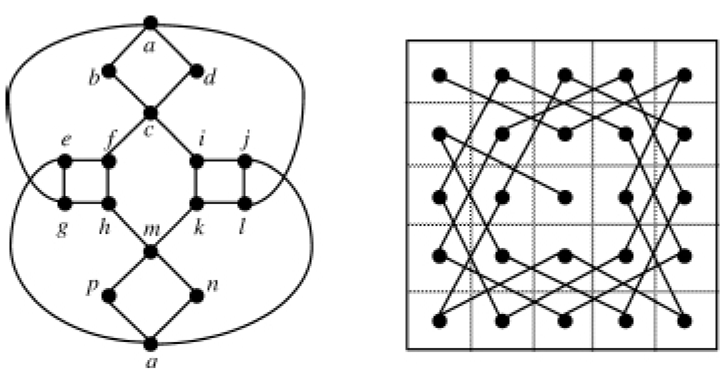
28.



29. Suppose at most two vertices have odd degree. Then by the Corollary, the graph has either 0 or 2 vertices of odd degree. These are precisely cases (1) and (2) of Euler's Theorem.

30. No. If the knight starts at square 5 it can't get to any other square. If the knight starts at any other square it can't get to 5.

31.



The knight's graph of the 4 × 4 board is shown above to the left. *A* knight at *b* and *d* can only move to *a* and *c*, so one of *b* and *d* must be an end of the tour. (Similar restrictions apply at *p* and *n*, so one of *p* and *n* is the other end of the tour.) If the knight leaves the square *a*, *b*, *c*, *d* from *a*, by symmetry we can assume it goes to *l*. Since the edges *cf* and *ag* cannot now be used, a knight at *g has* to go to *e* and *h*, as does the knight when it gets to *f*. This gives a small cycle in the knight's tour which is not possible.

A similar argument applies if the knight leaves the square *a, b, c, d* from *c*.

One knight's tour of the 5 × 5 board is shown above right. Is this the only possible tour of the 5 × 5 board?

32. Yes (if you try hard enough).

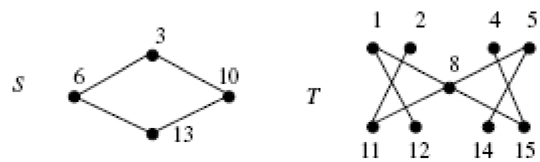
33. The program will vary depending on your machine and the language used. Can you prove there is a knight's tour for every $n \times n$ board with $n \geq 5$? (Rewrite the program to find all possible knight's tours.)

34. (a) The 3 × 4 board is just trial and error.

(Use the 3 × 4 knight's tour to show there is a 3 × 7 knight's tour. Use symmetry.)

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

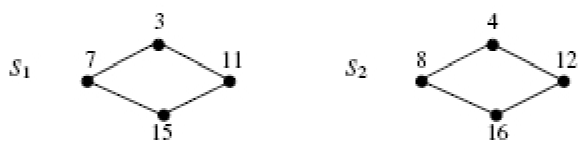
From the 3 × 5 board we see that part of the knight's graph is the graph S and part is T.



As the graph is not connected there can be no knight's tour.

(b) $n = 4$ and $n \geq 7$. For $n = 6$, there are two disconnected parts of the knight's graph which looks like S.

These are shown below.



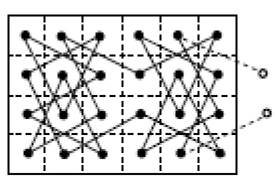
Hence there is no knight's tour on a 3 × 6 board.

We can use symmetry on the 3 × 4 board to get a 3 × 7 tour. For $n > 7$, a proof that $3 \times n$ is possible, needs some thought.

(c) Straightforward.

(d) 4 × 6 is O.K. Using the pattern shown in the diagram, the dotted lines show how the knight's tour can be extended to a 4 × 9 board.

In this same way we can do a 4 × 12 board and so on.



This argument shows that $4 \times 3m$ boards have knight's tours. What about the other $4 \times n$ boards?

(Flushed with all this success try $5 \times n$ and so on. This could make a nice project.)

35. By Exercise 31 there is no knight's tour of a 4 × 4 board so there can be no knight's cycle.

If there is a knight's cycle of a 5 × 5 board then when the knight gets in to a corner square it can come out in only one way. However these moves all join up to give a cycle 1, 8, 5, 14, 25, 18, 21, 12, 1. Hence if a knight starts on one of these squares it can never leave them. If it starts at any other square it can never get to square 1.

A 6×6 knight's cycle is shown in Figure 3.13. Is this the only one?

You can't find a knight's cycle on a 7×7 board.

There are lots of knight's cycles on an 8×8 board.

So what is your conjecture for a 9×9 board, and then for an $n \times n$ board?

Why can you never get a knight's cycle on a board with n odd?

Can you find a general method of construction for n even?

36. (a) No knight's tour means no knight's cycle, so $n \neq 3, 5, 6$, immediately. For $n = 4$, the cycle 1, 7, 9, 2, 8, 10, 1 is forced. Hence there is no knight's cycle for $n = 4$.

Ah, now, $3 \times$ n, for n odd is not possible for the same reason that odd $\times$ odd boards don't have knight's cycles. That's cut it down a bit.

What about $n = 8$ then? This does not have a knight's cycle. Is there a knight's cycle on any $3 \times n$ board for $n \geq 10$ and even?

(b) There is no $4 \times n$ board with a knight's cycle. (See R. Honsberger, *Mathematical Gems*, MAA, Providence, 1973, p. 145 or try the web.)

37. (a) For mn odd we have a different number of black and white squares, so clearly there is no knight's cycle.

(b) Try $m = 4$.

38. For $|VG| = n > 1$, $K_{1,n}$ is the only graph which has neither a Hamiltonian path nor a Hamiltonian cycle except for three graphs on 5 vertices.

39. (a) For all $n \geq 3$.

(b) For all $m = n \geq 2$.

40. We can go from any u to any v via the Hamiltonian cycle.

41. There are many such graphs. The simplest is obtained by joining 1 to 2 to 3 to 4 to 5 to 6 to 7 to 8 to 9 to 10.

42. Yes.

43. No. See Figure 3.21, p. 84. This graph is frequently referred to as the Petersen graph.

44. You can find him on the web. Try MacTutor.

45. (a) Yes, except for K_4.

(b) For all $n \geq 5$. This requires a little work though.

(c) For all $n \geq 6$.

46. Is it possible for n even? So prove it's true for n odd.

47. (a) 2 distinct edges and 3 distinct edges, respectively.

(b) A set of distinct edges.

(c) Aye, there's the rub!

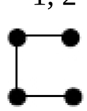
48. (a) Yes.

(b) Yes for K_6 and K_8. No for the others.

49. This time it's even n that works.

50. When you remove Hamiltonian paths from K_n for n odd, you get…distinct edges? How many?

51. (a) $K_{1,2}$.

(b)  and longer lists for 5 and 6.

(c) Two in (a); three for the four-vertex graphs; four for the five-vertex graphs; five for the six-vertex graphs.

(d) |EG| = |VG| - 1.

(e) Mathematical Induction is useful right here. (See Chapter 6.)

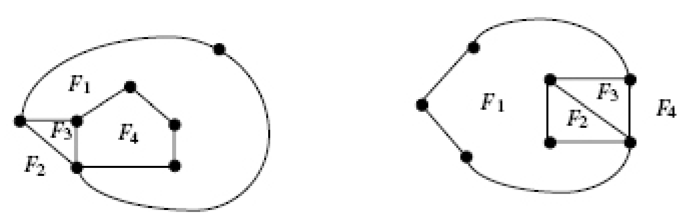
52. Just check them all out.
53. K_5, K_6, K_{33} and every subgraph of K_6 where you can "see" a K_5 or a $K_{3,3}$.
54. There is a planar drawing of K_4. So the answer is yes.
55. No. K_5 is not planar.
56. Yes. $K_{3,3}$ is not planar.
57. (a) Assume $m \leq n$, then K_{mn} for $m = 1$, 2 and any n.

(b) K_n for $1 \leq n \leq 4$.

(c) All trees are planar.

58. I leave this to your imagination.
59. This is the same as all graphs on four vertices.
60. Obvious? Let $G = (X, Y)$ be a bipartite graph with $|X| \leq m$ and $|Y| \leq$ n. Then G is a subgraph of $K_{m, n}$.
61. There are quite a few. Start with K_5 and $K_{3,3}$ and add vertices till the required number is found. Then add edges at will. (How can you be sure not to get two isomorphic graphs this way?)
62. Non-planar. You should be able to find a measly form of K_{33} by deleting a few edges.
63. There are plenty to choose from. One way to go is to successively add sets of 5 edges to the graph of Figure 3.20.
64. Kuratowski you can find on the web but maybe not the fact that Frank Harary was the inventor of the notation K_5 and $K_{3,3}$ in honour of K.K.

65.



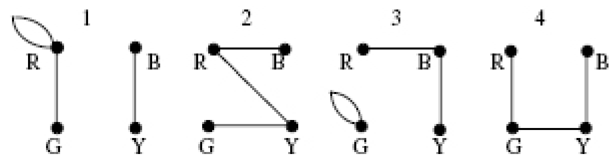
66. 4 faces.
67. $v - e + f = 2$.
68. Things go well till you get to the vertex of degree 5. Around this vertex at least one colour must be repeated. There is no guarantee that Kempe chain arguments will get rid of any particular colour.
69. See the web or a library with graph theory books.
70. Think of the 9 people as vertices of a graph and join those who know each other. Let u be a vertex with degree 6 and let u be adjacent to u for i = 1, 2, 3, 4, 5, 6. If v and w are the remaining vertices we know that one of the u has degree at least five. Hence u_i knows u_j for some j and we have our triangle (or three people who all know one another).
71. This problem relies on a theorem by the Hungarian mathematician Turán which says that if G contains no K_3, then $|EG| \leq |EB_v|$, where $v = |VG|$ and B_v is a bipartite graph

in which each part has as close to v/2 vertices as possible. It turns out that $|EB_v| = {}^{v-k}C_2 + {}^{k+1}C_2$, where $k = \lfloor \frac{n}{2} \rfloor$ (the integral part of 1/2n).

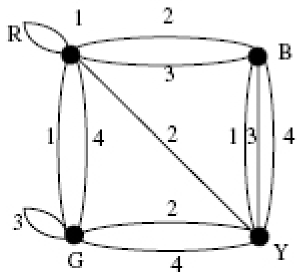
In the problem let G be the graph whose vertices are the people in the club and whose edges represent people who have not yet partnered each other. If this graph has no triangle, then when the new member arrives no game is possible.

Show that G contains 50 edges while the corresponding B_v contains only 49. Hence G contains a triangle by Turán's theorem.

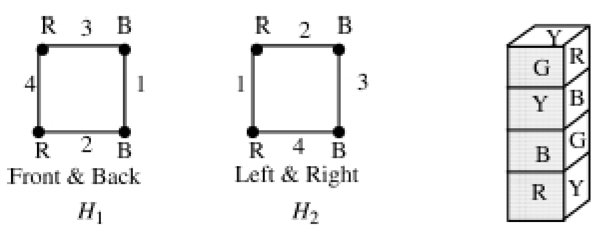
72. First produce a graph for each of the cubes. The vertices of the graph are the four colours and two vertices are adjacent if the corresponding colours are on opposite faces. These graphs are shown below.



Now put these graphs together to form the graph H below.



Now each solution of the puzzle has two faces of each colour on each of the two pairs of opposite sides of the tower of cubes, so the required solution is found by finding two edge-disjoint subgraphs H_1 and H_2 of H which are (i) regular of degree 2 and (ii) contain precisely one edge of each numbered cube. The graphs H_1 and H_2 then represent the colours appearing in the front and back and left and right side of the tower. The solution can be read off from these subgraphs.



73. If one point is joined to 4 points x_1, x_2, x_3, x_4 by blue lines, then since there are no "blue" triangles, all the 6 lines x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4 are red and the question is solved.

Assume then that each point is joined to at least 5 other points by red lines. There cannot be exactly 5 red lines at each point since $\frac{1}{2}(9 \times 5)$ is not an integer.

Therefore some point b is joined to 6 other points by lines which are all red, say by_1, by_2, by_3, by_4, by_5, by_6. At least 3 of the 5 lines y_1y_2, y_1y_3, y_1y_4, y_1y_5, y_1y_6 have the same colour. Let these be y_1y_2, y_1y_3, y_1y_4. If this colour is blue, then b, y_2, y_3, y_4 are four points with all 6 lines red.

If y_1y_2, y_1y_3, y_1y_4 are all red, then because one side at least of the triangle $y_2\,y_3\,y_4$ is red, we can assume y_2y_3 is red. Then b, y_1, y_2, y_3 are four points with all 6 lines red.

74. (a) In $K_{n,n}$ how many sets of n edges can be chosen so that no two edges in the set have a common end vertex?

Starting at any vertex we have a choice of n edges. The next vertex yields a possible $n - 1$ further edges, and so on. Hence we have n! pairings.

(b) Label the vertices of the two parts of $K_{n, n}$ $\{1,2,\ldots, n\}$ and $\{1', 2',\ldots, n'\}$, where i is married to i'. Then we want the number of assignments i to j' so that $j' \neq i'$ for any i.

(c) The number produced in (b) is the number of ways a postman can deliver n letters to n houses so that no house gets a letter addressed to it. This can be done in $n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!})$ ways. (This is known as the number of *derangements* of n things.)

75. For the squatter's sons produce the graph with the regions as vertices and two vertices adjacent if the regions have a common boundary. The graph is K_5 which is non-planar. So condition (i) and (ii) cannot both be fulfilled. Lucky troopers.

K_5 may be non-planar but it can be drawn on a doughnut (with a hole) so that no two edges cross. The daughters get the planetoid.

76. Draw a graph G with vertices A_i such that two vertices are adjacent if the corresponding sets A_i and A_j have an element in common.

Suppose $a_1 \in B$ and a_1 is also in *more* than two of the A_1. Then since deg $A_i = 2n$, A_i must contain one element a_2 which is not in A_j for any $j \neq i$. But this contradicts (c). Hence every element of B belongs to precisely two of the A_i.

Clearly $G = K_{2n+1}$. This is the disjoint union of n Hamiltonian cycles. If n is even, assign 0 to half of these Hamiltonian cycles and 1 to the other half. This gives the required assigned to elements of B by giving 0 to the elements defining the edges in the “0” Hamiltonian cycle.

If n is odd, and the assignment were possible, then consider the subgraph of G formed by the edges labelled 0. This graph has an odd number of vertices all of degree n, which is itself odd. No such graph exists.

Hence n must be even.

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[a]You might find this article of some interest: http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Ramsey.html.

Chapter 4
Number Theory 1

4.1. What is It?

Number Theory is about the theory of numbers. And the numbers we will talk about here are largely the *natural numbers*. N = {1, 2, 3, 4,…}. Any results which have anything to do with N are results in Number Theory. In this chapter we mainly look at four aspects of number: divisibility; the Euclidean Algorithm; Fermat's Little Theorem; and Arithmetic Progressions. However, Number Theory does cover more than this as we shall see in a later book.

To give some idea of the sort of problems that arise, the rest of this section will be devoted to problems about numbers that I think you might be able to manage without too much help. Have a go anyway and see how far you get.

Exercises

1. Each asterisk represents a digit. What are the two numbers being multiplied together in the following?

```
          * * *
          * 2 *
          -----
          * * *
      * * * * 0
      * 8 * 0 0
    -----------
    * * 9 * 2 *
```

2. In the following addition each letter stands for a different digit. Find the digits corresponding to each letter, given that there are no zeros.

```
      C R O S S
  +   R O A D S
  -------------
    D A N G E R
```

3. The number *739ABC* is divisible by 7, 8 and 9. What values can *A*, *B*, *C* take?
4. Now *M* = AB4 and *N* = 4AB. Further, *N* is as much bigger than 400 as *M* is smaller than 400. What is the number *M*?
5. (a) Find the five digit numbers whose digits are reversed on multiplying by 4.
   (b) Find all five digit numbers whose digits are reversed on multiplying by 9.
   (c) Find all five digit numbers whose digits are reversed on multiplying by 8.
6. Let *n* be a five digit number and let *m* be the four digit number formed from *n* by deleting the middle digit. Find all $\frac{n}{m}$ for which is an integer.
7. Find all positive integers with first digit 6 such that the integer formed by deleting this 6 is $\frac{1}{25}$ of the original integer.
8. An absent-minded bank teller switched the dollars and cents when he cashed a cheque for me. After buying a 5 ¢ stamp I discovered I had twice as much left as the original cheque. How much did I write the cheque out for?
9. Find the unique solution to the following long division.

```
                * * 8 * *
      * * * | * * * * * * * *
              * * *
              -----
                * * * *
                  * * *
                  -----
                    * * * *
                    * * * *
                    -------
```

10. In the division below all the 2's are shown. Find the other missing digits.

```
                2 * * * *
    * * 2 | * * * 2 * * 2
              * 2 *
              -----
                2 * 2
                * * 2
                -------
                * * * *
                * 2 * *
                -------
                      * * 2
                      * * 2
                      -----
```

11. A lone goose met a flock of geese flying in the opposite direction. He cried “Hello 100 geese!”

The leader of the flock replied “We aren't 100. If you take twice our number and add half our number and add a quarter of our number and finally add you. Then we *are* 100.”

How many geese in the flock?

12. Again each letter stands for a different digit in the following *addition.*

```
    H O C U S
    P O C U S
  -----------
  P R E S T O
```

13. Find all integer solutions of $2x^2 + 2xy + y^2 = 25$.
14. Prove that 121 is a square no matter what base it is written in.
15. What is the largest prime factor of one million minus one?
16. Find all integer solutions of $(x^2 — 3x + 1)^{X+1} = 1$.
17. How many integers satisfy the equation $x \cdot x^{\frac{3}{x}} = \frac{x^x}{x}$?
18. Let *N* = 1234567891011…998999 be the natural number found by writing the integers 1,2,3 999 in order. What is the 1988th digit from the left?
19. Prove that given any six consecutive numbers there is one which has no factor in common with any of the others.
20. What two digit number is twice the product of its digits?

4.2. Divisibility by Small Numbers

One of the main problems in number theory is to find out what numbers are divisible by what. This is not as easy as it sounds. There is an important type of code that depends on the fact that finding the factors of large numbers is computationally hard. Let's find out the easy tests for divisibility by small numbers.

On a first run through this section, the proofs are not important. What you do need to know though is how to test a number to see if it is divisible by 2, 3,4, 5,…

Divisibility by 2. This of course is easy. If a number is even it's divisible by two and vice-versa. To see whether a number is divisible by two or not just check that its last digit is even.

Given any two consecutive numbers it's clear that precisely one of them must be even.

Divisibility by 3. It always surprises me that if I add up all the digits of a number and if the sum is divisible by 3, then the original number is divisible by 3.

Have a look at this. Now 327 = 3 x 109, so clearly 327 is divisible by 3. But 3 + 2 + 7= 12 which is itself divisible by 3.

Similarly

246150 = **3** x 82050 and 2 + 4 + 6+1 + 5 + 0=18 = **3** x 6.

Why is this so? Well it's quite simple really — when you know how.

Suppose we have a number $a_n a_{n-1} a_{n-2}$ ∠ ∠ ∠ $a_1 a_0$. For instance, if the number was 246150, then $a_0 = 0$, $a_1 = 5$, $a_2 = 1$, $a_3 = 6$, $a_4 = 4$ and $a_5 = 2$. We can write 246150 as 2 x 10^5 + 4 x 10^4 + 6 x 10^3 + 1 x 10^2 + 5 x 10 + 0. In the same way $N = a_n a_{n-1} a_{n-2} \ldots a_1 a_0$ can be written as $a_n \times 10^n + a_{n-1} \times 10^{n-1} + \ldots + a_1 \times 10 + a_0$.

Now notice that 10 = 3 x 3 + 1, 10^2 = 3 x 33 + 1, 10^3 = 3 x 333+1 and so on. In fact every power of 10 can be written as 3 times a string of 3's plus 1. Check it for yourself. Actually every power of 10 is of the form $3ki + 1$, where ki is a string of i 3's.

So

$$\begin{aligned} 246150 &= 2(3 \times 33333 + 1) + 4(3 \times 3333 + 1) \\ &\quad + 6(3 \times 333 + 1) + 1(3 \times 33 + 1) + 5(3 \times 3 + 1) + 0 \\ &= 3 \text{ (some number)} + 2 + 4 + 6 + 1 + 5 + 0. \end{aligned}$$

Now I can't be bothered to write out the exact multiple of 3 because it isn't important, so I've just written "some number". This means that

246150 is divisible by 3 if and only if 2 + 4 + 6+1 + 5 + 0 is.

In general then, we have

$$\begin{aligned} N &= a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_1 10 + a_0 \\ &= a_n(3k_n + 1) + a_{n-1}(3k_{n-1} + 1) + \cdots + a_1(3k_1 + 1) + a_0 \\ &= 3(a_n k_n + a_{n-1} k_{n-1} + \cdots + a_1 k_1) + a_n + a_{n-1} + \cdots + a_1 + a_0 \\ &= 3K + (a_n + a_{n-1} + \cdots + a_1 + a_0). \end{aligned}$$

Clearly then since 3K is divisible by 3, N is divisible by 3 if $a_n + a_{n-1} + \ldots + a_1 + a_0$ is divisible by 3 and vice-versa. Since $a_n + a_{n-1} + \ldots + a_1 + a_0$ is just the sum of the digits of N then we have just proved a theorem.

Theorem 1. *N is divisible by* 3 *if and only if the sum of its digits is divisible by* 3.

Before you try your hand at this test to see which numbers are divisible by 3, I just want to take a moment to explain the mystic "if and only if" in Theorem 1.

When you see a statement "a if and only if b" it means "if *a* is true then *b* is true *and* if *b* is true then *a* is true". For instance, $a^2 = 25$ if and only if $a = \pm 5$. Clearly if $a^2 = 25$ then $a = \pm 5$ *and* if $a = \pm 5$ then $a^2 = 25$.

A problem which asks you to prove "c if and only if d" requires you to show first that if *c* is true then so is *d* and second that if *d* is true then so is c. So you've got two things to prove.

Now go back to the discussion just before Theorem 1, and first suppose *N* is divisible by 3. Since $N = 3K + (a_n + a_{n-i} + \ldots + ai + a_0)$, then the string in brackets has to be divisible by 3. Hence "if *N* is divisible by 3 then so is $a_n + a_{n-}i + \ldots + ai + ao$". On the other hand if $a_n + a_{n-}i + \ldots + ai + ao$ is divisible by 3 then so is $3K + (a_n + a_{n-i} + \ldots + a_i + a_0)$. As a result *N* is divisible by 3. Then we have "if $a_n + a_{n-i} + \ldots + a_i + a_0$ is divisible by 3 then so is *N*".

Now try some problems.

Exercises

21. Which of the following numbers are divisible by 3?
(i) 123456789; (ii) 555333111; (iii) 76543211234567.
22. Which of the following numbers are divisible by 6?
(i) 134567892; (ii) 433452254; (iii) 433254456.
23. Write down a test for numbers which are divisible by 6.
24. A number is said to be "fattened" if an arbitrary number of zeros is inserted. For instance 20300412090 is a fattened form of 234129.
Let *M* be a fattened form of *N*. Which of the following are true?
(i) *M* is divisible by 2 if and only if *N* is. (Remember there are two things to show here. (1) If *M* is divisible by 2 then *N* is. (2) If *N* is divisible by 2 then *M* is.)
(ii) *M* is divisible by 3 if and only if *N* is.
(iii) *M* is divisible by 6 if and only if *N* is.
25. Can we go further than Theorem 1? What can you say about the remainder we get on dividing *N* by 3?

Divisibility by 4. This isn't quite as easy as 2. After all, although 34 ends in a number divisible by 4, it is not itself divisible by 4. But then neither is 134, 1034 or any other number with 3, 4 as the last two digits. On the other hand, anything ending in 32 *is* divisible by 4.

The point here is that 10^2, 10^3, 10^4 and so on are *all* divisible by 4. Hence so are all multiples and sums of multiples of 100. For divisibility by 4, the crucial point is the last two digits. If they are divisible by 4 then so is the complete number and vice-versa.

Theorem 2. *N is divisible by* 4 *if and only if the last two digits of N taken as a 2-digit number is divisible by* 4.

Proof. Let $N = a_n10^n + a_{n-1}10^{n-1} + \ldots + a_210^2 + a_110 + a_0$. Then $N = 4K + a_110 + a_0$, since $10^r = (4 \text{ x } 25)^r = 4(4^{r-1} \text{ x } 25^r)$. So if *N* is divisible by 4, then so is $a_110 + a_0$, the last "two digit number of N". And if the last two digits of *N* together as a number are divisible by 4, then $a1\ 10 + a0$ is divisible by 4. Hence $4K + a1\ 10 + a0$ is divisible by 4, which means that *N* is. ?

Exercises

26. Which of the following numbers is divisible by 4?
(i) 1437640856; (ii) 433452254; (iii) 134567896.
27. Let *M* be a fattened form of *N* and let *N* be divisible by 4. For which *M* is it true that *M* is divisible by 4?
If *N* is divisible by 12, what restrictions must be placed on *M* so that it too is divisible by 12?

28. N is a 4-digit number comprised of the digits 1, 2, 3, 4, 5 used at most once each. How many such numbers are there which are multiples of 12?
29. Is it true that the remainder on dividing N by 4 is the same as the remainder on dividing the last two digits of N (taken as a 2-digit number) by 4?

Divisibility by 5. This is a cinch. Numbers which are divisible by 5 end in 0 or 5. End of story.

Divisibility by 8. This is a little harder than 4 but goes along the same lines. The first thing to observe is that 1000 is divisible by 8. Hence so is every multiple of 1000. Consequently we only have to worry about the last three digits.

Theorem 3. *N is divisible by* 8 *if and only if the last three digits of* N *taken as a 3-digit number is divisible by* 8.

Exercises

30. Whof the following integers are divisible by 15?
    (i) 47243535; (ii) 9871200; (iii) 7892305.
31. Which of the numbers in Exercise 26 is divisible by 8?
32. For what n is $\sum_{i=1}^{n} i!$ divisible by 5? (Recall from Chapter 2 that $\sum_{i=1}^{n} i!$ =1'! = 1! + 2! +…+ n!. Also recall that n! = n(n — 1)(n — 2)…3 . 2 . 1.)
33. Let M be a fattened form of N. If N is divisible by 5 is M divisible by 5?
34. Prove Theorem 3.
35. What do I want to know about remainders here?

Divisibility by 9. Think back to 3. Now 10 = 9 + 1, 100 = 99 + 1, 1000 = 999 + 1 and so on. Every power of 10 is one more than a multiple of 9. And the multiple of 9 is 9 times a string of 1's. So $10^n = 9k_n + 1$, where k_n is a string of n ones. Let $N = a_n a_{n-i} \ldots a_i a_0$. Then

$$\begin{aligned} N &= a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0 \\ &= a_n(9k_n + 1) + a_{n-1}(9k_{n-1} + 1) + \cdots + a_1(9k_1 + 1) + a_0 \\ &= 9\sum_{i=1}^{n} a_i k_i + \sum_{i=0}^{n} a_i. \end{aligned}$$

Hence the following theorem.

Theorem 4. *The remainder on dividing* N *by* 9 *is r if and only if the remainder on dividing the sum of its digits by* 9 *is r.*

Don't worry if you found the proof before the theorem hard. The important thing is to make sure you know how to test a number to see if it has 9 as a factor.

Divisibility by 10. Make my day!

Exercises

36. Which of the numbers of Exercise 22 is divisible by 9?
37. Let M be a fattened form of N. Which of the following statements is true? For those which are, prove them, for those which ain't, give a counterexample.
    (a) M is divisible by 9 if and only if N is. (Remember there are two things to be shown.)
    (b) M is divisible by 10 if and only if N is.
    (c) M is divisible by 18 if and only if N is.
    (d) M is divisible by 30 if and only if N is.
38. Let N = a6796. If N is divisible by 72, find a and b.

39. (a) Divisibility by 11. See if you can come up with your own theorem here. Look for a simple test for divisibility by 11 and prove that it always works.

(b) Divisibility by 7 is somewhat harder. There is no rule as simple as the ones we've produced for other numbers but you should manage something here.

Divisibility by 7. There is a very nice algorith[a] which will help you to decide whether or not a number is divisible by 7. The algorithm works by taking a number *N* and reducing it to a smaller number *M* in such a way that *N* is divisible by 7 if and only if *M* is divisible by 7.

You should be able to use the algorithm given as Theorem 5, to write a program to determine whether or not a given input is divisible by 7 or not.

Theorem 5. *Let* $N = a_n a_{n-1} \ldots a_1 a_0$ *and let* $M = (a_n a_{n-1} \ldots a_2 a_1) - 2 \times a_0$. *Then N is divisible by* 7 *if and only if M is divisible by* 7.

Before I prove the theorem let me show you how it works.

Let *N* = 31759. Then *M* = 3175 — 18 = 3157.

Keep repeating this process.

$$315 - 14 = 301$$
$$30 - 2 = 28$$

But 28 is a multiple of 7. Theorem 5 then claims that 301 is too, as is 3157 and finally 31759.

Now let's see why the theorem works.

Proof of Theorem 5. Let $L = a_n a_{n-1} \ldots a_2 a_1$. Then $N = 10L + a_0$ and $M = L - 2a_0$.

We first show that if *N* is divisible by 7 then so is *M*.

If *N* is divisible by 7, then so is $2N = 20L + 2a_0$. Obviously $21L + 7a_0$ is a multiple of 7 so $(21L + 7a_0) - (20L + 2a_0) = L + 5a_0$ is also a multiple of 7. But then so is $(L + 5a_0) - 7a_0 = M$.

Now we go the other way and show that if *M* is divisible by 7, then so is *N*.

If $L - 2a_0$ is divisible by 7, then so is $10(L - 2a_0) = 10L - 20a_0$. Clearly 21oo is a multiple of 7, so 10*L* — 20oq + 21oq = *N* is too. ?

Exercises

40. Which of the following numbers is divisible by 7? (Use the test developed above.)

(i) 231; (ii) 1988; (iii) 4965;
(iv) 31756; (v) 1234567; (vi) 471625;
(vii) 12030403.

41. Which of the following is divisible by 11? (Use the test developed in Exercise 39(a). Look at the solution if you need to.)

(i) 231; (ii) 1212398; (iii) 8282395.

42. Notice that 1001 = 7 x 11 x 13. We can use this to get a quick test for divisibility by 7. Now 31759 = 31 x 1000 + 759 = 31 x 1001 — 31 + 759. This means that 31759 is divisible by 7 if and only if —31 + 759 is. Now 759 — 31 = 728 = 7 x 104. Hence 31759 is divisible by 7.

(a) Use the above test to do Exercise 40 again. ((vii) is made easier if you do the following $12030403 = 12 \times 10^6 + 030 \times 10^3 + 403 = (12 \times 1001000 - 12000) + (30 \times 1001 - 30) + 403 = (12 \times 1001000 + 30 \times 1001) - (12 \times 1001 - 12) +$

(—30 + 403) = (12 x 1001000 + 18 x 1001) + (12 — 30 + 403). The original number if divisible by 7 if 12 — 30 + 403 is).

(b) Discover a “block of 3 digits” method for testing divisibility by 7.

(c) Discover a “block of 3 digits” method for testing divisibility by 11. Use this test on the numbers in Exercise 41.

(d) Which of the following numbers are divisible by 13?

(i) 123456; (ii) 123456789; (iii) 1123456789.

4.3. Common Factors

If we are given a number, one of its important properties is its factors. We can start to find small factors by the methods of Section 4.3. However, if we are given *two* numbers we often want to know what factors they have in common or more especially what is the *largest* common factor that they have. This number is known as the highest common factor (h.c.f.) or greatest common divisor (g.c.d.).

Naturally one way to find the h.c.f. of two numbers is to find all their factors and then compare the two sets of factors. Fortunately there is a quicker way.

First let's observe the *division algorithm.* This is just another step by step procedure. It's very simple actually and something you've known for a long time. For instance, you know that 31 = 4 x 7 + 3.

The Division Algorithm. *If we divide a number n by a smaller number q, then we can express n in the form n = aq* + r, where r is the remainder and 0 ≤ r q.

All this means of course is that when you divide a number *n* by a number you can organise things to get a remainder which is less than q. When we divided 31 by 7 we got a remainder of 3 which is less than 7.

It also means that you can express any number in terms of a multiple of another number plus a remainder. So, for instance, if q = 3, any number n can be written as 3a, 3a +1 or 3a + 2, because the remainder r is such that 0≤ r 3.

This way of writing numbers in terms of other numbers can be useful. Exercise

43. Find a and r for the following values of n and q.

(i) n = 25, q = 7; (ii) n = 87, q =11; (iii) n =149, q = 21.

Having mastered the simple division algorithm we extend it to the Euclidean Algorithm which does the job we set out to do — find what the g.c.d. of two given numbers is.

Example 1. Find the g.c.d. of 22 and 6. We do this by applying the division algorithm several times.

$$22 = 3 \times \underline{6} + \underline{4}$$
$$\underline{6} = 1 \times \underline{4} + 2$$
$$\underline{4} = 2 \times \underline{2}$$

Each time we use the “q” of the previous step as the “n” of this step and the “r” of the previous step as the “q” of this set. As this forces the next “q” to be smaller than the previous “q”, the remainder must get smaller. Finally one of them is zero. Then the last non-zero remainder turns out to be the required g.c.d.

In this example, then, the g.c.d. is 2. This is easily checked by finding all the factors of 22 and 6 and comparing them.

Example 2. Suppose we want to find the highest common factor of 125 and 90. The first step of the Euclidean Algorithm is

$$125 = 90 + 35.$$

(Assume that g is the highest common factor. Then g divides 125 and 90, so it must divide their difference. So g divides 35.)

The second step is

$$90 = 2 \times 35 + 20.$$

(Since *g* divides 90 and 35, it divides 90 and 2 x 35. Hence *g* divides 20.)

The third step is

$$35 = 20 + 15.$$

(Consequently *g* divides 15.)

Then the fourth step is

$$20 = 15 + 5.$$

(So *g* now divides 5.)

The last step is

$$15 = 3 \times 5 + 0$$

The algorithm has stopped (as it always must since the remainder continually decreases). The last positive remainder is 5, so the highest common factor of 125 and 90 is 5.

At this stage we haven't proved that 5 is the highest common factor of 125 and 90. We have only proved that the highest common factor of 125 and 90 is also a factor of 5. So *g* divides 5.

But we can work back the other way. From the last step 5 divides 15. From the fourth step, 5 divides 15 and 5 x (the terms on the right-hand side), so 5 divides 20. Repeating the argument at the third step we must have 5 dividing 35. Continuing to the second step gives 5 divides 90. The argument applied to the first step gives 5 divides 125.

We thus have 5 is a factor of both 90 and 125. It must therefore be true that 5 is a factor of *g* because *g* is the highest factor that divides 90 and 125.

Since 5 divides *g* and *g* divides 5, then *g* = 5.

This is the reasoning on which the Euclidean Algorithm is based. The argument given above can be applied in general to prove that the last nonzero remainder is the highest common factor of the original two numbers.

We use the notation (m, n) to denote the g.c.d (h.c.f.) of m and n. Hence (22, 6) = 2 and (125, 90) = 5.

Exercises

44. Use the Euclidean Algorithm to find the highest common factors of the following pairs of numbers.

(i) 21, 15; (ii) 28, 12;

(iii) 630, 132; (iv) 597, 330; (v) 1988, 236; (vi) 1987, 235.

45. Using the Division Algorithm repeatedly we get

$$
\begin{aligned}
m &= a_1 n + r_1 \\
n &= a_2 r_1 + r_2 \\
r_1 &= a_3 r_2 + r_3 \\
\dots & \quad \dots \quad \dots \\
r_{s-1} &= a_{s+1} r_s r_{s+1} \\
r_s &= a_{s+2} r_{s+1} + r_{s+2}.
\end{aligned}
$$

Prove that

(a) for some s, $r_s+_2 = 0$, and

(b) if $r_s+_2 = 0$, then r_{s+i} is the g.c.d. of *m* and n.

But the Euclidean Algorithm can be used to do more than this. We can actually find integers *a* and *b* such that*am* + *bn* = *g* ,
where *g* = (m, n) the g.c.d. of *m* and n.

Example 3. Find *a* and *b* such that *22a* + 6b = 2.

From Example 1 we know that

$$
\begin{aligned}
& 2 = 6 - 1 \times 4 \\
\text{But} \quad & 4 = 22 - 3 \times 6 \\
\therefore \quad & 2 = 6 - 1 \times (22 - 3 \times 6) \\
& = 4 \times 6 - 1 \times 22
\end{aligned}
$$

Hence *a* = —1 and *b* = 4.

Example 4. Find *a* and *b* such that 125a + 90b = 5.

From Example 2 we know that

$$
\begin{aligned}
& 5 = 20 - 15 \\
\text{But} \quad & 15 = 35 - 20 \\
\therefore \quad & 5 = 20 - (35 - 20) \\
& = 2 \times 20 - 35 \\
\text{Now} \quad & 20 = 90 - 2 \times 35 \\
\therefore \quad & 5 = 2 \times (90 - 2 \times 35) - 35 \\
& = 2 \times 90 - 5 \times 35 \\
\text{Finally} \quad & 35 = 125 - 90 \\
\text{So} \quad & 5 = 2 \times 90 - 5 \times (125 - 90) \\
& = 7 \times 90 - 5 \times 125.
\end{aligned}
$$

Hence *a* = —5 and *b* = 7.

Exercises

46. Use the Euclidean Algorithm to find a and *b* which satisfy *xa* + *yb* = g , where *g* = (x, *y*) the g.c.d. of x and *y* .

(i) x =15, y = 21; (ii) x = 12, y = 28;

(iii) x = 132, y = 630; (iv) x = 139, y = 72.

47. Note that 2 = 5 x 22 — 18 x 6. This means that there is not a unique value for *a* and *b* in the equation 2 = 22a + 6b.

Find all *a* and *b* such that 2 = 22a + 6b.

48. Find all *a* and *b* such that 5 = 125a + 90b.

It turns out that the following theorem can be proved. It's actually a generalisation of Theorem 1 of Chapter 1.

Theorem 6. *Let m and n be given integers with g* = (*m*, *n*).

(a) *There exist integers a and b such that am* + *bn* = *g*.

(b) *If g divides 7 then the complete solutions of mx* + *ny* = γ *are given by* $x = \frac{a\gamma}{g} + \frac{\alpha n}{g}$ *and* $y = \frac{b\gamma}{g} - \frac{\alpha m}{g}$, *where* $\alpha \in \mathbb{N}$.

(c) *If Y is not divisible by g, then mx* + *ny* = *7 has no integer solution.*

Example 5. Find all solutions of 22x + 6y = 70. Now here *m* = 22, *n* = 6 and γ = 70. We know from Example 1 that *g* = (22, 6) = 2. So because 2 divides 70, the equation does have solutions.

From Example 3, we know that —22 + 4 x 6 = 2, so *a* = —1 and *b* = 4. Using Theorem 6(b), we see that all solutions of 22x + 6y = 2 are given by

$$x = -\frac{70}{2} + \frac{6\alpha}{2} \quad \text{and} \quad y = \frac{4 \times 70}{2} - \frac{22\alpha}{2}.$$

In other words x = –30 + 3α and y = 140 –11α.

(Check: 22(—35 + 3a) + 6(140 — 11a) = —770 + 840 = 70.) *Exercise*

49. Find all solutions (if any exist) to the following equations (i) 10x + 35y =110; (ii) 24x + 63y = 99; (iii) 121x + 25y = 210; (iv) 68x + 17y =100.

Equations such as those in Exercise 49 are called *Diophantine Equations* after the Greek Mathematician Diophantus (see the web for more). They arise in a number of situations. When they relate to practical problems it is useful to note that *x* and *y* may need to be restricted to being positive, or at least non-negative.

Exercises

50. John collected an even number of insects in a jar — some were beetles, some were spiders. He counted 54 legs in all. How many spiders did he have?
51. A woman spent $29.60 buying drinks for a party. The largest bottle of Poke cost $1.70 while L&C cost $1.10. How many bottles of each did she buy?
52. An absent-minded bank teller switched the dollars and cents when he cashed a cheque for Mr Brown, giving him dollars instead of cents, and cents instead of dollars. After buying a 35 cent newspaper, Brown discovered that he had left exactly twice as much as his original cheque. What was the amount of the cheque? (No, you haven't seen this precise problem before. Use Diophantine equations to solve it.) (What reasonable amounts — other than 5 cent and 35 cents — can replace the cost of the newspaper to make this a sensible problem?)
53. A man goes to a stream with a 9litre container and a 16litre container. What should he do to get precisely 1 litre of water in the 16 litre container? (See Chapter 1.)
54. Prove that the fraction (*21n* + 4)/(4n + 3) is irreducible for every natural number n. (In other words show that no matter what value *n* has, 21n + 4 and 14n + 3 never have a common factor.)

But Diophantine equations don't have to be linear, that is, they don't have to be such that the variables are only to the power one as in *αx* + βy = γ. There may be quadratic (power 2) terms.

Example 6. Show that x^2 — y^2 = 2 has no integer solutions.

An answer to this relies solely on the factorisation x^2 — y^2 = (x — y) ∠ (x + y). Since *x* and *y* have to be integers we require either *x* — *y* = 2 and *x* +*y* = 1or *x* — *y* = 1 and *x* +*y* = 2 or the equivalent equations with —1 and —2. Solving the first equations gives *x* =$\frac{3}{2}$, *y* = —$\frac{1}{2}$ and solving the second equations gives *x* = *y* = (Solving the equations with —1 and —2 gives fractional answers too.) Hence x^2 — y^2 = 2 has no integer solutions.

Exercises

55. (a) Show that the equation x^2 — y^2 = 74 has no integral solutions.

(b) Is it true that $x^2 — y^2 = 2r$ has no integer solutions for any natural number r?

(c) For what *r* does $x^2 — y^2 = 2r$ have no integral solutions?

56. Find all solutions of $x^2 — y^2 = 27$.

57. For what integral values of *x* and *y* is $x^2 — y^2$ divisible by 4?

58. Without using mechanical or electronic aids, decide whether $112296^2 — 79896^2 = 13!$ (n! is defined in Exercise 32, p. 114.)

Actually $x^2 — y^2 = (x — y)(x + y)$ is the first of a series of similar factorizations. It turns out that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3)$$

and

$$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

Check these by multiplying out the right-hand sides of the equations.

In fact $x — y$ is always a factor of $x^n — y^n$.

(One day you might find this useful for differentiating x^n from first principles.)

Factorisation 1. For all natural numbers n,

$x^n — y^n = (x — y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 +\ldots+ xy^{n-2} + y^{n-1})$.

Exercises

59. (a) Show that $1^2 — 2^2 + 3^2 — 4^2 = —(1 + 2 + 3 + 4)$.

(b) Show that $1^2 — 2^2 + 3^2 — 4^2 + 5^2 = (1 + 2 + 3 + 4 + 5)$.

(c) Generalise the results of (a) and (b).

60. Prove that for all positive integers n, $N = 1^n + 8^n — 3^n — 6^n$ is divisible by 10.

For what *n* is *N* divisible by 20? Is *N* ever divisible by 40?

61. Prove that, for any positive integer n, $1492^n — 1770^n — 1863^n + 2141^n$ is divisible by 1946.

Make up similar problems where the answer (here 1946) is the current year.

62. (a) Show that $4n^3 + 6n^2 + 4n + 1$ is composite for all natural numbers n.

(b) Is $5n^4 + 10n^3 + 10n^2 + 5n +1$ always composite?

(c) What about $6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n +1$?

(d) Generalise.

63. What numbers divide $n^3 — n + 24$ for all values of n? Prove it.

Actually if *n* is odd we can factorise $x^n + y^n$ too. For instance,

$$x^3 + y^3 = (x + y)(x^2 — xy + y^2),$$

and

$$x^5 + y^5 = (x + y)(x^4 — x^3 y + x^2 y^2 — xy^3 + y^4).$$

Check these out by multiplying out the brackets and collecting like items.

In general we have the next result.

Factorisation 2. For all odd natural numbers n,

$$x^n + y^n = (x + y)\left(\sum_{i=0}^{n-1}(-1)^i x^{n-i-1} y^i\right).$$

Exercises

64. (a) Show that $M = 7^{2n+1} + 15^{2n+1}$ is divisible by 22 for all n e N U {0}.

(b) For what *n* is *M* divisible by 44?

(c) For what n is M divisible by 66?

65. (a) Repeat Exercise 64 with M replaced by $L = 6^{2n+1} + 16^{2n+1}$.

(b) If $T = a^{2n+1} + b^{2n+1}$ is such that a + b = 22, for what a, b and n is T divisible by 66?

66. Prove that $5^{2n+1} + 11^{2n+1} + 17^{2n+1}$ is divisible by 33 for every natural number n.

4.4. Fermat's Little Theorem

Fermat's (Big) Theorem finally is. In 1622, or thereabouts, Fermat made a name for himself by scribbling in a book. The librarian was not amused. Essentially he said that he could prove that, for no n > 2, did $x^n + y^n = z^n$ have integral solutions for x, y, z. He compounded his felony with the mathematicians by adding that the margin wasn't big enough to give the proof!

Suffice to say that most people believe he didn't have a proof. This is largely because it took until 1995 before a proof was found and the mathematics that was used in the proof hadn't been invented in 1622. It took a tours de force by Andrew Wiles, an Englishman working in the States, to produce the proof and settle other interesting, but not obviously related, problems. (For more details on the historical and mathematical aspects of this see Hilton, Holton and Pedersen, “Mathematical Vistas”, Springer-Verlag, 2002 or http://cgd.best.vwh.net/home/flt/flt01.htm or MacTutor.)

So what about Fermat's Little Theorem?

Fermat's Little Theorem. If p is a prime and 1 ≤ a p, then a^p has remainder a when divided by p.

Example 7.

(a) Let p = 5 and a = 2. Now $2^5 = 32 = 6 \times 5 + 2$.

(b) Let p = 7 and a = 3. Now $3^7 = 2187 = 312 \times 7 + 3$.

Example 8. Find the smallest value of n for which $2^n — 1$ is divisible by 41. (The following proof should be skipped the first time you read this chapter. This is because the method of proof is “Proof by Contradiction”. I don't explain this method until Chapter 6. However the important thing which follows from this Exercise is Remark 1. Make sure you know and understand this remark.)

Now by Fermat's L.T., 2^{41} has a remainder of 2 when divided by 41 since 41 is a prime. Hence 2^{41} = 41a + 2. Clearly a is even, so 2^{40} = 41b +1, where 2b = a. Hence 2^{40} — 1 is divisible by 41.

But is there a smaller value of n than 40?

Suppose c is the smallest number such that 2^c — 1 is divisible by 41. Now 40 = tc + r for r c by the Division Algorithm in Section 4.3.

Now 2^c = 41d + 1, so 2^{tc} = (41d +1) must be of the form 41 f + 1 — just apply the Binomial Theorem (see Chapter 2). But 2^{40} = 41g + 1, so let 2^r = 41h + s.

Hence

$$2^{40} = 41g + 1 = 2^{tc+r} = (41f + 1)(41h + s) = 41j + s.$$

Hence s = 1.

However this says that 2^r — 1 is divisible by 41. Since r c , this contradicts the assumption that c was the smallest number such that 2^c — 1 is divisible by 41. Hence r = 0 and c divides 40.

So c must be 1, 2, 4, 5, 8, 10, 20 or 40. Checking, we see that

$$2^1 - 1 = 1, \quad 2^2 - 1 = 3, \quad 2^4 - 1 = 15, \quad 2^5 - 1 = 31,$$

$$2^8 - 1 = 255 = 6 \times 41 + 9, \; 2^{10} - 1 = 1023 = 24 \times 41 + 40.$$

You do the rest. $2^{20} - 1$ is divisible by 41 and so 20 is the smallest number n for which 2n — 1 is divisible by 41. This is a lot of work for only a small gain but it seems to be the only way to get there.

Remark 1. Fermat's L.T. guarantees that $2^{p-1} - 1$ is divisible by p for p a prime. However, it is always possible that some divisor c of p — 1 also has the property that $2^c - 1$ is divisible by p.

Exercises

67. Show that $1^{241}+2^{241}+3^{241}+4^{241}$ is divisible by 5 but $1^{240}+2^{240}+3^{240}+4^{240}$ isn't.
68. For what n is $\sum_{i=1}^{4} i^n$ divisible by 5?
69. Find the smallest possible integer n such that $2^n - 1$ is divisible by 47.

Actually in this area of Number Theory we can make life a lot easier for ourselves if we use some better notation. Hence we introduce the concept of congruences.

We write a = b (mod c) (pronounced “a congruent to b modulo c”) to mean that a and b have the same remainder when we divide by c. For example, 7 = 3 (mod 4) and 8 = 2 (mod 6).

The notation is used because when we are dealing with remainders modulo c we can often get away with doing much less arithmetic.

Example 9. What are the remainders when 1988^2 and 1989^2 are divided by 4?

Well we could go straight to our calculator and find 1988^2 then get the remainder. But 1988 = 4.497 and so $1988^2 = 4^2 \cdot 497^2$. Obviously the remainder is zero.

Another way of writing this is 1988 ≡ 0 (mod 4), so $1988^2 \equiv 1988 \cdot 0 \equiv 0 \pmod 4$.

Now 1989 ≡ 1 (mod 4). Hence $1989^2 \equiv 1989\ 1 \equiv 1989 \equiv 1 \pmod 4$. So 1989^2 has a remainder of 1 when divided by 4.

To make life easier, here are a few lemmas (baby theorems) that help when dealing with congruences.

Lemma 1. If a = b (mod c), then ma = mb (mod c).

Lemma 2. If a = b (mod c), then $a^n = b^n$ (mod c).

Exercises

70. Find a in each of the following, where a is non-negative and as small as possible.
    (i) 1234 = a (mod 5); (ii) 4^{16} = a (mod 3);
    (iii) 2^{240} = a (mod 3); (iv) 2^{240} = a (mod 5).
71. Restate Fermat's Little Theorem in terms of congruences.
72. Redo Exercises 67, 68, 69 using congruences.
73. Prove Lemmas 1 and 2.
74. For which non-negative integers n and k is
$$(k+1)^n + (k+2)^n + (k+3)^n + (k+4)^n + (k+5)^n$$
75. Show that $\sum_{i=1}^{6} i^n \equiv 0 \pmod 7$ if and only if n is not congruent to 0 (mod 6).
76. Generalise the results of Exercises 68 and 73.
77. Find the smallest n such that $2^n - 1$ is divisible by 31.
78. For what primes p is $2^{(p-1)/2} \equiv 1 \pmod p$?

For what primes p is p — 1 the smallest positive integer n such that $2^n = 1$ (mod p)? (Beware!)

79. Find the smallest natural number N which has the properties:
   (i) it's decimal representation has 6 as the last digit;
   (ii) if the last digit is removed and placed in front of the remaining digits, the resulting number is 4N.

4.5. A.P.'s

So far we have looked at Number Theory problems involving division but perhaps addition is a more fundamental operation. In this section we try to find simple ways of adding numbers that form a well defined pattern.

Example 10. Find the 5th term, the 10th term and the general (nth) term of the following sequence[b] of numbers:

$$2,5,8,11,\dots$$

We notice that for each new term we are adding on 3. Since the 4th term is 11, then the 5th term is 14. To get the 10th term we can work our way up: 14, 17, 20, 23, 26, 29. The 10th term is therefore 29.

This isn't a very efficient way to proceed though if we're looking for the one million two hundred and thirty-four thousand, seven hundred and eighty-second term. So let's try to find an expression for the nth term, T_n.

If $n = 1$, that's easy $T_1 = 2$. Now $T_2 = T_1 + 3$, $T_3 = T_2 + 3 = T_1 + 2 \times 3$, $T_4 = T_3 + 3 = T_1 + 3 \times 3$. So we notice that the multiple of 3 is always one less than the number of the term we're looking at. Hence $T_n = T_1 + (n - 1) \times 3 = 2 + 3n - 3 = 3n - 1$.

If we test this out for T_1, T_2, T_3, T_4, T_5 and T_{10}, we see we've got the right expression for the general term. (After all $T_{10} = 3 \times 10 - 1 = 29$ as we found before.)

Exercises

80. Find the 5th, 10th and nth terms of the following sequences all of whose terms increase by a fixed constant:
   (i) 3,5,7,9,…; (ii) 3,11,19,27,…;
   (iii) 5,6,7,8,…; (iv) 4,10,16,22

81. Consider the sequence a, $a + d$, a+2d , a+3d ,…Here $T_1 = a$, $T_2 = a + d$, $T_3 = a + 2d$ and $T_4 = a + 3d$. Find an expression for T_n. Check your answers to Exercise 80 by using this most general T_n .

A sequence of numbers of the form a, a + d, a + 2d, a + 3d,…, where each new number is obtained from the previous one by adding the constant difference d, is called an Arithmetic Progression. (A.P. for short.)

The first term of the general arithmetic progression is a , the second a + d, and so on. The nth term is $a + (n - 1)d$. Just add on d each time.

We will now see how to add up consecutive terms of an A.P.

Example 11.[c] Find the sum $S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$. Well that's pretty easy. Obviously it's 55. But suppose we had wanted to add up a large number of consecutive integers. What would we have done then? Have a look at this trick.

$$\begin{array}{lrcl} & S & = & 1+\ 2+\ 3+\ 4+\ 5+\ 6+\ 7+\ 8+\ 9+10 \\ \text{Clearly} & S & = & 10+\ 9+\ 8+\ 7+\ 6+\ 5+\ 4+\ 3+\ 2+\ 1 \\ \therefore & 2S & = & 11+11+11+11+11+11+11+11+11+11 \end{array}$$

So as a result of these shenanigans we see that 2S = 10 *x* 11. From that we get *S* = 55 again.

Example 12. Find an expression for $\sum_{i=1}^{n} i$.

$$\begin{array}{lrcl} \text{Let} & S_n & = & 1 + 2 + \cdots + (n-1) + n \\ \text{So} & S_n & = & n + (n-1) + \cdots + 2 + 1 \\ \hline \text{Adding} & 2S_n & = & (n+1) + (n+1) + \cdots + (n+1) + (n+1) \end{array}$$

On the right-hand side of this last equation we have *n* terms of the form *n* +1. Hence $2S_n = n(n+1)$. So we have

$$S_n = \sum_{i=1}^{n} i = \frac{n}{2}(n+1).$$

Exercises

82. Find the sum of the first 100 natural numbers.
83. The sum of the first *n* natural numbers is 100 less than the sum of the next *n* natural numbers. Find *n*.
84. (a) Find the sum of the first 100 even natural numbers.
    (b) Find the sum of the first 100 odd natural numbers.
85. (a) Find the sum 1 + 4 + 7 + 10+…+ 121.
    (b) Find an expression for $\sum_{i=1}^{n}(3i-2)$, using the technique of Example 12.

So how about we try to add up the first *n* terms of a general A.P.? Remember that $T_1 = a$, $T_2 = a + d, \ldots, T_n = a + (n-1)d$.

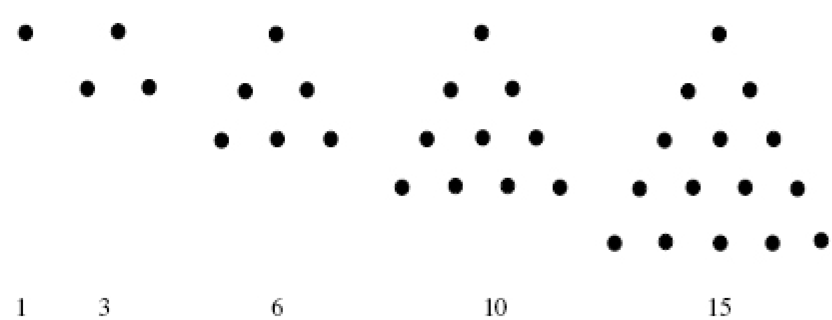
Theorem 7. Let $S_n = \sum_{i=1}^{n} a + (i-1)d$. *Then* $S_n = \frac{n}{2}[2a + (n-1)d]$.

Exercises

86. Find the sum of the first twenty terms of the following A.P.'s.
    (i) 2,5,8,…; (ii) 2,9,16,…;
    (iii) 15,21,27,…; (iv) —7,0,7,…;
    (v) —90, —80, —70,…; (vi) —2, —4, —6
87. Find the sum of all numbers less than 200 which are divisible by 3.
88. Use the technique of Example 12 to prove Theorem 7.

Show that S_n is the product of the number of terms and the average of the sum of the first and last term. That is $S_n = n\left(\frac{T_1+T_n}{2}\right)$.

89. The *triangular numbers* 1, 3, 6, 10, 15, 21, 28,…are the sums of the first *n* positive integers. They are called triangular numbers because of the triangular form shown below.



(a) Write down an expression for t_n , the nth triangular number.

(b) Notice that $t_3 = 2t_2$. Find another pair of triangular numbers such that one is twice the other.

(c) Are there triangular numbers t_r , t_s which satisfy $t_s = 3t_r$ or $t_s = 4t_r$?

(d) Show that for any triangular number t_s , $s > 1$, there is another, distinct, t_r , such that $t_s \div t_r$ is an integer.

But we can also add up powers of numbers too. For instance, we might we might to find $\sum_{i=1}^{n} i^2$ or $\sum_{i=1}^{n} i^3$.

Example 13. Find an expression for $S_n = \sum_{i=1}^{n} i^2$.

Now we do this by first writing that $(n+1)^3 = n^3 + 3n^2 + 3n + 1$, so

$$\begin{array}{ccccccccc}
(n+1)^3 & - & n^3 & = & 3n^2 & + & 3n & +1 \\
n^3 & - & (n-1)^3 & = & 3(n-1)^2 & + & 3(n-1) & +1 \\
(n-1)^3 & - & (n-2)^3 & = & 3(n-2)^3 & + & 3(n-2) & +1 \\
\cdots & & \cdots & & \cdots & & \cdots & \cdots \\
4^3 & - & 3^3 & = & 3\cdot 3^2 & + & 3\cdot 3 & +1 \\
3^3 & - & 2^3 & = & 3\cdot 2^2 & + & 3\cdot 2 & +1 \\
2^3 & - & 1^3 & = & 3\cdot 1^2 & + & 3\cdot 1 & +1
\end{array}$$

As in Example 12, we add up the left and right sides. On the left side we get $\{(n + 1)^3 — n^3\} + \{n^3 — (n — 1)^3\} + \{(n — 1)^3 — (n — 2)^3\} + +\{4^3 — 3^3\} + \{3^3 — 2^3\} + \{2^3 — 1^3\}$. This simplifies nicely to $(n + 1)^3 — 1^3$.

On the right-hand side we get $3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + n$. Now $\sum i^2 = S_n$ is what we're trying to find and $\sum_{i=1}^{n} i = \frac{n}{2}(n+1)$. So

$$(n+1)^3 - 1^3 = 3S_n + \frac{3}{2}n(n+1) + n.$$

If we simplify all this and rearrange we get

$$S_n = \frac{n(n+1)(2n+1)}{6}.$$

Exercises

90. Find the sum of the squares of the first 10 positive integers using the formula of Example 13. Check your answer by direct addition.

91. Note the following:

$$1^2 = 1\cdot 2\cdot 3/6$$
$$1^2 + 3^2 = 3\cdot 4\cdot 5/6$$
$$1^2 + 3^2 + 5^2 = 5\cdot 6\cdot 7/6.$$

Use the above to guess a formula for the sum of the squares of the first *n* odd integers. Prove this formula is correct.

92. Find a formula for the sum of the squares of the first *n* even integers.

93. Find an expression for the sum of the cubes of the first *n* natural numbers.

94. $\lfloor a \rfloor$ means the integer part of a. In other words $\lfloor 7.5 \rfloor = 7$, $[8.321] = 8$, $[\pi] = \lfloor 3 \rfloor$, $\lfloor e \rfloor = 2$, $\lfloor 9 \rfloor = 99$.

Find a formula for

$$S_n = \lfloor 1^{\frac{1}{2}} \rfloor + \lfloor 2^{\frac{1}{2}} \rfloor + \cdots + \lfloor (n^2-1)^{\frac{1}{2}} \rfloor = \sum_{i=1}^{n^2-1} \lfloor i^{\frac{1}{2}} \rfloor.$$

95. Find an expression for $S_n = \sum_{i=1}^{n^3-1} \lfloor i^{\frac{1}{3}} \rfloor$.

4.6. Some More Problems

We end as we started with twenty questions. They all use some aspect of the material in the previous sections or the pigeonhole principle (see Chapter 2). The problems are in no particular order. Some of the later ones are easier than the earlier ones.

Exercises

96. Find all n for which $n^2 + 2n + 4$ is divisible by 7.
97. The lengths of the sides of a right angled triangle are consecutive terms in an A.P. Prove that the lengths are in the ratio 3:4:5.
98. Calculate the sum of the numbers 6 + 66 + 666 +…+ 66…6, where the last number consists of n 6's.
99. Show that among any seven distinct natural numbers not greater than 126, there are two, m and n , such that
100. The product of three consecutive odd numbers is 357627. What is the smallest of the three?
101. Let k be even. Show that 48 is always a factor of $k^3 — 4k$.
102. Find all n for which n, $n + 2$, $n + 4$ are prime numbers.
103. Find all 2-digit numbers which are the square of the sum of their two digits. Are there any 3-digit numbers which are the square of the sum of their three digits?
104. (a) If the tens digit of a perfect square is 7, what is the units digit?
 (b) What is the longest string of 9's you can have at the end of a square number?
 (c) Can 33**6 or 301** be perfect squares, where the asterisks stand for digits?
 (d) Find all squares, all of whose digits are odd.
105. Show that $n(2n + 1)(7n + 1)$ is always divisible by 6. Is it ever divisible by 12?
106. Prove that $n^4 — n^2$ is divisible by 12.
107. Find all natural numbers n for which $n^2 + 80$ is a perfect square.
108. If n is odd and not divisible by 3, show that $n^2 — 1$ is divisible by 24. What are the last two digits of $2^{222} — 1$?
109. What are the last two digits of $2^{222} - 1$?
110. For what positive rational numbers $m = \frac{p}{q}$ is $m + \frac{1}{m}$ an integer?
111. Prove that for any number n,

$$\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+2}{6} \right\rfloor + \left\lfloor \frac{n+4}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+3}{6} \right\rfloor .$$

112. Show that there are no integers a, b, c for which $a^2 + b^2 — 8c = 6$.
113. Let a, b, c , d be fixed integers with d not divisible by 5. Assume that m is an integer for which $M = am^3 + bm^2 + cm + d$ is divisible by 5.
 Prove that there exists an integer n for which $N = dn^3 + cn^2 + bn + a$ is also divisible by 5.
114. (a) Determine all positive integers n for which $2^n + 1$ is divisible by 3.
 (b) Determine all positive integers n for which $2^n + 1$ is divisible by 5.
115. Prove that when 2x+3y is divisible by 17 then so is 9x+5y and vice-versa.

4.7. Solutions

1. 987 × 121 = 109427. This problem can be solved by systematic trial and error. (But note that, since 2 ×*** is a four digit number and "□" × *** is a three digit number, then "□" = 1.)
2. That D equals 1 follows fairly quickly, as does $A \geq 2$ and $C + R \geq 11$. But R is even. Now follow through the various cases. The summands are 92633 + 62513.
3. $A = 3$, $b = 6$, $C = 8$, or $A = 8$, $B = 7$, $C = 2$.

Note that $7 \times 8 \times 9 = 504$ and 739000 divided by 504 has a remainder of 136.

4. 364. (It boils down to using the basic subtraction algorithm or solving $4AB + AB_4 = 800$.)
5. (a) 21978; (b) 10989; (c) none.
6. Start with $n = 10^4x + 10^3y + 10^2z + 10u + v$ and show that $n = 10^3r$.
7. 625×10^a for $a \geq 1$.
8. Suppose the cheque was for \$y : x i.e., $100y + x$ cents. Then $100x + y - 5 = 2(100y + x)$. Now if x 50, then $2x = y - 5$ and $x = 2y$. But this leads to negative solutions. Hence $x \geq 50$ and $x = 2y + 1$, $2x - 100 = y - 5$. This gives $x = 63$ and $y = 31$ so the original cheque was for \$31:63.
9. In this problem 124 divides 10020316 to give 80809.

   (To get started note that 8 times the divisor is only a three digit number. So the divisor is less than 125. Further 9 times the divisor is a four digit number, so the divisor is greater than 111. The rest is careful detective work.) (Where did the 9 come from?)
10. 162 divides 3532572 to give 21806.
11. Let g be the number of geese. Then $2g + \frac{1}{2}g + \frac{1}{4}g + 1 = 100.$

    Hence $g = 36$.
12. HOCUS is 54867.
13. $x^2 + (x + y)^2 = 25$. Now this only has integer solutions if $x^2 = 0, 9, 16$ or 25. Hence (0, ±5), (3,1), (3, –7), (–3, –1), (–3, 7), (4, –1), (4, –7), (–4,1), (–4, 7), (±5,0) are solutions for (x, y).
14. $(121)_b = 1 \cdot b^2 + 2 \angle b + 1 = (1 + b)^2$.
15. $999{,}999 = 3^3 \times 7 \times 11 \times 13 \times 37$. Hence the answer is 37.
16. The left side is 1 if:

    (i) $x^2 - 3x + 1 = 1$, when $x = 0, 3$;

    (ii) $x^2 - 3x + 1 = -1$ and $x + 1$ is even, when $x = 1$; or

    (iii) $x + 1 = 0$ and $x^2 - 3x + 1 \neq 0$, when $x = -1$.

    Hence $x = -1, 0, 1$ or 3.
17. Here we get $x^{\frac{3}{x}+2-x} = 1$ if

    (i) $x = 1$; (ii) $x = -1$ and $\frac{3}{x} + 2 - x$ is even; or

    (iii) $x^2 - 2x - 3 = 0$. Hence $x = -1, 1$ or 3.
18. From 1 to 9 is 9 digits; from 10 to 99 is a further 180 (a total of 189 so far); from 100 to 698 is a further 1797 (a total of 1986 so far). We therefore want the second digit of 699. The answer is 9.
19. Let the six numbers be n, $n+1$, $n + 2$, $n + 3$, $n + 4$, $n + 5$.

First suppose n is even. Then so are $n + 2$ and $n + 4$. One of n, $n + 2$, $n + 4$ must be divisible by 3 as must one of the odd numbers $n + 1$, $n + 3$, $n + 5$. But two of these odd numbers are not divisible by 3 and at most one of them is divisible by 5. So at least one of the six numbers is not divisible by 2, 3 or 5 and so, is not divisible by 4 or 6 either. Hence this number is divisible by primes which are greater than or equal to 7. None of the other numbers can have this number as a factor (because there are only five of them). Hence the result follows.

If n is odd, then only one of n, $n + 2$, $n + 4$ is divisible by 3 and the result follows by the argument above (as applied to $n + 1$, $n + 3$, $n + 5$).

20. We require $10a + b = 2ab$. Hence $10a = b(2a — 1)$. Now since $10a$ and $2ab$ are even, b must be even. Let $b = 2k$. So $5a = k(2a — 1)$. Hence 5 divides k or $2a — 1$. If 5 divides k, 10 divides b. This is not possible since b is a digit. Hence 5 divides $2a — 1$, which gives $a = 3$ or 8. If $a = 8$, $40 = 15k$ which is not possible since k is an integer. If $a = 3$ then $k = 3$ and $b = 6$. Checking we see that 36 has the required property.

21. (i) and (ii) are.

22. (i) and (iii) are.

23. They must be even and divisible by 3. So they must have an even digit in the units column and the sum of their digits must be divisible by 3.

24. (i) If N is even, then M is (even if a zero is added at the end).
However, if M is even (when M ends in zero), N may be odd.
(ii) Yes. Adding zeros will not affect the sum of the digits.
(iii) What about 30 and 3?

25. Using the ideas of the proof of Theorem 1 we see that we can tell the remainders of the number from the remainders of the sum of its digits.

26. (i) and (iii).

27. Let the last two digits of N be ab and the last two digits of M be cd. If $b = d$ is 0, 4 or 8, then M is always divisible by 4 (whether $c = a$ or $c = 0$). If $b = d$ is 2 or 6, then $c = a$ for M to be divisible by 4. If $d = 0$, then M is divisible by 4 (if $c = 0$ or $c = b$, which is even).

For N divisible by 12 we have N divisible by 3 and 4. The sum of the digits of M is divisible by 3 so is M. From the first paragraph we know when M is also divisible by 4.

28. To be divisible by 4, N must be of the form **12, **32, **52, **24. Since N is divisible by 3 it can only be 4512, 5412, 1452, 4152, 1524, 5124.

29. Yes.

30. (i) and (ii).

31. (i) and (iii).

32. $1! = 1$; $2! = 2$; $3! = 6$; $4! = 24$; $i!$ for $i \geq 5$ is divisible by 5. Hence we only have to test $\sum_{i=1}^{n} i!$ for $n \leq 4$. However none of these sums is divisible by 5. The answer is none.

33. Yes — whether or not a zero goes on the end.

34. Basically, every power of 10 from 1000 is divisible by 8.

35. It's what you would expect for 5 and 8.

36. (i) and (iii).

37. (a) The sum of the digits in M is divisible by 9 if and only if the sum of the digits in N is.
(b) False. After all 10 is a fattened form of 1.
(c) False. 90 is divisible by 18 but 9 isn't.
(d) False. Look at 30 and 3.

38. Since N is divisible by 72 it is divisible by 8. Hence $79b$ is divisible by 8. So $b = 2$. Since N is divisible by 9 then so is $a + 6 + 7 + 9 + 2$. Hence $a = 3$.

Hence $a = 3$.

39. (a) Let $N = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$. Since $10 = 11 - 1$, $10^n = (11-1)^n = 11k + (-1)^n$ (by the Binomial Theorem, see Chapter 2, p. 45). So $N = 11t + (-1)^n a_n + (-1)^{n-1}a_{n-1} + \cdots + (-1)a_1 + a_0$.

This gives N is divisible by 11 if the alternating sum $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$ is divisible by 11.

This means that 92723752 is divisible by 11 since $2 - 5 + 7 - 3 + 2 - 7 + 2 - 9 = -11$ is divisible by 11.

(b) Perhaps use $10 = 7 + 3$ so $10^n = 7k + 3^n$.

40. (i), (ii), (vi) and (vii).

41. (i), (ii) and (iii).

42. (a) (i) 231 has to be tested directly;

(ii) for 1988 we need to look at 988 – 1 = 987. This is divisible by 7 so 1988 is;

(iii) 4965 requires 965 — 4 = 961. This is not divisible by 7;

(iv) 756 – 31 = 724 — not divisible by 7;

(v) –1234 + 567 = –667 — not divisible by 7;

(vi) 625 – 471 = 154 — yes;

(vii) 385 is not.

(b) 1234567876543218 is divisible by 7 if 218 – 543 + 876 – 567 + 234 – 1 is divisible by 7. So in general, break up the digits into blocks of 3, putting + and — signs on alternating blocks of 3. If the resulting sum is divisible by 7 then the original number was (and vice-versa).

(c) The same test holds for 11. Why?

(i) 231 – yes (directly);

(ii) for 1212398 think of 1 – 212 + 398 = 187 and 187 divisible by 11;

(iii) for 8282395 test 8 – 282 + 395 = 121 and it's yes again.

(d) The test is exactly the same as for 7 and 11.

(i) 456 - 123 = 333, no;

(ii) 123 - 456 + 789 = 456, no;

(iii) 789 - 456 + 123 - 1 is divisible by 13.

43. (i) 3 and 4; (ii) 7 and 10; (iii) 7 and 2.

44. (i) 21 = 15 + 6; 15 = 2 × 6 + 3; 6 = 2 × 3. Hence (21,15) = 3;

(ii) 4; (iii) 6; (iv) 3; (v) 4; (vi) 1.

45. (a) By the Division Algorithm $0 \geq \mathrm{r}_{i+1}\ r_i$. Hence at each step the quotient (r_i) decreases and so does the remainder (r_{i+1}). Eventually the remainder must become zero.

(b) If r_{s+2} is zero, then r_{s+1} is a factor of r_s. From the second last row, r_{s+1} is a factor of r_{s-1}. Working up the rows we see r_{s+1} is a factor of m and n and hence of (m, n).

On the other hand the g.c.d. g of m and n divides m, n and hence r_1. From the second row g divides n, r_1, and hence r_2. Working down we eventually see that g divides r_{s+1}. Hence since r_{s+1} is a factor of g and vice-versa, so $g = r_{s+1}$.

46. (i) a = 3, b = -2; (ii) a = -2, b = 1; (iii) a = 43, b = -9; (iv) a = -29, b = 56.

47. First note that $2 = 4 \times 6 + (-1) \times 22 = (4 + 22) \times 6 + (-1 - 6) \times 22 = (4 - 22) \times 6 + (-1 + 6) \times 22$ and so on. Hence we can insert as many multiples of 22 to multiply the 6 as we subtract multiples of 6 to multiply the 22. But since 2 divides 22 and 6, we can use 11 and 3. So $2 = 22(3n - 1) + 6(4 - 11n)$, where n is any integer.

48. $5 = 125(18n - 5) + 90(7 - 25n)$.

49. (i) $x = -66 + 7n$, $y = 22 - 2n$; (ii) $x = 12 + 21n$, $y = -3 - 8n$;

(iii) $x = 1260 + 25n$, $y = -6090 - 121n$;

(iv) there are no solutions since $g = 17$ does not divide 100.

50. Spiders have 8 legs and beetles 6. So you have to solve $8s + 6b = 54$ with s, b positive and $s + b$ even. Hence 3 spiders and 5 beetles.

51. Convert this to $17x + 11y = 296$. So $x = 9$ and $y = 13$.

52. If the original cheque is for $100x + y$ cents, we want to solve $98y - 199x = 35$, with x positive and $0 \le y \le 99$. Now $(-67 + 199n)98 + (33 - 98n)199 = 1$ (by the Euclidean Algorithm). We now need to find n such that $0 \le -67.35 + 199n$ 99. Here $n = 12$ to give $y = 43$. Then $33 \cdot 35 - 98 \cdot 12 = -21$. So the original cheque was for \$21.43.

Experiment with values other than 5 and 35.

53. $1 = 4 \times 16 - 7 \times 9$. Fill the 16 litre container 4 times and empty the contents into the 9 litre container. Throw away 7 lots of full 9 litre containers and you'll have 1 litre left.

54. Assume $g = (21n + 4, 14n + 3)$, then there exists a and b such that $(21n + 4)a + (14n + 3)b = g$. Hence $7n(3a + 2b) + (4a + 3b) = g$. Since this equation is true for all n, $3a + 2b = 0$ and $4a + 3b = g$. This gives $a = -2g$ and $b = 3g$. But then g^2 is a factor of $(21n + 4)a$ and $(14n + 3)b$. So g^2 is a factor of the sum of these which is g. Hence $g = 1$.

55. (a) $(x - y)(x + y) = 74$. So either $x - y = 1$, $x + y = 74$ or $x - y = 74$, $x + y = 1$ or $x - y = 2$, $x + y = 37$ or $x - y = 37$, $x + y = 2$ etc. with the factors of 74. None of these have integer solutions.

(b) No. Try $r = 4$.

(c) If r is odd, then one of $x - y$, $x + y$, has to be odd. Then there are no integral solutions. If r is even we can always split the factors of $2r$ so that $x - y$ and $x + y$ are both even. Hence they have integral solutions. So the complete answer is r odd.

56. $(\pm 14, \pm 13)$, $(\pm 6, \pm 3)$.

57. For integral solutions 2 is a factor of $x - y$ and $x + y$. Hence x and y are either both even or both odd.

58. $112296^2 - 79896^2 = (112296 - 79896)(112296 + 79896) = (32400) \cdot (192192)$. Now $32400 = 10 \times 5 \times 648 = 10 \times 5 \times 9 \times 72 = 10 \times 5 \times 9 \times 6 \times 12$. Further $192192 = 11 \times 17472 = 11 \times 7 \times 2496 = 11 \times 7 \times 8 \times 312 = 11 \times 7 \times 8 \times 3 \times 104 = 11 \times 7 \times 8 \times 3 \times 4 \times 26 = 11 \times 7 \times 8 \times 3 \times 4 \times 2 \times 13$. All the factors of 13! are present. (You should use the tests discovered in Section 4.3.)

59. (a) $(1 - 2)(1 + 2) + (3 - 4)(3 + 4) = -(1 + 2 + 3 + 4)$.

(b) $1 + (3 - 2)(3 + 2) + (5 - 4)(5 + 4) = 1 + 2 + 3 + 4 + 5$.

(c) $1^2 - 2^2 + 3^2 + (-1)^{n-1}n^2 = (-1)^{n-1}(1 + 2 + 3 + \ldots + n)$.

60. N is obviously even because $1^n—3^n$ is even. Then $(1^n—6^n)+(8^n—3^n) = (1 — 6)(1 + 6 + 6^2 +\ldots+ 6^{n-1}) + (8 — 3)(8^{n-1} + 8^{n-2}3 +\ldots+ 3^{n-1})$. Hence N has a factor of 5.

Now $N = –5(1+6+6^2 +\ldots+6^{n-1})+5(8^{n-1}+3 \cdot 8^{n-1} +\ldots+3^{n-1}) = –5[1 + 6(1 + 6k)] + 5[3^{n-1} + 8m]$. So N is divisible by 4 if and only if $5(3^{n-1} – 7)$ is divisible by 4. This holds for n even.

N is divisible by 40 if and only if $3^{n-1} – 43$ and $n < 2$ is divisible by 8. This is true for n even and $n \geq 4$.

How far can you go? 80? 160?

61. Since 2141 –1863 = 1770 –1492 = 278, the given expression is divisible by 278. Similarly, 2141 — 1770 = 1863 — 1492 = 371, which is relatively prime to 278, also divides the given expression. Hence (278)(371) = (53)(1946) is a divisor.

This means finding the factors of the current year and working them into an $a^n – b^n + c^n – d^n$ scenario.

62. (a) $4n^3+6n^2+4n+1 = (n+1)^4-n^4 = ((n+1)^2-n^2)((n+1)^2+n^2)) = (2n+1)[(n+1)^2+n^2]$.

(b) Try $n = 1$.

(c) $(n + 1)^6 – n^6 = [(n + 1)^3 – n^3][(n + 1)^3 + n^3]$.

(d) Conjecture: $(n + 1)^m – n^m$ is composite if n is even. It is not necessarily composite if n is odd (though it can be sometimes — when?).

63. Experiment. You should find that 6 does but 12 or 18 doesn't. Note that $n^3 – n = n(n – 1)(n + 1)$.

64. (a) $M =(7+15)(7^{2n} – 7^{2n-1} \cdot 15 +\ldots+(–1)^{2n}15^{2n})$ which is divisible by 2 for $n > 0$. The case $n = 0$ is OK.

(b) $E = 7^{2n} – 7^{2n-1} \cdot 15 +\ldots+(–1)^{2n}15^{2n}$ is the sum of an odd number of odd numbers. So it's odd and 44 is out.

(c) From (b), $E = 7^{2n} + 15k$ so E is divisible by 3 when 7^{2n} is. That is, never.

65. (a) See Exercise 64(a). You'll do better with 44 here but not with 66.

(b) When is $E = a^{2n} – a^{2n-1}b +\ldots+(–1)^{2n}b^{2n}$ divisible by 3? Never if a (or b) alone is divisible by 3. If $a = b = 11$, then $E = (2n+1)11^{2n}$which is divisible by 3 if and only if $2n + 1$ is. The same thing happens for $a = 2$, $b = 20$ (or vice-versa). For $a = 8$, $b = 14$ I think the answer is $n – 1$ needs to be divisible by 3.

66. From Exercise 64 the expression is clearly divisible by 11. Now $5^{2n+1} = (3+2)^{2n+1}$ which is of the form $3k+2^{2n+1}$ (by the Binomial Theorem). Similarly for 11^{2n+1} and 17^{2n+1}. Hence $5^{2n+1} + 11^{2n+1} + 17^{2n+1}$ is divisible by 3 if $2^{2n+1} + 2^{2n+1} + 2^{2n+1}$ is. But $3 x 2^{2n+1}$ is obviously always divisible by 3.

67. By Fermat $a^5 = 5k + a$ for some k. Now $a^{241} = a(5k + a)^{48} = 5t + a^{49}$(by the Binomial Theorem). But $a^{49} = a^4(5k + a)^9 = 5s + a^{13}$ and $a^{13} = a^3(5k + a)^2 = 5u + a^5 = 5v + a$.

Hence $E = 1^{241} + 2^{241} + 3^{241} + 4^{241}$ has remainder 1 + 2 + 3 + 4 when divided by 5. Hence E is divisible by 5.

The same argument gives a^{240} has remainder 1 on dividing by 5. Hence $F = 1^{240} + 2^{240} + 3^{240} + 4^{240}$ has remainder 1 + 1 + 1 + 1. Hence F is not divisible by 5.

68. $\Sigma^4_{i=1} i^n$ is divisible by 5 if and only if n is not a multiple of 4.

(Wait till you've read the congruences section before you try to prove this.)

69. By Remark 1 if d is the smallest number such that $2^d - 1$ is divisible by 47, then d divides 46. Hence d =1,2, 23 or 46. Clearly $d \neq 1,2$. However $2^{23} - 1$ is divisible by 47.

Is it true that $m = p(p - 1)/2$ always gives 2^m is divisible by p, a prime? If so, why didn't Fermat prove this?

70. (i) $a = 4$; (ii) $a = 1$ (since $4 \equiv 1 \pmod 3$);

(iii) $2^{240} \equiv (2^2)^{120} \equiv 1^{240} \equiv 1 \pmod 3$ (or use Fermat);

(iv) $2^{240} \equiv (2^4)^{60} \equiv 1 \pmod 5$.

71. $a^p \equiv a \pmod p$ for p a prime.

72. Exercise 67: $a^5 \geq a \pmod 5$. Hence $a^4 \equiv 1 \pmod 5$ for a not a multiple of 5. So $a^{241} \equiv (a^4)^{60}a \equiv 1^{60}a \equiv a \pmod 5$. Hence $1^{241} + 2^{241} + 3^{241} + 4^{241} \equiv 1 + 2 + 3 + 4 \equiv 0 \pmod 5$. Hence $1^{240} + 2^{240} + 3^{240} + 4^{240} \equiv 4$ which is not congruent to 0 (mod 5). Exercise 68: Let $n = 4k + r$. Then $a^n \equiv (a^4)^k a^r \pmod 5$. Hence $a^n \equiv a^r \pmod 5$. Hence we only have to consider $r \equiv 0, 1, 2, 3$.

$$r = 0: \quad 1^0 + 2^0 + 3^0 + 4^0 \equiv 4 \not\equiv 0 \ (mod\ 5)$$
$$r = 1: \quad 1^1 + 2^1 + 3^1 + 4^1 \equiv 0 \ (mod\ 5)$$
$$r = 2: \quad 1^2 + 2^2 + 3^2 + 4^2 = 30 \equiv 0 \ (mod\ 5)$$
$$r = 3: \quad 1^3 + 2^3 + 3^3 + 4^3 = 100 \equiv 0 \ (mod\ 5)$$

Hence $\Sigma^4_{i=1} i^n = 0 \pmod 5$ if and only if n is not divisible by 4. Exercise 69: We want d to be the smallest positive number such that $2^d \equiv 1 \pmod{47}$. Since d = 1, 2, 23 or 46 we only have to test the first three values. $2^1 = 2$ not congruent to 1 (mod 47). $2^2 = 4$ not congruent to 1 (mod 47). Now $2^9 \equiv 42 \equiv -5 \pmod{47}$. Hence $2^{18} \equiv 25 \pmod{47}$. So $2^{19} \equiv 3 \pmod{47}$ and $2^{23} \equiv 3\times 16 \equiv 48 \equiv 1 \pmod{47}$.

73. **Proof of Lemma 1.** If $a \equiv b$ (mod c) then $a - b = ck$ for some k. Hence $ma - md = mck$, so $ma = mb$ (mod c).

□

Proof of Lemma 2. If $a \equiv b \pmod c$ then $a = b + ck$. Hence $a^n = (b + ck)^n = b^n + ct$ (by the Binomial Theorem). Hence $a^n = b^n \pmod c$.

□

74. Since for all k, the five terms $k + 1$, $k + 2$, $k + 3$, $k + 4$, $k + 5$ are congruent, in some order to 1, 2, 3, 4, 5 (mod 5), then we only need consider $\Sigma^5_{i=1} i^n$. But $5^n = 0 \pmod 5$ for all n. Hence we only need to consider $\Sigma^4_{i=1} i^n$. Now go back to Exercise 66.

75. Again $a^n = a^{6k+r} = a^r \pmod 7$. We only need consider the cases r = 0, 1,2, 3, 4, 5 to see that the result follows.

76. When p is a prime, is $\Sigma^{p-1}_{i-1} i^n = 0 \pmod p$ if and only if n is not divisible by $p - 1$?

Does this work for composite p though?

77. The smallest n is 1, 2, 3, 4, 5, 6, 10, 15 or 30. Clearly 1, 2, 3 do not work. But $2^5 = 32 \equiv 1 \pmod{31}$. Hence $n = 5$.

78. For p odd, $2^{(p-1)/2} \equiv 1$ (mod p) if and only if $p \equiv \pm 1 \pmod 8$. I don't know a simple way of proving this.

"The smallest n is p – 1" problem is an, as yet, unsolved problem. It is not even known whether or not there are an infinite number of such primes. If you think you have a solution please let me know.

79. If n has last digit 6, then $n = 10N+6$. Condition (ii) gives $6 \times 10^k + N = 4(10N + 6)$. Hence $2 \times 10^k - 8 = 13N$, so $2 \times 10^k \equiv 8 \pmod{13}$. Thus $10^{k+1} = 40 \equiv 1 \pmod{13}$.

From Fermat, $k + 1 = 1, 2, 3, 4, 6$ or 12. Trial and error gives $k = 5$. Hence 13N = 199992 and so $n = 153846$.

80. (i) $T_5 = 11$, $T_{10} = 21$, $T_n = 2n + 1$;

(ii) $T_5 = 35$, $T_{10} = 75$, $T_n = 8_n - 5$;

(iii) $T_5 = 9$, $T_{10} = 11$, $T_n = n + 4$;

(iv) $T_5 = 28$, $T_{10} = 58$, $T_n = 6_n - 2$.

81. $T_n = a + (n - 1)d$.

(i) Here $a = 3$, $d = 2$, so $T_n = 2n + 1$;

(ii) $a = 3$, $d = 8$, $T_n = 8n - 5$;

(iii) $a = 5$, $d = 1$, $T_n = n + 4$;

(iv) $a = 4$, $d = 6$, $T_n = 6n - 2$.

82. $\sum_{i=1}^{100} i = \frac{100}{2}(101) = 5050$.

83. $\sum_{i=1}^{2n} i - \sum_{i=1}^{n} i = 100$. Therefore $\frac{2n(2n+1)}{2} - \frac{n(n+1)}{2} = 100$. Hence $3n^2 + n - 200 = 0$. Since $n \neq -25/3$, $n = 8$.

84. (a) $\sum_{i=1}^{100} 2i = 2\sum_{i=1}^{100} i = 10100$.

(b) $\sum_{i=1}^{100}(2i - 1) = \left(\sum_{i=1}^{100} 2i\right) - 100 = 10000$.

85. (a) $2S_n = (1 + 121) + (4 + 118) + (7 + 115) + \ldots + (121 + 1) = 41 \cdot 122$

$\therefore \quad S_n = 2501$.

(b) $\frac{n}{2}(3n - 1)$.

86. (i) $a = 2$, $d = 3$ $\therefore$ $S_{20} = 610$; (ii) $a = 2$, $d = 7$ $\therefore$ $S_{20} = 1370$;
(iii) 1440; (iv) 1190;
(v) 100; (vi) -420.

87. We want $\sum_{i=1}^{n} 3i$, for $3n < 200$. So $n = 66$. $\sum_{i=1}^{66} 3i = 6633$.

88.
$$S_n = a + (a + d) + \cdots + [a + (n - 1)d]$$
$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + \cdots + a$$
$$\therefore \quad 2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + \cdots + [2a + (n - 1)d]$$
$$\therefore \quad S_n = \frac{n}{2}[2a + (n - 1)d.$$

89. (a) $t_n = \sum_{i=1}^{n} i = \frac{n}{2}(n + 1)$.

(b) $t_3 = 6$, $t_2 = 3$. Let $t_s = \frac{s}{2}(s + 1)$ and $t_r = \frac{r}{2}(r + 1)$ be such that $t_s = 2t_r$. Hence $s(s + 1) = 2r(r + 1)$.

The only other solution I have found so far is $t_{20} = 2t_{14}$. Are there any others?

(c) $t_8 = 3t_3$, $t_9 = 3t_5$.
$t_s = 4t_r$? Hmmm! this requires $s(s + 1) = 4r(r + 1)$.
If $s \leq 2r$, then $s^2 + s \leq 4r^2 + 2r < 4r(r + 1)$.
If $s \geq 2r + 1$, then $s^2 + s \geq 4r^2 + 6r + 2 > 4r(r + 1)$.
Hence there is no solution to $t_s = 4t_r$. (Maybe there is a nicer proof of this.)

(d) Notice that $t_{2t_r} = \frac{2t_r(2t_r+1)}{2} = t_r(2t_r + 1)$. Hence t_r divides t_{2t_r} exactly.

90. $\sum_{i=1}^{10} i^2 = \frac{10 \cdot 11 \cdot 21}{6} = 385.$

91. $\sum_{i=1}^{n} (2i-1)^2 = \frac{(2n-1)(2n)(2n+1)}{6} = \frac{n}{3}(4n^2-1)$

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)^2 &= 4\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i + \sum_{i=1}^{n} i \\ &= 4\frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\ &= \frac{1}{3}(4n^3-n) \\ &= \frac{n}{3}(4n^2-1).\end{aligned}$$

92. $\sum_{i=1}^{n}(2i)^2 = 4\sum_{i=1}^{n} i^2 = \frac{2}{3}n(n+1)(2n+1).$

93. To show that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, start considering the differences of the form $(n+1)^4 - n^4$ and follow the method of Example 13.

94. First you will need to discover that

$$\lfloor k^{\frac{1}{2}} \rfloor = \begin{cases} 1 & \text{for } 1^2 \le k \le 2^2 - 1 \\ 2 & \text{for } 2^2 \le k \le 3^2 - 1 \\ 3 & \text{for } 3^2 \le k \le 4^2 - 1 \\ \cdots & \cdots \end{cases}$$

So in S_n there are $2^2 - 1$ ones, $3^2 - 2^2$ twos, $4^2 - 3^2$ threes and so on.

Hence

$$\begin{aligned} S_n &= 1(2^2-1^2) + 2(3^2-2^2) + \cdots + (n-1)[n^2-(n-1)^2] \\ &= -1^2 - 2^3 - 3^2 - \cdots - (n-1)^2 + (n-1)n^2 = \frac{n}{6}(n-1)(4n+1).\end{aligned}$$

95. Here you'll need

$$\lfloor k^{\frac{1}{3}} \rfloor = \begin{cases} 1 & \text{for } 1^3 \le k \le 2^3 - 1 \\ 2 & \text{for } 2^3 \le k \le 3^3 - 1 \\ 3 & \text{for } 3^3 \le k \le 4^3 - 1 \\ \cdots & \cdots \end{cases}$$

Hence $S_n = \frac{n^2}{4}(n-1)(3n+1)$.

96. Let $n = 7k+r$. Then $n^2 + 2n+4 = (7k+r)^2 + 2(7k+r)+4 \equiv r^2 + 2r+4 \pmod 7$. Checking $r = 0, 1, 2, 3, 4, 5, 6$ we see that $r = 1$ or 4. Hence n is of the form $7k + 1$ or $7k + 4$ (i.e. $n \equiv 1$ or $4 \pmod 7$.)

97. Let the sides be a, $a + d$, $a + 2d$. Then $a^2 + (a + d)^2 = (a + 2d)^2$. We solve the quadratic for d to give $d = \frac{a}{3}$ or $-a$. If $d = -a$ one side has negative length. Hence the sides are a, $\frac{4a}{3}, \frac{5a}{3}$ 4f, and are in the required ratio.

98.
$$\begin{aligned} S_n + \frac{1}{2}S_n + n &= (6+3+1) + (66+33+1) + \cdots + (66\cdots6 + 33\cdots3 + 1) \\ &= 10 + 100 + \cdots + 10^n \\ &= \frac{10}{9}(10^n - 1) \quad \text{(Sum of Geometric Progression)}\end{aligned}$$

Hence $S_n = \frac{20}{27}(10^n - 1) - \frac{2}{3}n$.

99. For the pigeonhole principle, see Chapter 2, p. 29*ff*. If we divide {1,2,…, 126} into 6 sets, then one of these contains at least two of the chosen 7 numbers. If we can now find 6 sets such that the largest number is at most twice the smallest we will have solved the problem.

The following sets will do:

{1,2}, {3,4,5,6}, {7,8,9,…,14}, {15,16,…,30}, {31,32,…,62}, {63,64,…,126}.

100. Let the odd numbers be $a - 2$, a, $a + 2$. (This simplifies the algebra.) Hence we have to find the solutions of $a^3 - 4a - 357627 = 0$. The cube root of 357627 is about 70 and a is odd so we find $a = 71$ is a possible root. Then $(a - 71)(a^2 + 71a + 5037) = 0$. Since a is positive $a^2 + 71a + 5037$ is never zero. The only solution is 71. Hence the smallest odd number required is 69.

101. Let $E = k^3 - 4k = k(k - 2)(k + 2)$. Since k is even, $k - 2$, k, $k + 2$ are consecutive even numbers, so one of them (at least) is divisible by 4. Hence E is divisible by 16. Further, since $k - 2$, k, $k + 2$ are consecutive, one of them is divisible by 3. Hence E is divisible by 48.

102. Now n must be odd, since otherwise $n + 2$ is not prime. Since n, $n + 2$, $n + 4$ are consecutive odd numbers, one of them is divisible by 3. But since they are all primes, one of them *is* 3. This prime has to be n , since 1 is not a prime. Hence the three primes are 3, 5, 7.

103. Let the required number be $N = 10a + b$. Then we have to solve $10a + b = (a + b)^2$. Now this gives $a^2 + a(2b - 10) + (b^2 - b) = 0$ which has solutions

$$a = \frac{(10 - 2b) \pm \sqrt{(2b - 10)^2 - 4(b^2 - b)}}{2}.$$

Now a is an integer, so $d = (2b-10)^2 - 4(b^2 - b)$ is a square. However, $d = 100 - 36b$, so clearly b 3. If $b = 0$, $a = 10$ and so isn't a digit. If $b = 1$, $a = 8$. If $b = 2$, d isn't a square. Hence 81 is the only number with the required property.

There's a nice problem for you!

104. (a) 6. This arises when squaring numbers congruent to 24, 26, 74 or 76 (mod 100).

(b) If $b^2 \equiv 9 \pmod{10}$, then $b \equiv 3, 7 \pmod{10}$.

If $(10a + b)^2 \equiv 99 \pmod{100}$, then $20ab + b^2 = 99 \pmod{100}$.

For $b = 3$, $60a + 9 \equiv 99 \pmod{100}$. This has no solutions for a.

For $b = 7$, $140a + 49 \equiv 99 \pmod{100}$. Again this has no solutions.

(c) Let $N = 33{*}{*}6 = M^2$. Then $M \equiv 4, 6 \pmod{10}$. Further 180 $M < 190$, so $M = 184$ or 186. Now 186 is too large, so $N = 33856$.

Let $P = 301{*}{*} = Q^2$. Then $170 < Q < 180$, but $173^2 = 29929$ and $174^2 = 30276$, so P isn't a square.

(d) Take 1 and 9 for free.

Let $N = (10a + b)^2$ have all odd digits. Then $N = 100a^2 + 20ab + b^2$ and $b^2 = 1, 5, 9 \pmod{10}$ and $20ab + b^2 =$ odd number (mod 100). Now in fact $b^2 = 1, 9, 25, 49, 81$ and all of these cause the tens digit to be even.

105. Let $n = 6q + r$. Then $n(2n + 1)(7n + 1) \equiv r(2r + 1)(7r + 1) \pmod{6}$. Hence the expression is divisible by 6 if it is for $n = 0, 1, 2, 3, 4, 5$. It is.

If $n = 4$, we get 36×29 which is divisible by 12.

106. Now $n^4 - n^2 = (n - 1)n^2(n + 1)$. One of $n - 1$, n, $n + 1$ is divisible by 3. If n is even we are finished. If n is odd, both $n - 1$ and $n + 1$ are even and we are finished.

107. If $n^2 + 80 = m^2$, then $m^2 - n^2 = 80$. So $(m - n)(m + n) = 80$. We take only even factors of 80 to give $n = 1, 8, 19$.

108. Let $N = (n - 1)(n + 1)$. Since n is odd, $n - 1$ and $n + 1$ are consecutive even numbers, so one of them is divisible by 4. Hence N is divisible by 8. Since n is not divisible by 3, one of $n - 1$, $n + 1$ is. Thus N is divisible by 24.

109. Now $2^{10} = 1024 \equiv 24 \pmod{100}$ and $2^{20} \equiv 24^2 = 576 \equiv 76 \equiv -24 \pmod{100}$. Hence $2^{30} = -24^2 = -24 \pmod{100}$, $2^{40} \equiv -24 \pmod{100}$ and so on. Since $222 = 220 + 2$, $2^{222} \equiv -24 \times 2^2 \equiv -96 \equiv 4 \pmod{100}$. Hence $2^{222} - 1 = 3 \pmod{100}$. The last two digits of $2^{222} - 1$ are 03.

110. Let $m = \frac{p}{q}$, where p and q are natural numbers with no common factor. Then $m + \frac{1}{m} = \frac{p^2+q^2}{pq}$. If this is an integer, then p and q both divide $p^2 + q^2$. Hence p is a factor of q^2. But p and q have no factors in common. Hence $p = 1$. Similarly the fact that q divides p^2 implies $q = 1$. Hence $m = 1$.

111. Now $n \equiv 0, 1, 2, 3, 4, 5 \pmod 6$. Testing all these values shows the result holds. Hence it holds for n in general.

112. Now $n \equiv 0, 1, 2, 3 \pmod 4$, then $n^2 \equiv 0, 1, 4 \pmod 8$. So $a^2 + b^2 = 0, 1, 2, 4, 5 \pmod 8$. But $a^2 + b^2$ is not congruent to 6 (mod 8). Hence $a^2 + b^2 \neq 8c + 6$.

113. Since $M \equiv 0 \pmod 5$ and d is not congruent to 0 (mod 5), then m is not congruent to 0 (mod 5). Hence $m = 5k + r$ for $r = 1, 2, 3, 4$.

Now $Mn^3 - N = (mn - 1)[a(m^2n^2 + mn + 1) + bn(mn + 1) + cn^2]$. We now attempt to choose n so that the right-hand side of this equality is divisible by 5.

This can be done by choosing n such that $mn - 1$ is divisible by 5. If $m = 5k + r$ and $n = 5t + s$ then $mn - 1 \equiv rs - 1 \pmod 5$. So if wecan find an s for each r, $1 \leq r \leq 4$, then we can find an n for every $m \equiv 0 \pmod 5$.

If $r = 1$, $s = 1$; if $r = 2$, $s = 3$; if $r = 3$, $s = 2$; if $r = 4$, $s = 4$.

(Actually for each $m \not\equiv 0 \pmod 5$ there are an infinite number of n which make $N \equiv 0 \pmod 5$.)

114. (a) Let $n = 2q + r$. Then $2^{2q+r} \equiv (2^2)^q 2^r \equiv 2^r \pmod 3$. Hence since $2^r + 1 = 0 \pmod 3$ for $r = 1$, then $2^n + 1 = 0 \pmod 3$ for all n odd.

(b) Let $n = 4q + r$. Then $2^n \equiv 2^r \pmod 5$. Now $2^0 \equiv 1 \pmod 5$, $2^1 \equiv 2 \pmod 5$, $2^2 \equiv 4 \pmod 5$ and $2^3 \equiv 3 \pmod 5$. Hence $2^n + 1 \equiv 0 \pmod 5$ for $n \equiv 2 \pmod 4$.

115. Using Theorem 6, if $2x + 2y = 17n$, then $x = -17n + 3k$ and $y = 17n - 2k$. Hence $9x+5y = 9(-17n+3k)+5(17n - 2k) = 17(-4n) + 17k$. Hence $9x + 5y$ is also divisible by 17.

Now suppose $9x + 5y = 17n$. Again by Theorem 6 we have $x = -17n + 5k$ and $y = 34x - 9k$. Hence $2x + 3y = 17(6n) - 17k$. So $2x + 3y$ is also divisible by 17.

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[a]An algorithm is a step by step procedure that eventually finishes.

Chapter 5
Geometry 1

5.1. Introduction

Geometry is a vast area that it would take many books to get close to uncovering. I have only written two chapters in this book but I hope that will be enough to get you started. In this first chapter I've done small amounts on squares, triangles, circles and their properties as well as some ruler and compass constructions.

Many of the problems here can be generalised. That means there are bigger problems that contain my problems as special cases. You should always be looking out for generalisations. That way, in one fell swoop you can solve a lot of little problems as a result of solving one big problem.

You should also be thinking of *extending* a problem. For instance, if something works for squares, does it work for something similar?

Keep asking questions; ask yourself, ask your friends, ask your teacher. In mathematics asking questions (the right questions) is half the battle. Getting the right answer is usually the result of a process of asking a sequence of the right questions.

5.2. Squares

One of the simplest shapes is a square so let's start there. Just in case you have never seen one of these we show one in Figure 5.1.

A *square* is a four-sided animal all of whose sides are equal and such that adjacent sides are perpendicular. So in Figure 5.1, $AB = BC = CD = DA$ and $\angle ABC = \angle BCD = \angle CDA = \angle DAB = 90°$. Naturally squares come in all sizes from the side-of-a-house-size squares to postage-stamp-size squares and even smaller and even bigger.

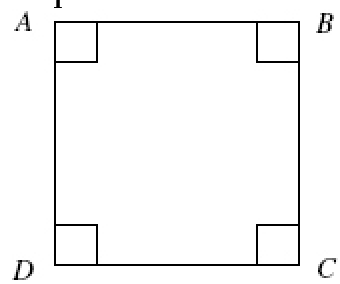


Figure 5.1.

For the record, if the side *AB* is of length a, then the perimeter of the square is 4a (that's just the length round the outside) and the area is a^2 (that's just the stuff inside).

That's all pretty dull really and perhaps so is the fact that the poor square invariably goes unnoticed as it is squashed under foot or stuck on the wall and splashed on. But, from our point of view, the fact that a square, along with an infinite gang of its mates all of whom are of the same size, fits together without gaps to completely cover the plane, is quite useful. Such stuff are tiles made of. We say that squares *tile* or *tessellate* the plane. This is shown in Figure 5.2.

Although squares are great to tessellate, they aren't the only shape that'll do it. We can see this by starting with a square and adding an arc of a circle on one side. (See Figure 5.3.)

Now add the same arc on the inside of the opposite side. Throw away the shaded area in Figure 5.3 and you've got another shape that'll tessellate.

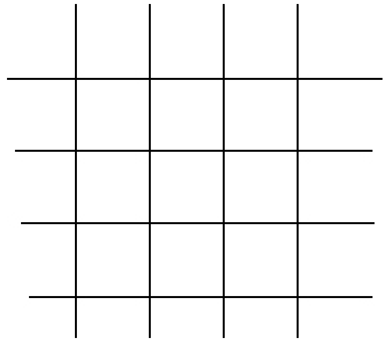


Figure 5.2.

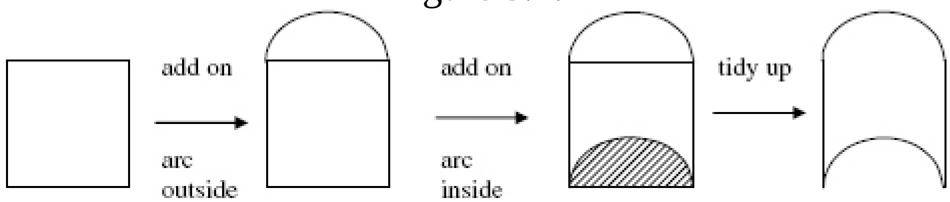


Figure 5.3.

Exercises

1. Experiment with various additions and subtractions from a square to produce irregular shapes that tile the plane. Can you produce animal, bird or fish shapes in this way, which tessellate?
   (What has this got to do with M.C. Escher? Who you won't find on MacTutor. Or will you?)
2. Starting from a square, use the idea of Figure 5.3 to show that there are hexagons (six-sided shapes), which tile the plane.
   The hexagons here are, of course, not *regular*. That is they don't have all sides equal and all angles equal. Do regular hexagons tessellate? Show that there are eight-sided figures which tessellate. Do regular octagons (all sides equal, all angles equal) tessellate?
   Show that there are 2n-sided figures which tessellate, for all natural numbers n. Are there ($2n + 1$)-sided figures which tessellate?
3. A square is a special type of quadrilateral— a shape with four sides. A square has two properties
   (i) all sides are equal; and
   (ii) all angles are right angles.
   Show that there is an infinite number of different quadrilaterals with property (i). What types of quadrilaterals have property (ii)?
4. Do all quadrilaterals tessellate?

A square not only tessellates it also does it in a self-replicating way. If we put four squares of the same size together we produce another square whose side length is twice that of the original square (see Figure 5.4). A *self-replicating* shape is one that, by putting enough copies of itself together, can produce a larger copy of itself.

Any self-replicating shape must tessellate the plane.

So we've found that a square can make another square. On the other hand, any square can be broken down into smaller squares. Clearly the large square of Figure 5.4 can be broken down into four smaller squares. It should also be easy to see that there are *five* squares in Figure 5.4.

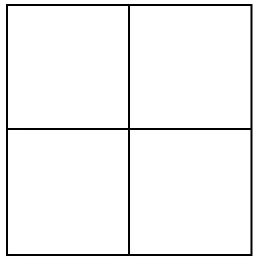


Figure 5.4.

Exercises

5. Find four self-replicating shapes.
6. Why does a self-replicating shape tessellate?
7. (a) How many squares are there in Figure 5.5(a)?

   (b) How many squares are there in Figure 5.5(b)?

   (c)Take a square of side length n, which is made up of n^2 smaller squares. Imagine that we've drawn the picture of this. How many squares are there in the picture? (This count is to include all 1 by 1, 2 by 2,…squares.)
8. So we can see how to *square* a square (make up a square from smaller squares) using squares all of which are of the same size.

   (a) Can you square a square with *two* different sizes of squares?

   (b) Can you square a square with *three* different sizes of squares?

   (c) Is it possible to square a square with squares all of whose sizes are different?

   (d) For what *m* is it possible to square a square with *m* squares?

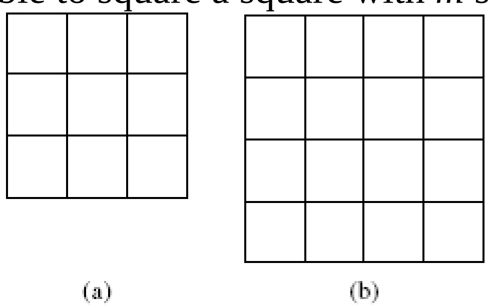


Figure 5.5.

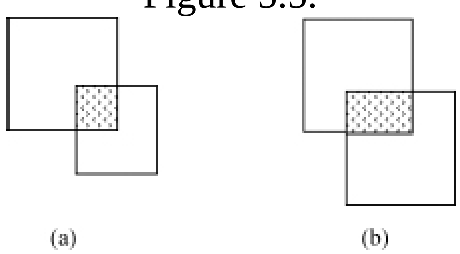


Figure 5.6.

   (e) Given one square each of size 1,2,…, *n*, is it possible to put them together to form a square?

(Take your time over this problem. It was not meant to be easy. You can learn a lot by sticking at it till you've got it out. If you can't solve it, don't look up the answer. Try it out on a friend.)

Take any two squares of any size and plonk them down on any plane that happens to be handy. Now have a good look at how they overlap. What sorts of intersection can we get?

Clearly if we put the squares a long way away from each other they won't intersect at all. Their intersection will be the empty set.

But if you look at Figure 5.6 you can see that we can get a square (area shaded in Figure 5.6(a)) or some other four-sided figure (see Figure 5.6(b)).

Exercises

9. (a) Can two squares overlap (intersect) in a four-sided figure whose angles are not *all* right angles?

(b) Can two squares intersect in a four-sided figure with (i) precisely three right angles; (ii) precisely two right angles; (iii) precisely one right angle; or (iv) no right angles?

(c) Can two squares intersect in n-gons (n-sided figures) for (i) $n = 3$; (ii) $n = 5$; (iii) $n = 6$; (iv) $n = 7$; (v) $n = 8$; (vi) $n = 9$; (vii) $n \geq 10$?

(d) Describe carefully all possible n-gons which arise by intersecting two squares. (Concentrate on the kind of angles the n-gons can have.)

(Again this question is meant to be thought provoking. Take your time over it. Try it out on your friends. Try it out on your poor unsuspecting long-suffering teacher. Only then look at the answer.)

10. Now take a cube-shaped piece of cheese. Cut it straight through with a knife. What shaped faces can you produce? (See the Bright Sparks section of the site www.nzmaths.co.nz.)

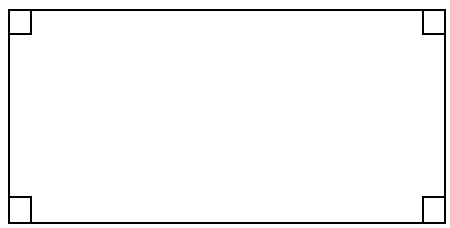


Figure 5.7.

5.3. Rectangles and Parallelograms

A *rectangle* is a four-gon (four-sided figure) all of whose angles are right angles and whose opposite sides are equal in length.

We show a rectangle in Figure 5.7. Obviously a square is a special type of rectangle.

Exercises

11. Do rectangles tessellate the plane?
12. Did Escher ever start one of his “tessellations” from rectangles?
13. Are rectangles self-replicating?
14. Is every shape that tessellates the plane a self-replicating shape?
15. Can you square a rectangle

(i) with squares of equal size;

(ii) with squares of unequal size;

(iii) with squares which are all of different sizes;

(iv) with m squares;

(v) with one square each of side length 1,2,…, n?

16. Divide a rectangle of side lengths 6 and 9 into squares of side length one. How many squares are there?

Generalise.

17. Can you rectangle a rectangle? That is, can you make up a rectangle from smaller rectangles? In what different ways can this be done?

18. Take any two rectangles and plonk them down anywhere in the plane. In how many different shapes will the two rectangles intersect?

So now we get to parallelograms. A *parallelogram* is a gram made of parallels. Take two pairs of parallel lines. The four-sided figure ("gram") they make is a parallelogram (see Figure 5.8). So a parallelogram is a foursided figure with both pairs of opposite sides parallel.

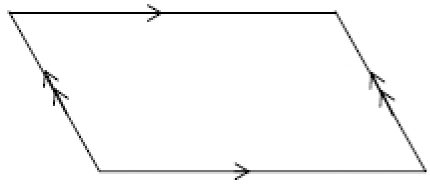


Figure 5.8.

We represent the parallel property by the insertion of arrows. Because the top and bottom sides of the parallelogram in Figure 5.8 are parallel we put an arrow on each of them. Because the left and right sides of the parallelogram are parallel (but not parallel to the top and bottom sides) we put two arrows on each of them.

In general the angles between adjacent sides of a parallelogram are not equal. However, when they are we get a rectangle or a square. Squares and rectangles are just special parallelograms.

Exercise

19. Repeat Exercises 11-18 with the words "rectangle" and "square" replaced everywhere by "parallelogram".

It's worth picking up a few tips about parallel lines and angles.

In Figure 5.9, it should be clear that the angles *b* and *c* are equal. As you rotate the horizontal line *BC* about *B* till it aligns with *AB*, the angles of size *b* and *c* are both traced out together. So $b = c$.

These angles are called *vertically opposite.* So vertically opposite angles are equal in size.

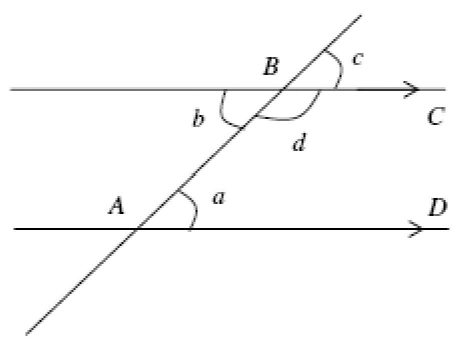


Figure 5.9.

Further $a = b$. These two angles are *alternate angles* on the parallel lines *BC, AD.*

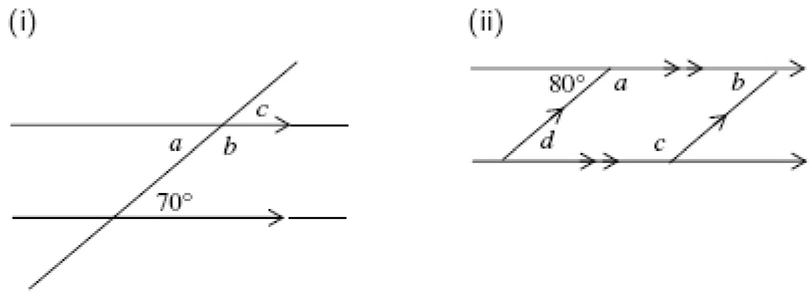
Since $a = b$ and $b = c$, then $a = c$. The angles *a* and *c* are *corresponding angles* on the parallel lines *BC, AD.*

Finally since $c + d = 180°$(*AB* is a straight line) and $a = c$ then $a + d = 180°$. Angles like *a* and *d* on a pair of parallel lines always add up to 180°.

Incidentally, angles that sum to 180° are called *supplementary*.

Exercises

20. Find the size of a, *b, c, d* in each of the following diagrams.



21. In any parallelogram show that opposite angles are always equal. Are two neighbouring angles in a parallelogram supplementary? What is the sum of the interior angles of any parallelogram?

5.4. Triangles

A *triangle* is a figure with three sides (or three angles). Triangles are much more varied than squares or even rectangles. The only limit to their variety is the one fact that they all have in common, apart from the three angles or three sides. This fact is that the sum of the angles of any triangle is 180°. We show a collection of triangles in Figure 5.10.

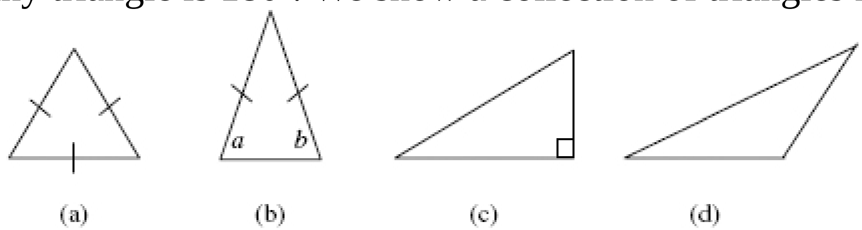


Figure 5.10.

In Figure 5.10(a) we have a triangle all of whose sides are equal and all of whose angles are equal. Such triangles are called *equilateral*. This comes from the Latin "equi" for equal and "latus" for side. An equilateral triangle is therefore equal sided. We show this in Figure 5.10(a) by putting a little line in the middle of each side.

Hence we can see the triangle in Figure 5.10(b) has only two sides equal. Such triangles are known as *isosceles* triangles. This comes from the Greek "iso" for equal and the "skelos" meaning leg. (You can't say that reading these booklets is not a cultural experience now can you?) If you've got two equal legs then you can make an isosceles triangle with the ground.

Figure 5.10(c) shows a *right angled triangle*. The side opposite the right angle is called the *hypotenuse*. Everyone knows that the square on the hypotenuse is equal to the sum of the squares on the other two sides. This is called Pythagoras' theorem.

Figure 5.10(d) is just another old triangle that doesn't have any particular name. Or does it?

Exercises

22. (a) What is the size of each angle in an equilateral triangle?
    (b) Are any angles in an isosceles triangle equal? What is the biggest number of degrees the angle at the feet of an isosceles triangle can be?
    (c) Is every isosceles triangle equilateral or vice-versa?
    (d) Can a right angled triangle be isosceles? If so, what are the sizes of its angles?
    (e) Can a right angled triangle be equilateral?
    (f) What is a scalene triangle? What is an obtuse angled triangle? What is an acute angled triangle?
23. (a) Do equilateral triangles tessellate the plane?

(b) Can the same be said of all other triangles?
(c) Are all triangles self-replicating?
(d) Did Escher ever start one of his "tessellations" from some sort of triangle?

24. (a) Divide an equilateral triangle of side length 2 into equilateral triangles of side length 1. How many equilateral triangles of side length 1 are there?
Repeat with an equilateral triangle of side length 3.
Generalise.
(b) Now ask how many equilateral triangles there are of any side length in an equilateral triangle of side length n.

25. The last exercise shows that you can "equilaterally triangle" an equilateral triangle. Is it possible to form an equilateral triangle using equilateral triangles all of which have sides that are of a different size?

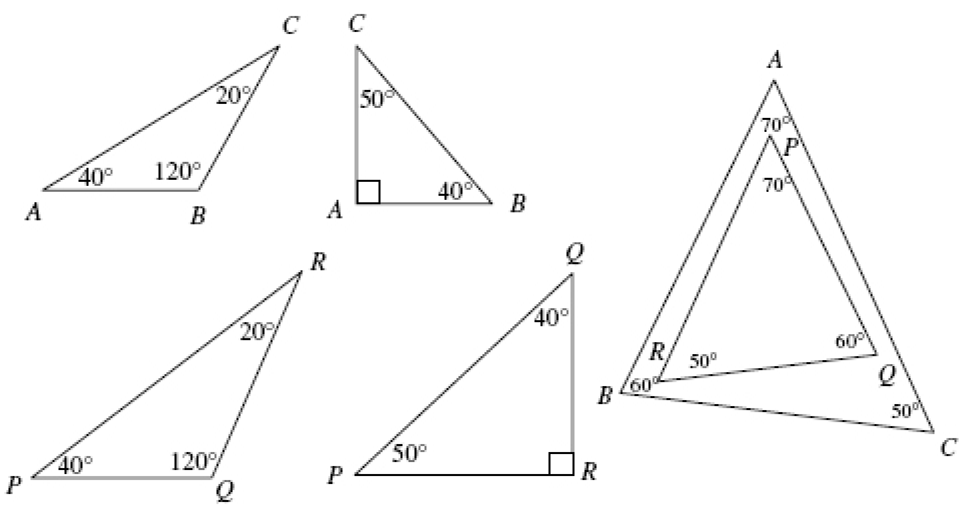
26. (It's plonk time again.) Plonk two equilateral triangles of arbitrary size down in the plane. What possible shapes are the intersections?
Repeat with various shaped triangles.

As a result of all the above activity we know that any triangle is a selfreplicating figure. How did we know that four copies of a triangle can be put together to form the same sort of triangle? The basic assumption was that two triangles were "the same" if all their angles were the same. Now that does seem a reasonable assumption. We'll use it to define similar triangles.

Two triangles are *similar* if corresponding angles are equal. Figure 5.11 shows three sets of similar triangles.

It looks as if the larger of any pair of similar triangles can be obtained from the smaller by "pumping it up". In actual fact that is pretty well what's going on. For each pair of similar triangles *ABC*, *PQR* (in Figure 5.11 and anywhere else) the ratio of corresponding sides is fixed. Hence, for some fixed r,

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = r.$$



Exercises

27. Draw a pair of similar triangles *ABC*, PQR. Measure *AB*, *BC*, *CA*, *PQ*, *QR* and *RP*. Check that $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP} = r$. What value of *r* did you get? Now draw a pair of similar triangles with *r* = 2.5.

28. Draw a pair of equilateral triangles. Measure the appropriate lengths to find *r*. Why are all equilateral triangles similar?

29. Assume that *Δ's ABC, PQR* are similar. What can be said about the values $\frac{AB}{BC}, \frac{PQ}{QR}$? Which of $\frac{BC}{CA}, \frac{PQ}{RP}, \frac{QR}{RP}$ are equal and why?

Now if $r = 1$, we can pick up one triangle and fit it exactly on top of the other one. In that case we say that the two triangles are *congruent.*

The next example is typical of a whole series of proofs in geometry.

Example 5.1. Show that the opposite sides of a parallelogram are equal.

Proof. Let the parallelogram be *ABCD* (see Figure 5.12). Join *B* to D. Now consider *Δ*'s *ABD* and *CDB.*

Now $\angle ABD = \angle CDB$, alternate angles on the parallel lines AB, *CD.* Similarly $\angle ADB = \angle CBD$. Since the angles in any triangle add up to 180°, these two angle equalities imply that $\angle BAD = \angle DCB$.

So *Δ*'s *ABD* and *CDB* are similar. Hence $\frac{AB}{BD} = \frac{CD}{BD}$ because of the fact that the ratio of corresponding sides in similar triangles is equal. We must therefore have $AB = CD$.

But $\frac{AD}{BD} = \frac{BC}{BD}$. Hence $AD = BC$.

So opposite sides of a parallelogram are equal.

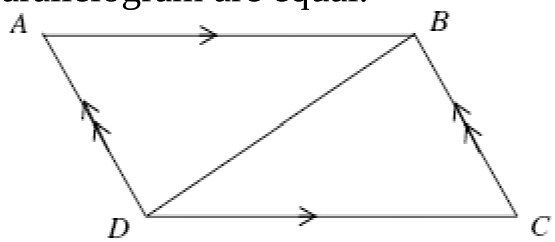
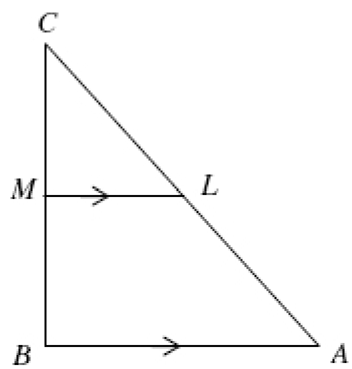


Figure 5.12.

Exercises

30. Prove that the diagonals of a parallelogram bisect each other. (In other words, show that in Figure 5.12, if *AC* and *BD* meet at the point *E*, then $AE = EC$ and $BE = ED$.)

31.



In the figure *M* is the midpoint of *BC* and *LM* is drawn parallel to *AB.* Show that (i) *L* is the midpoint of *AC* and (ii) *LM* is half the length of *AB.*

32. If in the figure of Exercise 31, we change the position of *M* so that $\frac{BM}{CM} = r$ what can be said about (i) the position of *L* along *AC* and (ii) the relative sizes of *LM* and *AB*?

5.5. Circles

A *circle* is a set of points in the plane all of which are the same distance from a fixed point in the plane. The fixed point is the *centre* of the circle and the constant distance is its *radius.*

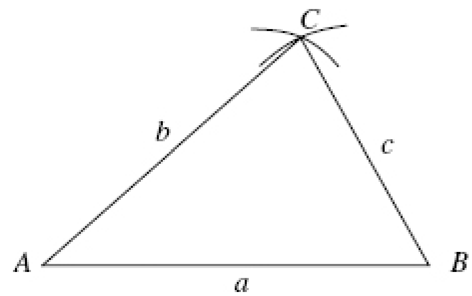
All circles look the same. They're sort of, well, round.

Just about now you should be expecting my usual onslaught of questions involving tessellating, self-replicating, circling the circle and so forth. However none of those work. You just can't fit two circles close enough together to tessellate or self-replicate or whatever. So let's try something else.

Exercises

33. Cover the plane with an infinite number of non-overlapping circles all of which have the same radius. If this is done as efficiently as possible what fraction of the plane is covered?

34. (a) Three circles of radius 1 just fit together inside a circle of radius *r* without overlapping. Find *r*.
    (b) Four circles of radius 1 just fit together inside a circle of radius *r* without overlapping. Find *r*.
    (c) Generalise.



35. A circle of radius 1 has area π and a square of side length 2 has area 4. So it might be possible to place two overlapping circles of radius 1 so that they completely cover a square of side length 2. Can this be done?
    If it can, show how. If it can't, find the smallest number of circles of radius 1 required to cover a square of side length 2.

Now we've got circles, we can construct triangles. Suppose we want to produce a triangle with sides *a*, *b* and c. Then first we draw a line of length *a*. Call the ends *A* and *B* (see Figure 5.13).

Now measure out a radius of length *b* on your compasses and draw an arc of the circle, centre A, and radius b. Repeat this process with the arc of a circle, centre B, and radius c. These two arcs intersect at *C* and ΔABC has sides *a, b, c* as required.

One thing to note here is that given *a*, *b*, *c* there is only one triangle that can be constructed with side lengths *a*, *b*, and *c*. (You can see this because the construction of Figure 5.13 produced only one triangle— of course there is another meeting point on the other side of *AB* but it produces a congruent triangle.) Hence all triangles with sides *a*, *b*, *c* are congruent. We say that they are congruent *SSS* (meaning side, side, side) since corresponding sides are equal in length.

Exercises

36. Construct triangles with the following side lengths.
    (i) 5, 12, 14; (ii) 3, 4, 5; (iii) 6, 7, 8; (iv) 6, 7, 18.

37. For what a, *b*, *c* is there *no* triangle of sides *a*, *b*, *c*? (Assume *a*, *b*, *c* are all positive real numbers.)

Why is $a + b > c$ known as the *triangle inequality?*

If two triangles agree *AAA* are they congruent? By this I mean are two triangles congruent if they have three corresponding angles equal?

Figure 5.14.

The answer to this might be obvious by now. Look back at Figure 5.11. So what combination of correspondingly equal angles and sides gives congruent triangles?

For a start we know from Chapter 2 that there are six ways of arranging three letters which can either be A's or S's. These are

SSS, *SAA*, *ASA*, *AAS*, *ASS*, *SAS*, *SSA*, *AAA*.

We've already dealt with *SSS* and *AAA*. Consider *SAA*. The two triangles in Figure 5.14 have two equal angles and a common side equal. Since the angles of a triangle sum to 180° we have $a + b + c = 180°$ and $a + b + d = 180°$. Hence c = *d*.

So *SAA* triangles are clearly AAA. In other words they're at least similar. However, they're similar with one equal side. Hence they must be congruent.

In exactly the same way a pair of *ASA* and a pair of *AAS* triangles are also congruent.

What about *ASS* then?

Exercises

38. In the following situations is it possible to construct more than one triangle *ABC*? (The units are in centimetres.)
 (i) $\angle ABC$ =60°, *BC* =10.0, *CA* = 9.5;
 (ii) $\angle ABC$ =60°, *BC* =10.0, *CA* = 9.0;
 (iii) $\angle ABC$ =60°, *BC* =10.0, *CA* = 8.66;
 (iv) $\angle ABC$ =60°, *BC* =10.0, *CA* = 8.0.
39. Under what conditions is it possible to construct a unique triangle, given an angle α, a side *b* and a side *c* in that order round the triangle?
40. (a) Can a pair of *ASS* triangles ever be congruent?
 (b) Repeat (a) for *SAS* and SSA.

Having covered all six possibilities we now have a complete set of tests for congruence. Two triangles are congruent if they agree *SSS, SAA, ASA , AAS, SAS* or *RHS.*

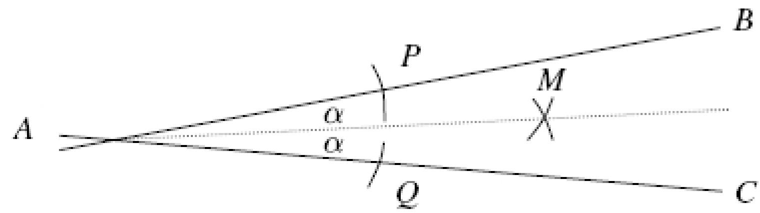
The last condition comes from *ASS* when we know we have a **R**ight angle and we're also given the **H**ypotenuse and a **S**ide. (See Exercise 40(a).)

Exercises

41. (a) Show that a diagonal divides a square into two congruent triangles.
 (b) Repeat (a) with “square” replaced by “rectangle”.
 (c) Is the result still true if “square” is replaced by “parallelogram”?
42. Show that the diagonals of a square intersect at right angles.
 For what other parallelograms is this true?

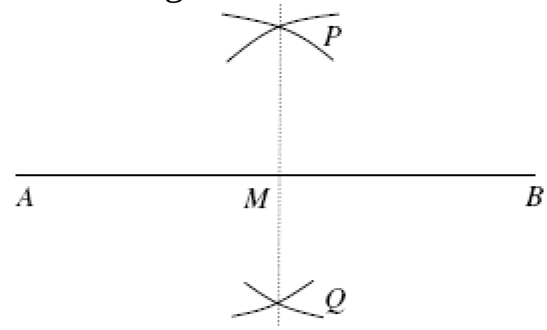
As the Greeks knew a couple of thousand years ago, circles and straight lines are good for making shapes with. So let's run through some ruler and compass constructions.

Construction 1. To bisect an angle. In the diagram, *P* and *Q* lie on a circle, centre A. *M* is the point of intersection of a circle with centre *P* and one of the same radius with centre Q. We claim that *AM* bisects angle BAC. So why does this work?



Join *P* to *M* and *M* to *Q* by line segments. Then consider Δ's *APM, AQM*. Now *AP* = *AQ* since these are both radii of the circle that we drew first which was centred at A. Similarly *PM* = *QM* — equal radii again. And of course *AM* = *AM*. Hence Δ's APM, *AQM* are congruent SSS. Hence ∠*PAM* = ∠*QAM*.

Construction 2. To bisect the line segment *AB*.



Arcs of circles centred at *A* and *B* with the same radius (greater than ½*AB*) meet at *P* and *Q*. The line *PQ* meets *AB* at *M*. We claim that *M* is the midpoint of *AB*.

Exercises

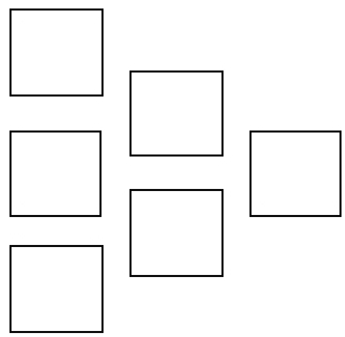
43. Use congruent triangles to prove that *M* is the midpoint of *AB* in Construction 2. Further prove that *PQ* is perpendicular to *AB*.
44. Use ruler and compasses to construct the altitude from *A* to *BC* in some Δ*ABC*. (That is, construct *AH* such that *AH* is perpendicular to *BC* and *H* is on *BC*.)
45. Use ruler and compasses to construct the median from *C* to *AB* in Δ*ABC*. (That is, construct *CM* such that *M* is on *AB* and *MA* = *MB*.)
46. Use ruler and compasses only to perform the following
    (i) Construct a square with a given side length.
    (ii) Find the centre of a given circle.
    (iii) Given a circle construct a square which lies outside the circle so that the sides of the square are tangents to the circle.
    (iv) Given a square, construct a circle which passes through its vertices.
    (v) Given a square, construct a circle which has the four sides of the square as tangents.
    (vi) Given an angle *α*, construct an angle at a given point equal to *α*.
    (vii) Repeat (iii), (iv) and (v) replacing "square" by "regular hexagon" and then "regular pentagon".
    (viii) Construct a triangle which has the same area as a given quadrilateral.
47. Show that there is an infinite number of circles which pass through two points.
    How many circles pass through three given points?
    How many circles pass through four given points?
48. Construct a square with two vertices on one side of a given triangle and the other two vertices one on each of the other two sides.
49. (a) Find the sum of the internal angles of a hexagon.
    (b) Find the sum of the internal angles of an n-gon.

(c) A *concave* polygon has some interior angles bigger than 180°. Find the sum of the interior angles of a concave quadrilateral.

(d) Repeat (c) for a concave pentagon.

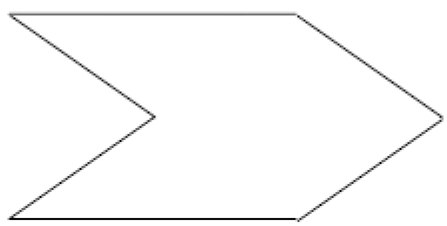
(e) Repeat (c) for a concave n-gon.

50. What is the size of an interior angle of a regular n-gon?
51. Which regular n-gons tessellate the plane?
52. Which regular n-gons are self-replicating?
53. Is it possible to divide any square up into n squares for any $n \in \mathbb{N}$?
54. The drawing below is an equilateral triangle of squares pointing to the right. What is the fewest number of squares that need to be moved so that the triangle is facing to the left?

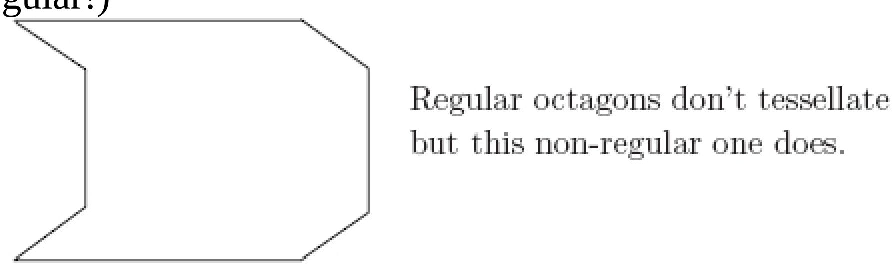


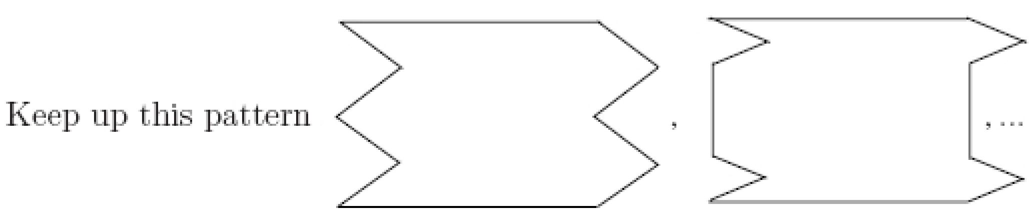
5.6. Solutions

1. Escher made many prints based on tessellations. He was able to produce birds, fish, horses and riders, etc. that came from tilings. How did you do? (See *The Graphic Work of MC Escher* published by Pan, London, 1973 or check on the web.)
2.

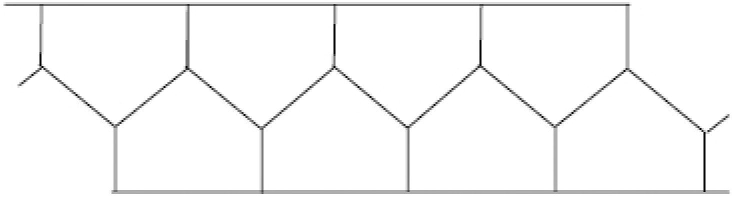


Regular hexagons do tessellate (ask your local bees how). (Not that the hexagon to the left is not regular!)

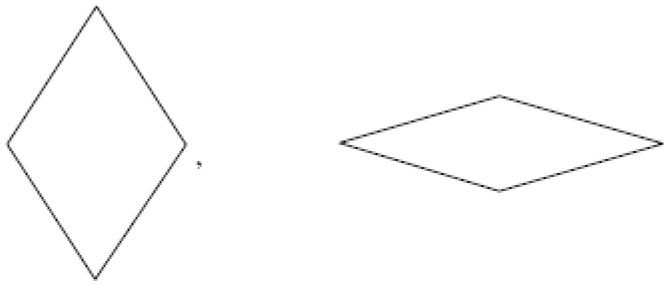




You can tessellate with odd-gons but you have to be a bit tricky. Start with pentagons and work your way up.



3. (i)

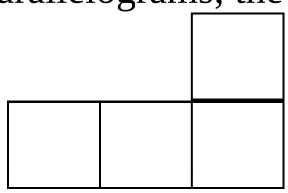


, etc.

(ii) Squares and rectangles.

4. Yes, but this takes a little bit of work.

5. Rectangles (take four copies); parallelograms; the shape below; for more see later.



6. It can be made bigger and bigger to eventually cover more and more, and eventually all of the plane.

7. (a) 14 = 9 + 4 + 1; (b) 30; (c) $1^2+2^2+3^2+\cdots+n^2=\frac{n}{6}(n+1)(2n+1)$ (see Chapter 4, Example 13, p. 130 for the simplification).

8. (a) Of course. Use one 2 × 2 and five 1 × 1 squares in Figure 5.5(a).

(b) You should manage to square a 5 × 5 square with one 3 × 3 and an assortment of 2 × 2's and 1 × 1's.

(c) Yes. It's done somewhere in this chapter but I'm not saying where.

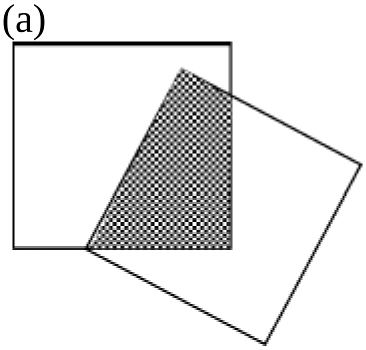
(d) This can be done for all *m* except *m* = 2, 3 and 5. Once you get 6, 7 and 8 you can successively divide a square up into four smaller squares and so get all the remaining values of *m*.

(e) Suppose this were possible. Then the smaller squares could make up an *m* × *m* square. Calculating areas gives $m^2=\sum_{i=1}^{n} i^2=\frac{n}{6}(n+1)(2n+1)$.

It is a non-trivial result in Number Theory to show that the only solutions of $6m^2 = n(n+1)(2n+1)$ are *n* = —1, 0, 1, 24. Did you get this far?

But can you actually square the 70 × 70 square with different squares ?

9. (a)



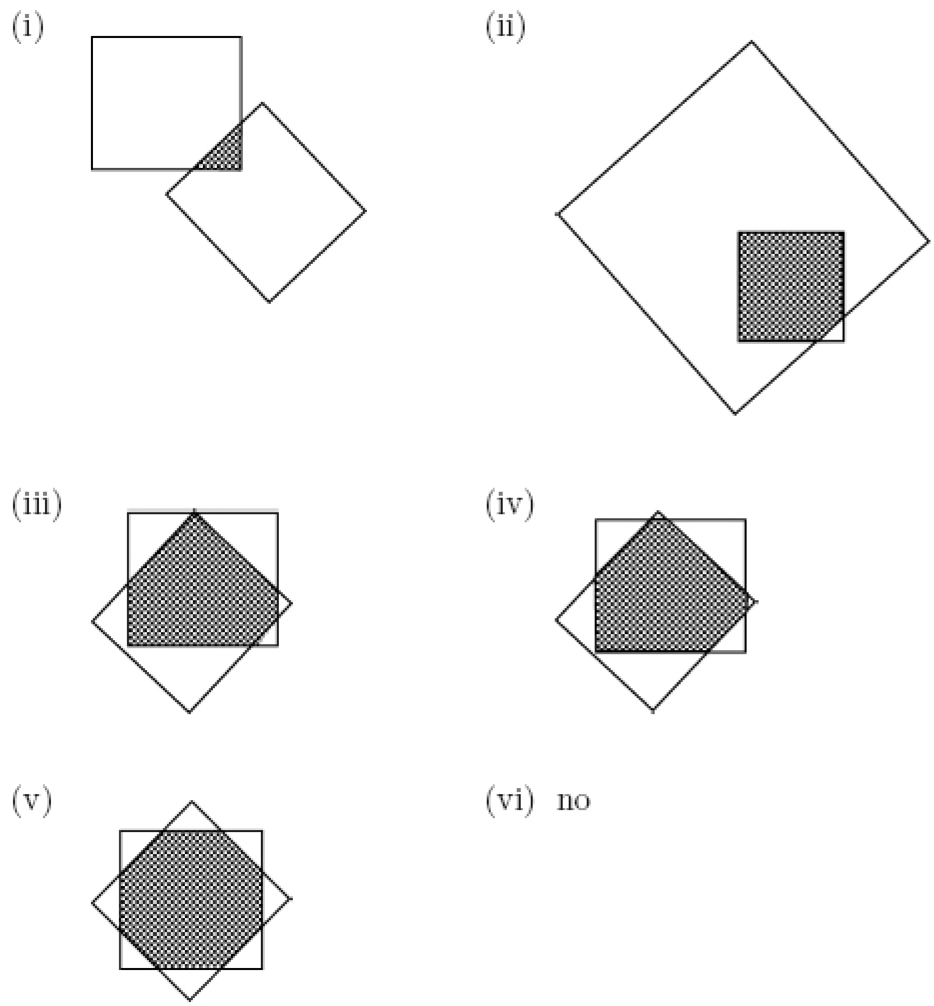
(b) (i) If three are right angles then the fourth one *has* to be, whether the figure is formed by squares or not;

(ii) See (a). Can the right angles be adjacent?

(iii) No. If the right angle is produced at the intersection of two sides of distinct squares, then the intersection has four right angles. The one right angle of the intersecting 4-gon must therefore be from one of the squares. All other corners of this square must lie outside the region of intersection. This forces one corner of the other square to be in the intersection.

(iv) This would mean that all the corners of the squares would be outside the region of intersection. This forces a polygon with more than four sides.

(c)



(vii) The most that can be done is 8, because no side of a square can be intersected more than twice.

(d) Must a triangular intersection contain a right angle?

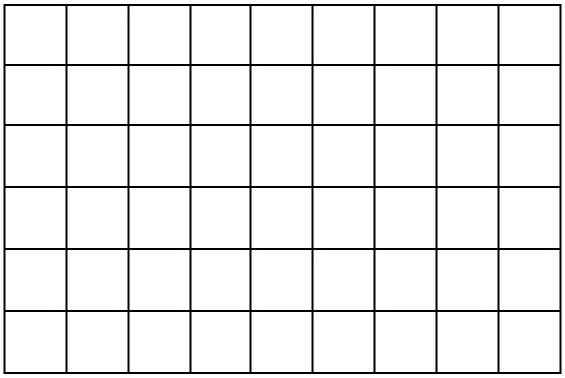
Must a 5-gon have three right angles? Must a hexagon have two right angles? Must a heptagon have only one and an octagon none?

10. See Bright Sparks at www.nzmaths.co.nz.
11. Yes. (If you can't see this, tessellate with squares and then let neighbouring pairs of squares join to form a rectangle.)
12. Most certainly. Look at his horsemen for instance.
13. Take four and put them together.
14. No. Try self-replicating the pentagons of Exercise 2.
15. (i) easily; (ii) see Exercise 8;

(iii) put the squares of Exercise 8(c) together;

(iv) this might depend on the size of the rectangle;

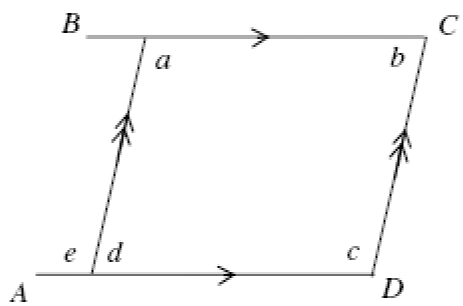
(v) I have no idea.



1×1 squares :	$6 \times 9 =$	54
2×2 squares :	$5 \times 8 =$	40
3×3 squares :	$4 \times 7 =$	28
4×4 squares :	$3 \times 6 =$	18
5×5 squares :	$2 \times 5 =$	10
6×6 squares :	$1 \times 4 =$	4
	Total	154

For an *m x n* rectangle where *m* < *n* we have $\sum_{i=0}^{m-1}(m-i)\times(n-i)$ squares. This simplifies to $\frac{1}{6}m(m+1)(3n-m+1)$. (What happens if *n* = m?) (Now see how many cubes there are in a box.)

17. It's easy enough to see that you can put the same rectangle together several times to build up another rectangle. Is it possible to divide a rectangle into unequal rectangles though? Have a look at Exercise 8 and then take a stretch.
18. Is there any difference between the sort of shapes you can get here and the ones you got in Exercise 8?
19. Ex. 11: Yes. Ex. 12: Possibly. Ex. 13: Yes. Ex. 15: Push rectangles out of shape. Ex. 16: You clearly can't divide all parallelograms up into rectangles. What about dividing a 6 × 9 parallelogram up into parallelograms of size 1 × 1 though? Generalise. Ex. 17: Push rectangles out of shape. Ex. 18: Does this give anything new? (Clearly you can drop the right angle restrictions.)
20. (i) *a* = c =70°, *b* =100°. (ii) *a* =100° = c, *b* = 80° = d.
21.



Now *e* = *a*, alternate angles. Then *e* = *c* corresponding angles. Hence *a* = *c*.

Since e + *d* = 180°, c + *d* = 180°. Hence *c* and *d* are supplementary. Since *a* = c, then *a* and *d* are supplementary.

Clearly *a* + *b* + c + *d* = 360°.

22. (a) 60

(b) Equal angles are opposite equal sides. In Figure 5.10(b) then, *a* = b. Since *a* + *b* < 180°, and *a* = b, then *a* and *b* are both less than 90°. However, they can be as close to 90° as you care to make them without ever equalling 90°.

(c) Every equilateral triangle is isosceles but not vice-versa.

(d) Yes. The angles of such a triangle are 45°, 45°, 90°.

(e) No! Definitely not!

(f) I hope you've looked these up in an old geometry book or on the web. If not, a *scalene triangle* has all its sides (and therefore angles) different sizes; an *obtuse* *angled* triangle has an obtuse angle — one bigger than 90°; and an *acute angled* triangle has no obtuse angles.

23. (a) Yes. (b) Yes. (c) Yes. (d) Yes.

24. (a) Four. Nine. In general an n-sided equilateral triangle has 2n + 1 triangles along a side. So the total number of small equilateral triangles is 1 + 3 + 5 +…+ (2n +1) = $\sum_{i=1}^{n}$ (2*i*+ 1). This turns out to be n^2. (For a proof see Chapter 6, Exercise 13(ii).)

(b) Now the side length 2 triangles give us some problems. First there are 1 + 2 + 3+ …+ (*n* — 1) oriented this way: Δ and a further 1 + 2 + 3+ -- - + (n — 4) oriented this way: V. This gives n^2 — 4n + 6 of these.

The other side lengths work in the same way.

For side length 3 we have [1 + 2+ + (n — 2)]+ [1 + 2+ + (n — 6)];

For side length 4 we have [1 + 2+…+ (n — 3)] + [1 + 2 +…+ (n — 8)] and so on.

Do these all add up to a simple formula? If they do, can the formula be obtained by some more efficient method?

25. I really don't know the answer to this. Can anyone help me? Is it on the web somewhere?

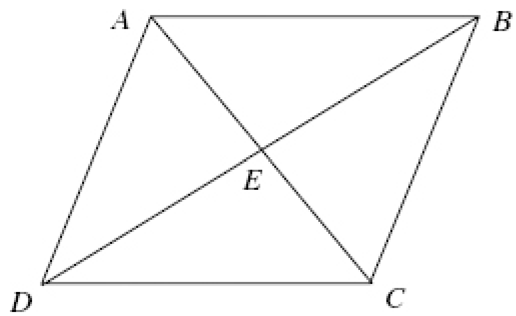
26. All sized n-gons from *n* = 3 to 6 inclusive can be obtained, whether or not we stick to equilateral triangles.

27. What did you get? Can you now make similar triangles with *r* to order? (It might be easy to do this with some geometry software.)

28. They all have the same angles.

29. They are equal. $\frac{BC}{CA} = \frac{QR}{RP}$ since $\frac{BC}{QR} = \frac{CA}{RP}$.

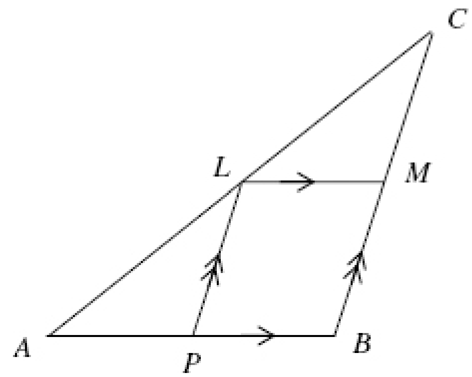
30.



In parallelogram *ABCD* we want to show that *AE* = *EC* and that *BE* = *ED*.

Consider Δ's *ABE*, *CDE*. Now ∠*BAE* = ∠*ECD* alternate angles and ∠*ABE* = ∠*EDC* for the same reason. Hence Δ's *ABE*, *CDE* are similar since all the angles are the same. (Of course we only need to prove two angles are equal. The third angle follows since the angles in any triangle sum to 180°. Actually in the present case ∠*AEB* = ∠*CED* since they are opposite angles.)

But we know from Example 1 that *AB* = *CD*. Hence Δ's *ABE*, *CDE* are congruent and so *AE* = *EC* and *BE* = *ED*.



We know that $CM = MB$ and $LM||AB$ (LM is parallel to AB). Draw LP so that P is on AB and $LP||CB$.

First we show that Δ's ALP, LCM are congruent. They are similar since they have the same angles. This can be seen by noting that $\angle LAP = \angle CLM$ (corresponding angles on parallel lines) and $\angle ALP = LCM$ (for the same reason). But $MB = LP$ by Example 1. Hence $LP = CM$. So Δ's ALP, LCM are indeed congruent.

(i) It follows immediately that $AL = LC$, so L is the midpoint of AC.

(ii) Further $AP = LM$ since Δ's ALP, LCM are congruent and $PB = LM$ by Example 1. Hence $LM = \frac{1}{2}AB$ as required.

32. By the initial part of the argument of the last exercise, Δ's ALP, LCM are similar.

(i) Now if $\frac{BM}{CM} = r$, then $\frac{PL}{CM} = r$ also ($PL = BM$ by Example 1). Since Δ's ACP, LCM are similar and $\frac{PL}{CM} = r$, then $\frac{AL}{LC} = r$. This gives the position of L and AC.

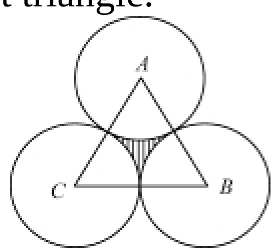
(ii) Now $\frac{AP}{LM} = r$ the similarity of Δ's ACP, LCM.

Hence $\frac{LM}{AB} = \frac{LM}{AP+PB} = \frac{LM}{AP+LM}$ by Example 1.

Hence $\frac{LM}{AB} = \frac{1}{\frac{AP}{LM}+1} = \frac{1}{r+1}$.

33. The closest packing of circles occurs when their centres are on equilateral triangles.

The fraction of the area of the plane not covered is the ratio of the shaded part of the triangle ABC to the total area of that triangle.

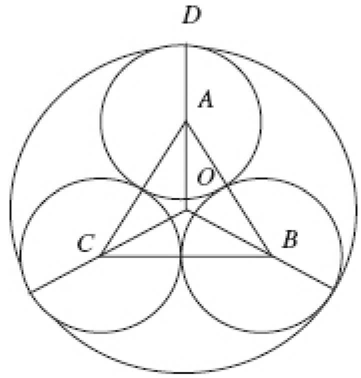


Suppose the radius of each circle is one. Then the length of each side of the triangle is 2 and the altitude is √3 (an application of Pythagoras). Hence the area of ΔABC is √3.

The sector of each circle in ΔABC subtends an angle of 60° at the centre. Hence its area is $\frac{1}{2}(\frac{2\pi}{6}) = \frac{\pi}{6}$.(The area of a sector is $\frac{1}{2}r^2\theta$. If you want to use degrees for the angle C then the area is $\frac{1}{2}r^2(\frac{2\pi}{360})\theta°.)$

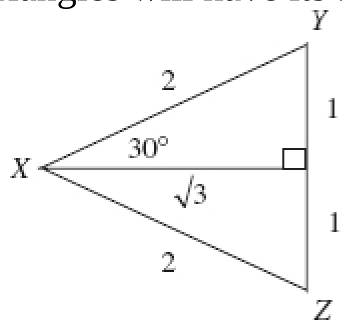
The shaded area is therefore $\left(\sqrt{3} - \frac{\pi}{2}\right) \div \sqrt{3}$ The proportion of the plane not covered is therefore $\sqrt{3} - \frac{3\pi}{6} = \sqrt{3} - \frac{\pi}{2}$. and the ratio covered is $\frac{\pi}{2\sqrt{3}}$. About 90% of the plane is covered by circles.

34. (a)



If we can find *OD* we will have found *r*. Now first consider the equilateral triangle *ABC*. Each side is of length 2 and the altitude is $\sqrt{3}$.

Inside triangle *XYZ* we have two right angled triangles with 30° and 60° angles. By similarity any such right angled triangles will have its sides in the ratios $1:\sqrt{3}:2$.

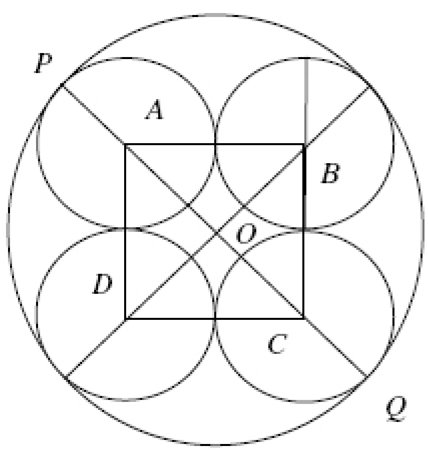


Now go back to the circles. Extend *DO* to intersect *BC* at *E*. Then ΔOEC is a right angled triangle and $\angle OCE = 30°$.

By what we said above $\frac{CE}{OE} = \frac{\sqrt{3}}{1}$. But *CE* = 1 since it is the radius of the circle. Hence $OE = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Now $AO = AE - OE = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2}{3}\sqrt{3}$. Thus $r = AD + Ao = 1 + \frac{2}{3}\sqrt{3}$.

(b).



PQ is a diameter of the circle, so *PQ* = *2r*. But *PQ* = *PA* + *AC* + *CQ*. Since *AC* is a diagonal of the square *ABCD* which has side length 2, then $AC = 2\sqrt{2}$ (use Pythagoras' Theorem). Hence $2r = 2 + 2\sqrt{2}$ or $r = 1 + \sqrt{2}$.

(c) It's probably easier to do an even number of circles first. Is the answer for 6 circles simply 3?
Show what happens to *r* as the number of circles gets larger and larger. Does *r* approach a limit?

35. If every corner of a square lies inside a circle then four different circles are required. This is because no circle can cover two corners in its interior.

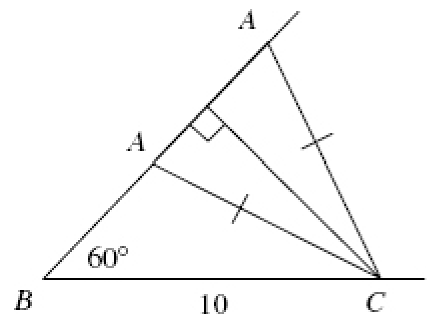
Two corners can only be covered if the side of the square is a diameter of a circle. The two other corners require two more circles (and then there is still some square not covered) unless we again use circles whose diameters are the sides of the square. Hence four circles whose diameters are the sides of the squares are needed.

36. You should be OK till (iv). Then, big trouble!

37. You should have worked out from the last exercise that, for a triangle to exist, the sum of any two sides is greater than the third. Hence we get $a + b > c$, $b + c > a$, and $c + a > b$. If for some side lengths a, b, c, $a + b < c$ then no triangle exists.

This can be seen when you try to construct such a triangle using compasses. So I should have mentioned in the construction of triangle *ABC*, that the construction won't work if you don't have the triangle inequality holding.

38. (i)

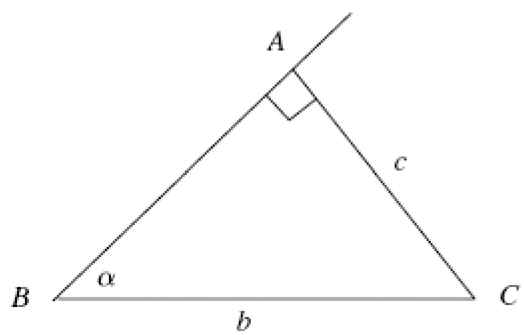


(ii) 2 triangles as in (i);

(iii) one triangle;

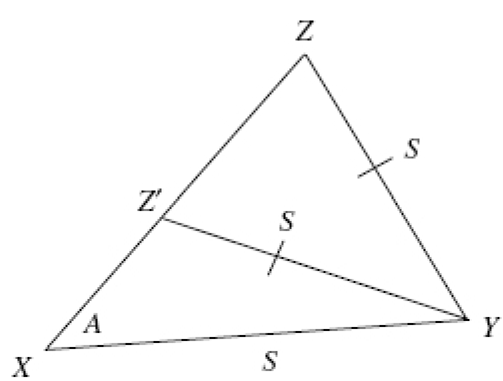
(iv) no triangles since, 8.0 < 5 3.

39.



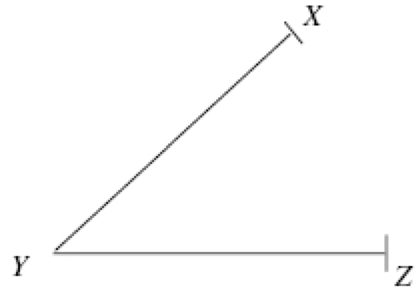
If *AC* is perpendicular to *BA* then the triangle is unique.

40. (a)



In the diagram, we have *ASS*. If we know $\angle ZXY$ and sides *XY*, Z'*Y* = ZY, then we can construct *two* triangles. So *ASS* is *not* a test of congruence unless $\angle XZ'Y = 90°$, or unless $\angle Z'XY = 90°$. (Draw the diagram in both cases.)

(b) *SAS* is OK.

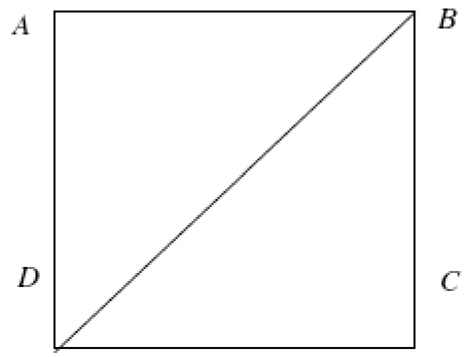


Suppose we know *XY*, *YZ* and ∠*XYZ*. Then *XZ* is uniquely *Y* defined. The quickest proof of this is by the cosine rule. But then we are in an *SSS* situation.

(c) This is the clockwise version of (a) above. *SSA* is not a test for congruence unless we have one of the right angle situations mentioned before.

41. This is essentially Example 1.

(a)



We wish to prove that Δ's *DAB*, *BCD* are congruent.

∠*CDB* = ∠*ABD* (alternate angles *AB* | |CD).

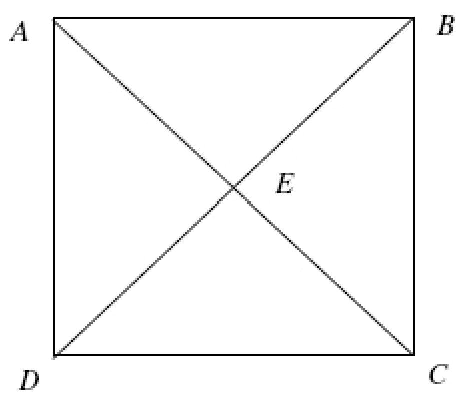
∠*ADB* = ∠*CBD* (alternate angles *AD* | |BC) and *BD* is common.

So Δ's *DAB, BCD* are congruent by the *ASA* test.

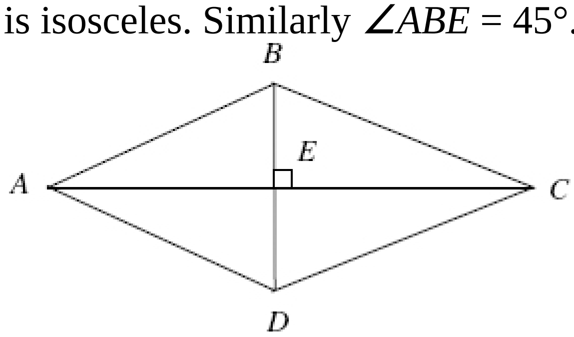
(b) If you label your rectangle in the right way you can use the proof of (a) with no change.

(c) See (b).

42.



∠*EAB* = 45° since Δ*ABC* is isosceles. Similarly ∠*ABE* = 45°. Hence ∠*AEB* = 90°.



Suppose the diagonals of *ABCD* intersect at right angles. Now we know that *AE* = *CE* since the diagonals of a parallelogram bisect each other (Exercise 30).
Consider Δ's *AEB, CEB*. From above *AE* = *CE*. Clearly *BE* = *BE*. Finally ∠*AEB* = ∠*CEB* = 90°. So we have congruence by SAS.

This means that *AB* = *CB*.

Precisely the same argument shows that *AB* = *AD* (= CD).

Hence if the diagonals of a parallelogram intersect at right angles, the parallelogram has all sides equal.

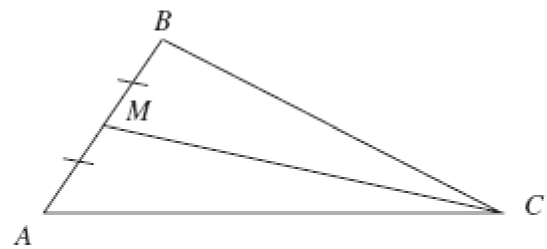
(Is it true that in a parallelogram with all sides equal, the diagonals intersect at right angles?)

43. Δ's *APB*, *AQB* are congruent *(SSS)* and isosceles. Hence ∠*PAM* = ∠*PBM* = ∠*QAM* = ∠*QBM*. Similarly Δ's *APQ, BPQ* are congruent (SAS) and isosceles. We can use these facts to show that A's *APM*, *BPM* are congruent (ASA). Hence *AM* = *MB*.

Since ∠*AMP* = ∠*BMP* (A's *APM*, *BPM* are congruent) and ∠*AMP* + ∠*BMP* = 180°, then *PM* (and hence *PQ)* is perpendicular to *AB*.

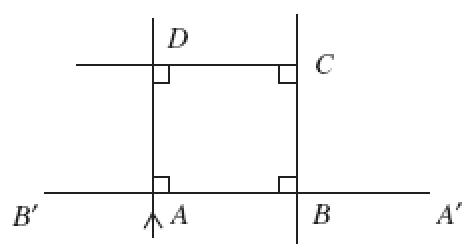
44. Draw circular arcs with centre *A* to intersect *BC* (produced if necessary) at *P* and Q. Now use Construction 2 to determine the midpoint *M* of *PQ*. Since *AM* is perpendicular to *PQ* it is perpendicular to *BC*. Hence *AM* is the required altitude.

45



Construction 2 enables us to find *M* the midpoint of *AB*. Join *C* to *M*.

46. (i)



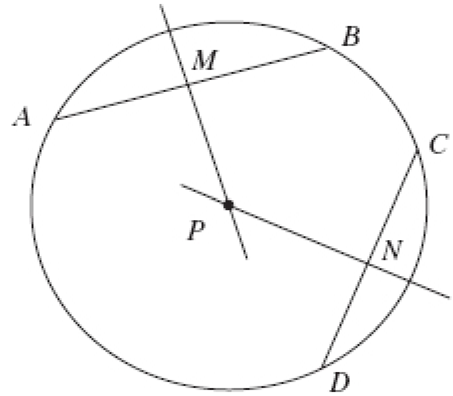
Draw the line segment *AB* with the required length.

Use the compasses to locate *A′* such that *AB* = *BA′*. By Construction 2, construct the perpendicular to *AB* at *B*.

Similarly construct the perpendicular to *AB* at *A*.

Use the compasses to locate *C* on the perpendicular at *B* so that *BC* = *AB*. Construct the perpendicular to *BC* at *C*. This perpendicular meets the perpendicular to *AB* at *A* in the point *D*. The points A, B, *C*, *D* are the vertices of the required square. (ii) (This is not to be done by looking for the hole in the paper.)

Draw two arbitrary chords *AB, CD* to the given circle. The perpendicular bisectors of these two lines meet at the centre, *P*, of the circle.



Proof. Suppose *O* is the centre of the circle.

Then since *OA* = *OB* (radii of the circle) A's AMO, *BMO* are congruent *SSS* (*AM* = *BM* and *MO* = *MO*). Hence *O* lies on the perpendicular bisector of *AB*.

Similarly *O* lies on the perpendicular bisector of *CD*. Since *O* is common to two perpendiculars, it must be at their point of intersection, namely *P*.

(iii) Through the centre *O* of the circle (found via (ii) if necessary) draw the line *AB* so that *A*, *B* are two points on the circumference. Construct perpendiculars to *AB* at *A* and *B*.

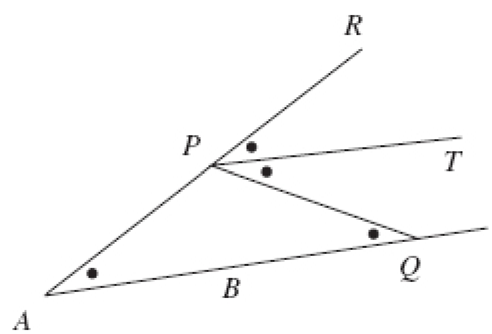
Now construct the perpendicular to *AB* at O. Let this perpendicular meet the circle at *C* and *D*. Construct perpendiculars to *CD* at *C* and D.

The perpendiculars at *A*, *B*, *C*, *D* meet to form the required square.

(iv) Construct the diagonals of the square. They meet at a point which is equidistant from each vertex of the square. This is the centre of the required circle. The radius of the square can be taken from the diagram.

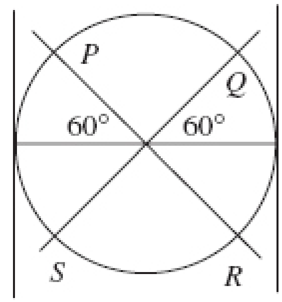
(v) The centre of the required circle is at the intersection of the perpendicular bisectors of adjacent sides of the square. The radius of the square can be taken from the diagram.

(vi) It is enough to show that I can construct, through a given point, a line parallel to a given line.



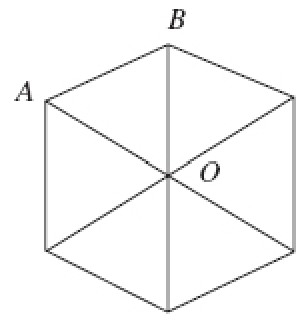
Let *AB* produced be a given line and *P* be a given point. Draw *AP*. Using *AP* as radius and *P* as centre find the intersection of this circle with *AB* produced. Let the new point be Q. Since ΔAPQ is isosceles, $\angle PAB = \angle AQP$. Now $\angle RPQ = \angle PAQ + \angle PQA = 2\angle PAQ$. Now bisect $\angle RPQ$. Then $\angle TPQ = \angle PQA$, so $PT || AB$.

(vii) *Hexagon around a circle.* First we need to be able to construct an equilateral triangle. But, of course, that's easy. Just use compasses to produce a triangle all of whose sides are equal. This also allows you to construct an angle of 60°.



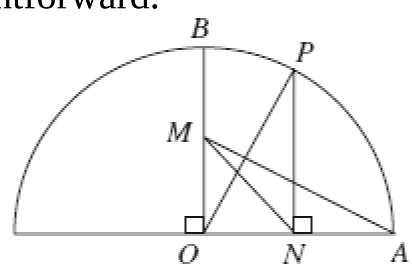
So take your circle and draw a diameter. Construct perpendiculars at the ends of the diameter. Now construct 60° angles as shown. Constructing perpendiculars (tangents to the circle) at *P, Q, R, S* to complete the regular hexagon.

Construct a circle around a hexagon. The lines joining opposite vertices of the hexagon intersect at the centre of the required circle. The radius of the circle can be taken from the diagram.



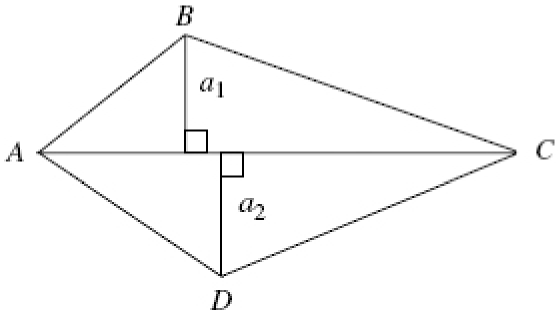
Construct a circle inside a hexagon. The same centre is required for this circle. Then construct the perpendicular from *O* to *AB*. This provides the radius for the circle.

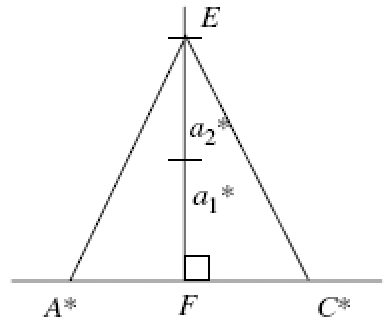
Pentagon around a circle. To do this we need to construct an angle of 72°. Once this has been done the rest is straightforward.



Draw the circle with radius *OA*. Construct *OB* perpendicular to *OA* with *B* on the circle. Then bisect *OB* to find the midpoint *M* of *OB*. Bisect the angle *OMA*. The point *N* is on the line of bisection and on OA. Construct the perpendicular at *N*. This perpendicular meets the circle at *P*. The angle *POA* is 72°. (The big question though is why. Can you prove this? Find an expression for the cosine of angle *POA*.)

(viii) Take quadrilateral *ABCD* and then construct diagonal *AC* (if *ABCD* is an arrowhead, concave quadrilateral, then take the internal diagonal). You now have two triangles — *ABC* and *ACD*. Now construct altitudes from *B* to *AC* and *D* to *AC*. Label them a_1 and a_2. Now draw a line segment *A* C** equal in length to *AC*. Through *F* draw a line perpendicular to *A*C** and lay off on it a segment equal in length to a_1 (labelled a_1*) and then at a_1*'s farthest end part, lay off a segment equal in length to $a_2(a_2*)$. Now connect *A** to the farthest end point of a_2* and *C** to the farthest end point of a_2^*. Label this last point *E*. Triangle A^*EC^* is the triangle wanted.





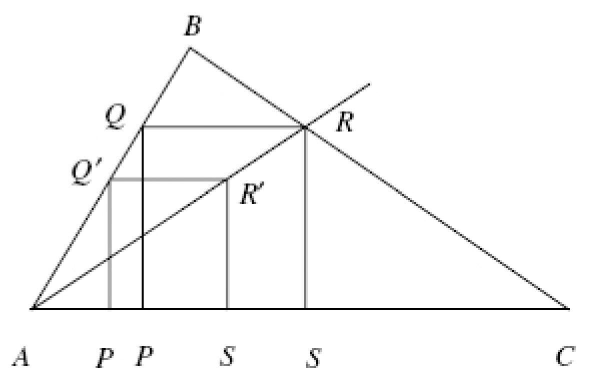
Proof. The quadrilateral *ABCD* was dissected into two triangles and had an area of ½ × (*AC*) × (a_i) + ½(*AC*)(a_2) which by the distributive law is equal
to i(AC) x (01 + 02). The area of triangle *A*EC** is i x (A*C*) x (*EF*). *A*C** is equal to *AC*. *EF* is equal to (ai + a_2). Therefore the areas of *ABCD* and *A*EF** are equal. ?

47. Construct the perpendicular bisector ℓ of the line segment between the two points. Any point on ℓ is the centre of a circle which passes through the two points.

    If the three given points are on a straight line, then no circle goes through them. If the three points *A, B, C* are not on a straight line, then the perpendicular bisectors of *AB, BC, CA* meet at a unique point. This point is the centre of the unique circle through A, B, *C*. (To prove this use a similar argument to that of Exercise 46(ii)).

    It is always possible to choose four points which do not lie on a circle or four points which lie on a unique circle.

48.

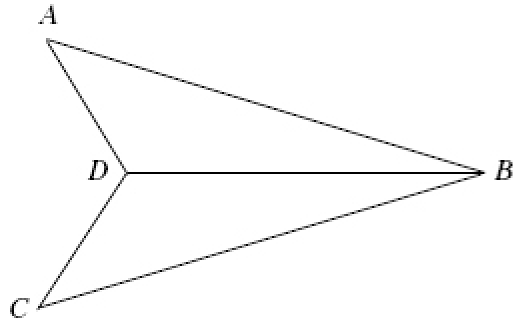


Let *PQRS* be the required square. If *P'Q'R'S'* is any square with one vertex on *AB* and two on *AC*, then *R'* is on the line *AR*. (You can prove this using similar triangles.) Hence to construct *PQRS*, first construct any square *P'Q'R'S'*. The point *R* is the intersection of *AR'* and*BC.*

The rest of the vertices of *PQRS* are then easily found.

49. (a) Take any point *P* inside the hexagon and construct the six triangles with *P* as one vertex and the sides of the hexagon as the sides of the triangle opposite to *P.* Hence the sum of the internal angles is the sum of the angles of six triangles minus the angle around *P.* This is 6 × 180° - 360° =4 × 180° = 720°.

    (b) In general this is *n x* 180° - 360° = (n - 2)180°. Now prove this without introducing a point *P* in the middle of the n-gon.



If we join *B* to *D* we see that the quadrilateral is composed of two triangles. Hence the sum of the internal angles is 360° (the same as a convex quadrilateral).

(d) Again divide the polygon into triangles and get three of them to give 540°.

(e) $(n - 2)180°$. But you have to make sure all your triangles can be inside the polygon.

50. We know that the interior angles sum to (n —2)180°. There are *n* angles, so in a regular polygon they are all equal to $\left(\frac{n-2}{n}\right) 180°$.

51. To tessellate the plane we need *k* of them to fit around a point. Hence $k\left(\frac{n-2}{n}\right) 180° =$ 360°. So $k(n - 2) = 2n$ or $k = \frac{2n}{n-2} = 2 + \frac{4}{n-2}$. But *k* has to be an integer, so *n* - 2 divides 4. There are only three possibilities.

Case 1. $n - 2 = 1$. Here $n = 3$ and the polygon is a triangle. We know equilateral triangles tessellate.

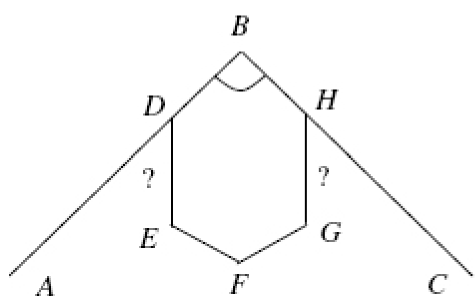
Case 2. $n - 2 = 2$. Here $n = 4$ and the polygon is a square. We know that squares tessellate.

Case 3. $n - 2 = 4$. Here $n = 6$. Regular hexagons do tessellate.

Hence only equilateral triangles, squares and regular hexagons tessellate.

52. If a regular polygon is self-replicating it will tessellate. Hence we only have three candidates for self-replicating regular polygons.

We already know that equilateral triangles and squares are selfreplicating. What about regular hexagons?



Suppose regular hexagons are selfreplicating. We know from Exercise 48 that their interior angles are 120°. Hence if *AB, BC* represent two sides of a “large” regular hexagon, $\angle ABC = 120°$.

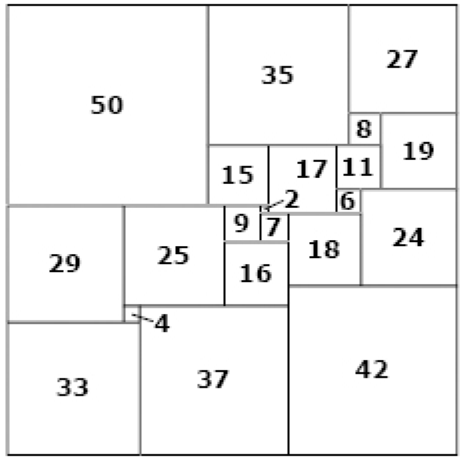
This means that if the figure is self-replicating, a smaller regular hexagon must fit exactly into the corner near B. But the $\angle ADE = 30°$. We cannot fill this angle using a regular hexagon. Hence regular hexagons are not self-replicating.

Are there *any* self-replicating hexagons?

53. Not too easy for 2, 3 or 5?

54. Just 2. That's not so hard but generalise this to equilateral triangles that have *n* squares on each side.

8. (c) Sorry you had to wait for so long.



(http://en.wikipedia.org/wiki/Squaring_the_square)

Chapter 6
Proof

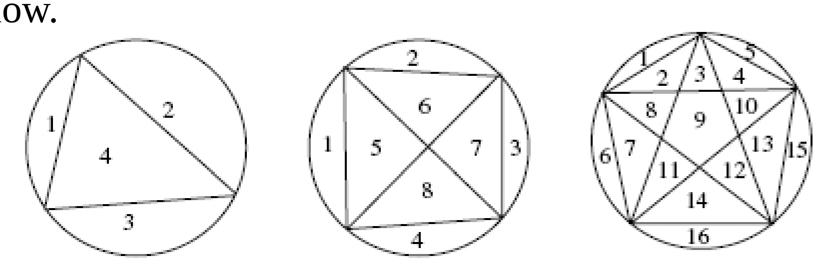
6.1. Introduction

This chapter looks at three things: A problem about regions in a circle; Proof by Contradiction; and Proof by Induction. The main reason for the problem is to show that you can never take anything in mathematics for granted. You think that you may have found a pattern or discovered a nice answer to a problem but until you are able to *prove* what you think then you can't be sure. Of course we have been proving things throughout this book so far but here we say why we have been doing that. In addition we show in this chapter that there are set types of proof and we talk about two of these that may come in handy.

6.2. Why Proof?

Consider the following problem. If n points are placed on the circumference of a circle and the ${}^{n}C_2 = \frac{1}{2}n(n-1)$ chords drawn so that no three have a common point of intersection, then how many regions is the circle divided into?

As with any problem, if you can't see the answer, try a few examples. I've done that in the diagram below.



$n = 3$ $n = 4$ $n = 5$

It's probably useful now to draw up a table. (It should be clear where the values for n = 1 and n = 2 come from.)

Number of points	1	2	3	4	5	⋯	n
Number of regions	1	2	4	8	16		?

The pattern is now perfectly obvious. The number of regions must be 2^{n-1}. So what's difficult about that? Surely nothing. Why don't you just check out the case n = 6?

Why proof? Well it's one thing to *discover* a pattern, it's another to be absolutely certain that you've discovered the *correct* pattern. In the example above, everything's behaving nicely, at least up to n = 5. It might also go on behaving well in the n = 6, 7, 8 cases. However, how can we be absolutely sure that by the time we get to n = 573 the pattern still holds?

We can't. So that's why proof comes in. And this is why Mathematics is different from Physics and Chemistry. Once a mathematical fact has been established by rigorous proof it is true for all time. This is not the case with the other sciences that seek to explain the Universe and what lies in it. For instance, the “truth” about the Solar System has changed as our ability to investigate it has changed. Originally Ptolemy convinced us that the Sun and the planets revolved around the Earth. That was the truth till Copernicus got to measuring and put the Sun at the centre with the planets moving in circular orbits. As measurement and theories got more sophisticated we gradually built up the picture we have today.

Now at this moment we may or may not fully understand the Solar System. The point is that the "truth" about the Solar System has been a function of time. Don't for one moment doubt that people believed they had it right. People were willing to kill to defend their views on the matter while, reciprocally, others were willing to die for their beliefs.

So there's a difference between Maths and Physics but before Maths gets too carried away by itself we should stop and reflect. The reason Maths is able to be rigorous is that it chooses its own ground rules.

Take Euclidean geometry, for example. By assuming certain axioms we can produce results about space. Those results are never wrong *but*, and this is a capital BUT, they may have nothing whatever to do with *actual* space. If things don't tie up in Mathematics with reality, then we go back and change the axioms and start again.

So maybe Mathematics isn't too different from the other sciences after all.

Oh, I think after all this philosophising we should come back to our original problem. Sorry folks! The number of regions into which the lines divide the circle is not 2^{n-1}. Go and work out what happens when $n = 6$. You should get 31 not 32. We didn't have to go as far as $n = 573$ after all.

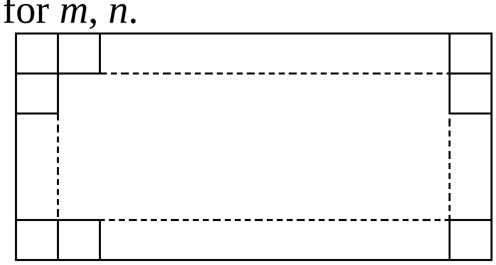
Exercises

1. Try to find the number of regions for $n = 7, 8$.
2. Conjecture the number of regions for n points.
3. Prove/disprove your conjecture.

(In the case of a disproof, GO TO 2.)

You are all probably coming to realise that the sort of problems we have been looking at require a *proof*. It is not enough to just come up with the answers. One way to see this is to consider an example.

Example 1. The floor of a rectangular room is covered with square tiles. The room is m tiles wide and n tiles long with $m \leq n$. If exactly half of the tiles are on the perimeter, then find all possible values for m, n.



Comment. A bit of work with pencil and paper will probably convince you that there are two solutions: $m = 5$, $n = 12$ and $m = 6$, $n = 8$. Try it. A bit of hand waving will *suggest* to you that there are no other solutions. But how can you be *sure*? In order to make things watertight we require a sound argument that these are indeed the *only* solutions. The matter must be *proved.*

Proof. The total number of tiles is mn. The total number of edge tiles is $2n+2(m-2)$. Because half of the tiles are on the perimeter $mn = 4n+4m-8$. Actually it's not totally obvious the first time you see it, how to solve this equation. Perhaps surprisingly, we have to rely heavily on the fact that m and n are integers. First factor the expression to give this

$$(m-4)(n-4)=8.$$

And now play the integer card. Now $m-4$ and $n-4$ are both integers. Further, the only pairs of integer factors of 8 are 1 x 8, (– 1) x (–8), 2 x 4 and (–2) x (–4). For physical reasons we can discard the negative factors. So, since $m \le n$, $m-4$ must be 1 and $n-4$ must be 8 or $m-4$ must be 2 and $n-4$ must be 4. Thus we have the solutions $m = 5$, $n = 12$ or $m = 6$, $n = 8$.

Notice that as far as the *answer* goes, providing a proof at first sight didn't seem to help. With hand waving and fast-talking we might have been happy with the two answers and gone off to do other, more interesting things like kicking a football or reading a book.

The point of the proof, however, is to bring total satisfaction, to eradicate all doubts, to make you feel you really understand and have complete control over the problem.

Once given "the proof", anyone can see what the solutions are, how they were obtained (in the case above, they were obtained systematically) and that there are no more, nor can there possibly be any more, solutions.

It is important in virtually all problem solving, to produce a proof because you will then know that the problem is settled. The proof should first convince *you* and second convince *everyone else.*

Proofs are often not common in school mathematics. Usually anything you do in school only involves a few steps that are often simply mechanical use of an algorithm. (Solve this quadratic equation, factorise this polynomial, and so on.)

As a result you may find writing proofs a little difficult. They certainly take a bit of practice. At the start of proof writing it may not be quite clear to you when you have a proof and when you haven't or whether you have included enough in the proof for it to be watertight. Overcoming these difficulties is important. Like everything else it involves a lot of work. Remember the old proverb "Practice makes proofs". And people who run competitions, especially the IMO, are looking to proofs to give points to.

Now friends, teachers, and family are all laid on for you to practice your proof presenting. When you think you have a proof to a problem write it out. Then ask a friend if she (or he) is convinced. If she isn't, then find out why and redraft your proof. Keep this up till she is convinced. Then put the proof aside for a day or two. After that period read it yourself to see if you are still happy with it. If you aren't, fix it up.

6.3. Proof by Contradiction

A proof is just a logical chain of statements which in total reaches some conclusion. There are some recognisable proof types. One of these is Proof by Contradiction, or, to give it a grander sounding name, *Reductio Ad Absurdum.*

The idea of this kind of proof is to assume the opposite of what you are trying to prove (which sounds a crazy thing to do). Then proceed via the logical chain of argument till you reach a demonstrably false conclusion. Since all the reasoning was correct and you've reached this false conclusion, then the only thing that could be wrong is the initial statement. What you are trying to prove *must* have been true.

Confused? Let's try the argument. First, let me remind you that a *rational number* is one of the form $\frac{m}{n}$ where m and n are integers. So ¾ is a rational number and so is $\frac{999}{3001}$

A number which is not rational is called *irrational.* We now give the classical proof that $\sqrt{2}$ is irrational.

Example 2. Required to Prove: $\sqrt{2}$ is irrational.

Proof. Assume $\sqrt{2}$ is rational.

If $\sqrt{2}$ is rational, then, $\sqrt{2} = \frac{m}{n}$ for some integers *m* and *n*. Actually we can say more than this. We can even assume that *m* and *n* have no factors in common because if they did, we could cancel the factors without changing the value of the fraction $\frac{m}{n}$

So $\sqrt{2} = \frac{m}{n}$. Hence $n\sqrt{2} = m$. This leads to $2n^2 = m^2$.

This means that m^2 is an *even* number. *Hence m is an even number.*

(The square of an odd number is odd — $(2n + 1)^2 = 4n^2 + 4n + 1$.) So we can write $m = 2p$ for some integer *p*.

Thus $2n^2 = (2p)^2 = 4p^2$. Hence $n^2 = 2p^2$. But this means that n^2 is even and so *n* must also be even.

However, if *m* and *n* are both even, then they must both have a factor of 2. Thus we contradict the assumption that they have no common factors.

Since all the steps in the argument are sound, the only reason for this contradiction is the fact that our original statement is wrong. Hence $\sqrt{2}$ is irrational.

In the above proof everything went well until we found that two numbers that didn't have a common factor, did. Every step of this proof has been correct. Therefore the original statement must have been false.

Now try your hand at the following questions.

Exercises

4. (a) Show that if 3 divides n^2, then 3 divides *n*.
   (Hint. *n* can only be of the form $3a$, $3a + 1$ or $3a + 2$).
   (b) Show that if 5 divides n^2, then 5 divides *n*.
   (c) For what *q* is it true that if *q* divides n^2, then *q* divides *n*.
5. Where possible, use Proof by Contradiction to settle the following. In each case below, *b* and *c* are integers.
   (i) Prove that $\sqrt{3}$ is irrational;
   (ii) Prove that $\sqrt{5}$ is irrational;
   (iii) Prove that $\sqrt{p}$ is irrational for any prime *p*;
   (iv) For what values of *b* is $\sqrt{b}$ rational?
   (v) Is $\sqrt{2} + \sqrt{3}$ irrational?
   (vi) If $\sqrt{b}$ and $\sqrt{c}$ are irrational is $\sqrt{b} + \sqrt{c}$ always irrational?
   (vii) If $\sqrt{b}$ and $\sqrt{c}$ are irrational is $\sqrt{b} - \sqrt{c}$ always irrational?
   (viii) For what values of *b* is $\sqrt[3]{b}$ rational?
   (ix) Is the sum of a rational number and an irrational number irrational?
   (x) Is the product of a non-zero rational number and an irrational number rational?
6. Prove that there is no largest integer. Is there a smallest integer?
7. Prove that there are infinitely many prime numbers.
8. Prove that for all a, $b \geq 0$, $\frac{1}{2}(a + b) \geq \sqrt{ab}$.
9. Prove that $3^{2n} + 5$ is never divisible by 8, no matter what value the natural number *n* takes.
10. Prove that the highest common factor of *n* and $n + 1$ is 1.
11. Prove that the decimal expansion of an irrational never terminates nor has a section which repeats continuously.

12. Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which satisfy the triangle inequality. (IMO 1968.)
13. Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if $f(n+1) > f(f(n))$ for each positive integer n, then $f(n) = n$ for each n. (IMO 1977.)

6.4. Mathematical Induction

How do you teach a robot to climb a ladder? There are really only three steps involved. These will enable the robot to get to the *n*th rung, where *n* is any natural number.

Step 1. Get the robot on the first rung.

Step 2. Assume that the robot can make it to the kth rung.

Step 3. If the robot can get to the kth rung it can move to the $(k + 1)$th rung.

Let's assume we've programmed our robot to follow the three steps above. Can it climb the ladder?

Well it can certainly get somewhere. Step 1 puts the robot on the ladder.

Ah! But don't you see, Step 1 has accomplished Step 2 for $k = 1$.

Now we can use Step 3. With $k = 1$, Step 3 tells us that the robot will go from the 1st rung to the $(1 + 1)$th rung. The robot has successfully got itself to the 2nd rung.

At this stage we can go back to Step 2. Clearly Step 2 is true for $k = 2$ now. So it's on to Step 3 which gets the robot from the 2nd rung to the $(2 + 1)$th or 3rd rung.

About now you ought to see what's going on. No matter how big *n* is, by alternating Steps 2 and 3 we can get our robot to the *n*th rung of the ladder. We've taught our robot to climb any ladder of any length.

Of course if it's not an infinite ladder the poor thing's going to fall off the top but you can work on that problem for the next prototype.

How do you make dominoes fall? You've all seen, on TV if nowhere else, strings of dominoes tumbling and making interesting patterns. How does this work? Well it's the old domino principle of course. Here's how to get the *n*th domino to fall.

Step 1. Push over the first domino.

Step 2. Assume that the *k*th domino has fallen.

Step 3. If domino *k* falls, then domino $k + 1$ falls.

How do your dominoes fall?

Apply Step 1 and you're off. Step 2 is now true for $k = 1$, so moving to Step 3 we see the second domino falling. Back to Step 2. This is now true for $k = 2$. So moving on to Step 3, the third domino goes.

Then it's back to Step 2, then Step 3, then 2, then 3,…And they all fall down.

OK. If you're on top of that you're ready for, roll on the drums, fanfare of trumpets, the *Principle of Mathematical Induction.* This is a simple three step proof which is good for proving a variety of results which are true for all positive integers.

First the three steps, which you will note are amazingly (what a coincidence) like robot ladder climbers and falling dominoes.

Step 1. Show the result is true for $n = 1$.

Step 2. Assume the result is true for *n* equal to some integer *k*.

Step 3. Prove that if the result is true for *k*, it is true for $k + 1$.

Once again it is easy to see *why* the proof method works. If the result is true for $n = 1$, then Step 2 is OK for $n = 1$ so Step 3 tells us it's OK for $n = 2$. Back to Step 2. This is fine for $n = 2$ so Step 3 gives the result for $n = 3$. We keep this up until we've covered all the integer rungs on the real number ladder or equivalently, all the integer dominoes have fallen.

This Principle of Mathematical Induction then enables us to prove results which are true for all integers.

Now you understand the idea, let's try an example or two.

Example 3. Prove that the sum of the first n positive integers is + $\frac{1}{2}n(n+1)$

Proof. We've seen this already when we did Arithmetic Progressions in Chapter 4, Section 4.5, p. 128. Now let's do it another way.

Step 1. Show the result is true for $n = 1$.

Now the first 1 integer adds up to 1.

If we put n = 1 in the expression \n{n-\-1) we get 1. So the result is certainly true for $n = 1$.

Step 2. Assume the result is true for $n = k$.

This step says $1 + 2 +\ldots k = \frac{1}{2}k(k + 1)$.

Step 3. If the result is true for k, it is true for $k + 1$.

This is the step that usually causes problems.

We now have to show that the result is true for $k + 1$. In other words we have to show that

$$1 + 2 + \cdots + (k+1) = \frac{1}{2}(k+1)(k+2)$$

Start with the LHS (left-hand side).

$$\begin{aligned} \text{LHS} &= 1 + 2 + \cdots + (k+1) \\ &= 1 + 2 + \cdots + k + (k+1) \\ &= \frac{1}{2}k(k+1) + (k+1) \quad \text{by Step 2} \\ &= (k+1)\left[\frac{1}{2}k + 1\right] \\ &= \frac{1}{2}(k+1)[k+2] \\ &= \text{RHS.} \end{aligned}$$

This completes Step 3.

Hence by the Principle of Mathematical Induction $1 + 2 +\ldots+ n =\frac{1}{2}n(n + 1)$, for all positive integers n.

The Principle of Mathematical Induction always works like this. Let's have a look at another example.

Example 4. Prove that $2^n > n$ for every natural number n.

Proof.

Step 1. If $n = 1$, $2^1 = 2$. Now $2^1 > 1$ and so, for $n = 1$, the inequality is certainly true.

Step 2. Assume $2^k > k$.

Step 3. If $2^k > k$, then we have to prove that $2^{k+1} > k +1$.

Now $2^{k+1} = 2 \times 2^k > 2 \times k = 2k$. (by Step 2)

But $2k \geq k + 1$. Hence $2^{k+1} > 2k \geq k +1$.

So $2^{k+1} > k + 1$ as required and Step 3 is completed.

Hence by the Principle of Mathematical Induction we know that $2^n > n$ for all natural numbers n.

Exercises

(Throughout, N is the set of natural numbers {1, 2, 3,…}.)

14. Use Mathematical Induction to prove

(i) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$;

(ii) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$;

(iii) $1 + 4 + 7 + \ldots + (3n - 2) = \frac{1}{2} n(3n - 1)$;

(iv) $2 + 7 + 12 + \ldots + (5n - 3) = \frac{1}{2}n(5n - 1)$;

(v) $a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d] = \frac{1}{2}n[2a + (n - 1)d]$, where n is a natural number and d is real.

15. Use Mathematical Induction to prove

(i) $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$;

(ii) $3 + 9 + 27 + \cdots + 3^n = \frac{3}{2}(3^n - 1)$;

(iii) $4 + 12 + 36 + \cdots + 4 \times 3^{n-1} = 2(3^n - 1)$;

(iv) $a + ar + ar^2 + \cdots + ar^{n-1} = a(1 - r^n)/(1 - r)$ for $r \neq 1$ where n is a natural number and r is real;

16. Use Mathematical Induction to prove

(i) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$;

(ii) $1^2 + 4^2 + 7^2 + \cdots + (3n - 2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$;

(iii) $2^2 + 5^2 + 8^2 + \cdots + (3n - 1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)$;

(iv) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$;

(v) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

17. Use Mathematical Induction to prove for all natural numbers n that

(i) $3^n > 2^n$; (ii) $\left(\frac{1}{4}\right)^n < \left(\frac{1}{3}\right)^n$

Does the result hold if n is an integer?

Here's another type of use for Induction.

Example 5. Prove in two ways that $n^2 + n$ is even.

Proof.

(1) First we'll use the Principle of Mathematical Induction. Let $S(n) = n^2 + n$.

Step 1. If $n = 1$, then $S(1) = 2$ which is even.

Step 2. Assume that $S(k) = k^2 + k$ is even.

Step 3. Now $S(k + 1) = (k + 1)^2 + (k + 1) = (k^2 + k) + (2k + 2) = S(k) + 2k + 2$. By Step 2, $S(k)$ is even and clearly $2k + 2$ is even. Hence $S(k + 1)$ is even.

Thus $S(n)$ is even for all natural numbers n.

(2) The second method is quicker.

Now $S(n) = n(n + 1)$. But any number or its successor is even. Hence $S(n)$ is even.

Exercises

18. Prove the following by two methods.

(i) $n^3 - n$ is divisible by 6;

(ii) $6^n + 4$ is divisible by 10.

19. If $f(n) = 3^{2n} + 7$, where n is a natural number, show that $f(n + 1) - f(n)$ is divisible by 8. Hence prove by Induction that $3^{2n} + 7$ is divisible by 8.

Here are a few harder questions that you can easily leave out the first time through.

Exercises

20. If m, n ℕ ℕ, where m is fixed, prove by Induction on n that

$$1+\frac{m}{1!}+\frac{m(m+1)}{2!}+\cdots+\frac{m(m+1)\cdots(m+n-1)}{n!}$$
$$=\frac{(m+1)(m+2)\cdots(m+n)}{n!}.$$

21. Prove by Induction that a set with n elements has exactly 2^n subsets.

22. (Euclid c. 300 BC). If the primes are written in ascending order of magnitude, $p_1 < p_2 < p_3 \ldots$, i.e. $2 < 3 < 5 < \ldots$, then

(i) prove that $P_{n+1} \leq 1 + (p1p2\ldots P_n)$, for n ℕ ℕ

(ii) use Induction to prove that $p_n \leq 2^{2n}$;

(iii) what do you think of the conjecture: $1 + (p_1p_2\ldots p_n)$ is a prime, for every n ℕ ℕ

23. Prove that there are infinitely many primes. (Again!!)

24. (Bernoulli's inequality 1686). For $x \geq -1$, prove that $(1 + x)^n \geq 1 + nx$, where n ℕ ℕ. Show, by choosing particular values of x and n, that the inequality is not necessarily true if n is not a positive integer.

25. If $\sin x \neq 0$, prove that for n ℕ ℕ

$$\cos x \cdot \cos 2x \cdots \cos 2^{n-1}x = (\sin 2^n x)/2^n \sin x.$$

(*Hint.* You may need to find an expression for $\sin 2A$ first.)

26. Find the flaw in the following "proof" that all positive integers are equal: "The proof is by Induction. For each n ℕ ℕ, consider the following statement: If r, s ℕ ℕ and $\max\{r, s\} = n$, then $r = s$.

(i) When $n = 1$ the statement is true because if $\max\{r, s\} = 1$, then $r = s = 1$.

(ii) Assume the statement is true for n. Let r, s ℕ ℕ with $\max\{r, s\} = n + 1$. Then $\max\{r - 1, s - 1\} = n$ and hence by the Induction hypothesis, $r - 1 = s - 1$, that is, $r = s$. Hence the statement is true for $n + 1$ and by Induction true for all n ℕ ℕ .

To finish off the proof, let r and s be positive integers. Then $\max\{r, s\} = n$, for some n ℕ ℕ, and hence $r = s$.

27. Prove that $7^{2n} - 48n - 1$ is divisible by 2304, for every n ℕ ℕ.

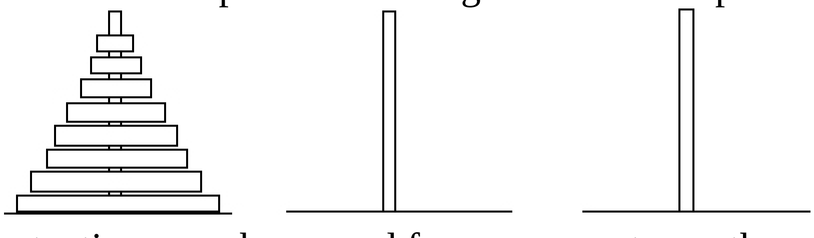
28. Prove that $\sum_{r=1}^{n} 1/\sqrt{r} \geq \sqrt{n}$, for all $n \in \mathbb{N}$

29. For every positive integer n, show that the Fibonacci number

$$u_n = [(1+\sqrt{5})^n - (1-\sqrt{5})^n]/(2^n\sqrt{5})$$

is a positive integer.

30. *The Towers of Hanoi.* This is a toy which consists of 3 pegs and n circular discs of different sizes with holes in their centres so that they fit over the pegs. At the beginning of the game the discs are all on one of the spindles, as shown in the diagram, the smallest at the top and increasing in size as one proceeds down the pole.



Rules. (i) One disc at a time may be moved from one peg to another.

(ii) No disc may be placed on top of a smaller disc.

Object. To move all the discs from one peg to another, subject to these rules.

(a) Prove by Induction that this can be done in $2^n - 1$ moves.

(b) Can you say anything about the *smallest* number of moves needed?

Sometimes of course someone has broken the first few rungs of the ladder. Our ladder-walking robot can still climb the ladder if only he can get on to it. Suppose the first five rungs are broken. Then we change Step 1 to **Step 1'.** Get the robot onto the 6th rung.

That's enough to get the robot going. From here, along with Steps 2 and 3, the robot can climb the ladder.

Sometimes the same sort of thing happens to a mathematical proof. An expression happens to be true *from some natural number onwards.* To cope with this situation we use the following modified version of the Principle of Mathematical Induction.

Step 1'. Show that the result is true for $n = a$.

Step 2'. Assume the result is true for n equal to some integer k greater than or equal to a.

Step 3'. Prove that if the result if true for k it is true for $k + 1$.

Example 6. Prove that for all sufficiently large natural numbers n, $n! > 3^n$.

Proof.

Step 1'. After some trial and error we see that $7! = 5040 > 3^7 = 2187$. So we will prove that $n! > 3^n$ for all $n \geq 7$.

Step 2'. Assume that $k! > 3^k$, for $n = k > 7$.

Step 3'. We must prove that if $k! > 3^k$ for $k \geq 7$, then $(k + 1)! > 3^{k+1}$.

Now $(k + 1)! = k!(k + 1) > 3^k(k + 1)$ (by Step 2). But $k \geq 7$, so $k + 1 \geq 7 > 3$. Hence $3^k(k + 1) > 3^k \times 3 = 3^{k+1}$.

We have now shown that $(k + 1)! > 3^{k+1}$ and Step 3 is complete.

By the Principle of Mathematical Induction, $n! > 3^n$ for all $n \geq 7$.

Exercises

31. Prove by Induction that for certain sufficiently large n,

(i) $n! > 2^n$;
(ii) $n! > n^2$;
(iii) $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$;
(iv) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{2}{n}$.

In each case, state the smallest value of n for which the statement is true.

In the remaining questions Induction may be used as part of the solution.

32. Given a $(2m+1) \times (2n+1)$ chessboard in which the four corners are black squares, show that if one removes any one red square and any two black squares, the remaining board is coverable with dominoes (1×2 rectangles).

33. Observe that

$$1^2 = 1 \cdot 2 \cdot 3/6$$
$$1^2 + 3^2 = 3 \cdot 4 \cdot 5/6$$
$$1^2 + 3^2 + 5^2 = 5 \cdot 6 \cdot 7/6$$

Guess a general law suggested by these examples, and prove it.

34. Let f be a function with the following properties:

(1) $f(n)$ is defined for every positive integer n;
(2) $f(n)$ is an integer;
(3) $f(2) = 2$;

(4) $f(mn) = f(m)f(n)$ for all m and n;

(5) $f(m) > f(n)$ whenever $m > n$.

Prove that $f(n) = n$ for $n = 1,2,3\ldots$

35. Let n be a positive integer and let $a_1, a_2\ldots a_n$ be any real numbers greater than or equal to 1. Show that

$$(1+a_1)\cdot(1+a_2)\cdot\ldots\cdot(1+a_n)\geq\frac{2^n}{n+1}(1+a_1+a_2+\cdots+a_n).$$

36. Prove that, for each positive integer n,

$$1-\frac{1}{2}+\frac{1}{3}-\cdots+\frac{1}{2n-1}=\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}.$$

37. State and prove a generalisation of the following set of equations.

$$1=1,$$
$$2\cdot1-\frac{1}{2}=1+\frac{1}{2},$$
$$3\cdot1-3\cdot\frac{1}{2}+\frac{1}{3}=1+\frac{1}{2}+\frac{1}{3},$$
$$4\cdot1+6\cdot\frac{1}{2}+4\cdot\frac{1}{3}-\frac{1}{4}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}.$$

38. Let n be a positive integer. Prove that the binomial coefficients ${}^nC_1, {}^nC_2, {}^nC_3\ldots{}^nC_{n-1}$ are all even, if and only if n is a power of 2.

39. Prove that $9\times\underbrace{109\cdots989}_{n}=\underbrace{989\cdots901}_{n}.$

40. Suppose that $0 \leq x_i \leq 1$ for $i = 1,2\ldots n$. Prove that

$$2^{n-1}(1+x_1x_2\cdots x_n)\geq(1+x_1)(1+x_2)\cdots(1+x_n),$$

with equality if and only if n — 1 of the x_i's are equal to 1.

41. Prove that there is a unique infinite sequence $\{u_0, u_1, u_2,\ldots\}$ of positive integers such that, for all $n \geq 0$,

$$u_n^2=\sum_{r=0}^{n}{}^{n+r}C_r u_{n-r}.$$

42. Determine all continuous functions f such that, for all real x and y

$$f(x+y)f(x-y)=[f(x)f(y)]^2.$$

43. Show that there exist infinitely many sets of 1983 consecutive positive integers each of which is divisible by some number of the form a^{1983}, where a is a positive integer greater than 1.

44. Show that if $x^2 + y^2$ is divisible by 7, then it is divisible by 49.

45. (a) What is the smallest number which has remainder 2 on dividing by 7 and remainder 4 on dividing by 9?

(b) Show that there is no number which has a remainder 2 on dividing by 7 and a remainder 6 on dividing by 9.

6.5. Conclusion

We started out with the problem of trying to find the maximum number of regions into which a circle can be divided by joining pairs of points from a set of size n with straight lines. The thing about this problem is that it does not behave as one starts to expect. From the table on p. 181, it starts to look as if the number of regions is 2^{n-1}. However, those of you who tried 6 points will have discovered only 31 regions — not the 32 you might have hoped for.

So just because patterns start off heading in one direction there is no guarantee that they won't veer off in another direction on the merest whim.

And that's why in Maths we have to prove things.

Let me finish by nailing down the problem we started with. First though, recall Euler's formula from Chapter 3, Section 3.9. It says that in a connected planar graph, the number of vertices, v edges, e and faces, f are connected by $v - e + f = 2$.

We can now prove that the largest number of regions in our circle problem is

$$^nC_4 + {}^nC_2 + 1 \quad \left(\text{where } {}^nC_4 = \frac{n!}{4!(n-4)!} \quad \text{and} \quad {}^nC_2 = \frac{n!}{2!(n-2)!}\right).$$

Before we start the proof, which is not by contradiction or Induction, we note that the formula governing the number of regions, f, formed *inside* our circle is $v - e + f = 1$. We get this using Euler's Formula and throwing away the outside face.

Proof. That $^nC_4 + {}^nC_2 + 1$ is the number of regions.

The result is clearly true for $n = 1,2,3$. So we work with $n \geq 4$. Now each subset of 4 from the given n points will contribute one intersection point in the circle. Conversely, each intersection-point arises from just one subset of four points, namely those at the ends of the two chords through it. Hence the number of intersection-points is equal to the number of ways of choosing 4 of the n given points, i.e., $^nC_4$.

Consider the graph formed by the n given points considered as vertices, the intersection points considered as vertices, and the “natural” lines joining these two types of vertices. Now each of the given points, considered as vertices, has degree $n - 1$. Further each of the $^nC_4$ internal vertices is of degree 4. Since the sum of the degrees equals twice the number of edges we have

$$2e = 4\,{}^nC_4 + n(n-1) \quad \text{or} \quad e = 2\,{}^nC_4 + {}^nC_2,$$

while

$$v = {}^nC_4 + n.$$

By Euler's Formula, $f = e - v + 1$. So that $f = {}^nC_4 + {}^nC_2 + 1 — n$.

But the regions that we want to count includes some regions snuggling between the graph and the circle. Adding these extra n regions the total number of regions is given by

$$^nC_4 + {}^nC_2 + 1.$$

The proof I've given here can be found in the *Mathematical Gazette*, May, 1972, pages 113–115. The article is by Timothy Murphy and is called “The dissection of a circle by chords”.

6.6. Solutions

1. $n = 7$ gives 57; $n = 8$ gives 99. Bang goes the 2^{n-1} conjecture.

2&3. What did you get? My solution appears in Section 6.5.

4. (a) Now $n = 3a$, $3a + 1$ or $3a + 2$. Hence $n^2 = 9a^2$, $9a^2 + 6a + 1$, $9a^2 + 12a + 4$. Since 3 divides n^2, then $n^2 = 9a^2$ and $n = 3a$. Hence 3 divides n^2.
   (b) Use a similar proof to (a). Let $n = 5a$, $5a + 1$, $5a + 2$, $5a + 3$, $5a + 4$. The only square of the form $5b$ comes from $n = 5a$.
   (c) q has to be square-free.

5. (i) Assume $\sqrt{3} = \frac{m}{n}$ where m and n have no common factors. Then use Exercise 4(a).

(ii) In a similar way if t^2 is divisible by 5 then so is *t*.

(iii) If s^2 is divisible by the prime *p* then *s* is divisible by *p*.

This follows by assuming that the prime decomposition of *s* is $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. If *p divides* s^2, then *p* has to divide $p_i^{2\alpha_i}$ for some *i*. Hence, since *p* is a prime, $p = p_1$. Therefore *p* is a factor of *s*.

The rest follows in the usual way.

(iv) Clearly *b* can be anything but a perfect square. The proof is as in the earlier parts of the question if *b* is square free. So assume $b = c^2 d$, where *d* is square free. Then we find that if $\sqrt{b} = \frac{m}{n}$ with *m* and *n* having no common factors, then $m^2 = n^2 b = n^2 c^2 d$. For $d \neq 1$ the usual proof technique shows that *d* divides *m* and *n*.

(v) Yes. If $\sqrt{2} + \sqrt{3} = \frac{m}{n}$, then $2\sqrt{6} + 3 = \frac{m^2}{n^2}$ Hence $\sqrt{6} = ½(\frac{m^2}{n^2} – 5)$. This is a contradiction since $\sqrt{6}$ is irrational (by (iv)).

(vi) Yes.

(vii) No. Try $b = c$. Is that a surprise? What if $b \neq c$

(viii) *b* must be a perfect cube. The proof follows along the lines of (iv).

(ix) Yes.

(x) No.

6. Let *m* be the largest integer. Then *m* +1 is an integer larger than *m*. This is clearly a contradiction.

   What do you mean by the smallest integer?

7. Assume there are a finite number of primes $p_1, p_2, \ldots, p_n$. Form $t = p_1 p_2 \ldots p_n + 1$. Now either *t* is a prime or contains a prime factor other than $p_1, p_2, \ldots, p_n$. This is the contradiction here.

8. Use the facts that (i) if $p > q$, then $p^2 > q^2$ and (ii) $r^2 \geq 0$ for every real number *r*. You might get the contradiction $0 > (a – b)^2$.

9. Assume $3^{2n} + 5$ is divisible by 8. Actually if you look at congruences modulo 8 you mightn't need contradiction.

   Blow it. The wretched thing's not even divisible by 4! After all, $3^{2n} + 5 \equiv (–1)^{2n} + 1 \pmod 4$.

10. Suppose $k > 1$ is the highest common factor of *n* and $n + 1$. Then $n = kq$ and $n + 1 = kr$. Hence $1 = k(r – q)$. For $k > 1$ this provides the contradiction.

11. Suppose the decimal expansion of the irrational number *b* terminates at the *r*th decimal place. Then $b = b' + 0 . b_1 b_2 \ldots b_r$, where $b', b_1, \ldots, b_r$ are integers and $0 \leq b_i \leq 9$.

    Clearly $b = b' + \frac{b_1}{10} + \frac{b_2}{10^2} + \cdots + \frac{b_r}{10^r}$ is a rational.

    Suppose *b* has continuous repetition of a section of *r* digits, then $10^r b — b$ is a terminating decimal.

12. Recall that the triangle inequality says the sum of the lengths of any two sides exceeds the length of the third. Equivalently, the three segments are *not* the sides of a triangle if and only if the longest of them is greater than or equal to the sum of the other two.

    Denote the vertices of any tetrahedron by *A, B, C, D* and let *AB* be the longest side. Suppose there is no vertex such that the edges meeting there are the sides of a triangle. Consider vertex A with attached edges AB, AC, AD. Then $AB \geq AC + AD$ by the above remarks. Similarly, by considering vertex B, we conclude that $BA \geq BC + BD$. Adding these inequalities, we get

$$2AB \geq AC + BC + AD + BD.$$

But from the triangular faces ABC and ABD we get $AB < AC + BC$ and $AB < AD + BD$; and if we add these two inequalities we get

$$2AB < AC + BC + AD + BD,$$

a contradiction.

13. First note that $f(1)$ is the unique minimum of f. For suppose that for some $j > 1$, $f(j)$ is minimum. Then $f > f(f(j - 1))$ and if $f(j - 1) = k$, this shows that $f(j) > f(k)$. Thus we contradict the claim that $f(j)$ is minimum.

The same reasoning shows that the next smallest value is $f(2)$, etc.

Thus

$$f(1) < f(2) < f(3) < \cdots.$$

Since $f(n) \geq 1$ for all n, we have, in particular, $f(n) \geq n$. Suppose that, for some positive integer k, $f(k) > k$. Then $f(k) \geq k + 1$; and since f is an increasing function, $f(f(k)) \geq \geq f(k + 1)$, contradicting the given inequality. Therefore $f(n) = n$ for all n.

[In the Mathematical Induction proofs that follow, we give only the key steps. Step 1 must always be tested but we omit it here because of space.]

14. (i) $2+4+6+\cdots+2k+2(k+1) = k(k+1)+2(k+1) = (k+1)(k+2)$;
(ii) $1+3+5+\cdots+(2k-1)+(2k+1) = k^2+(2k+1) = (k+1)^2$;
(iii) $1+4+7+\cdots+(3k-2)+(3k+1) = \frac{k}{2}(3k-1)+(3k+1) = \frac{1}{2}(k+1)(3k+2)$;
(iv) $2+7+12+\cdots+(5k-3)+(5k+2) = \frac{k}{2}(5k-1)+(5k+2) = \frac{1}{2}(k+1)(5k+4)$;
(v) $a+(a+d)+\cdots+[a+(k-1)d]+(a+kd) = \frac{k}{2}[2a+(k-1)d]+(a+kd) = \frac{(k+1)}{2}(2a+kd)$.

15. (i) $1+2+4+\cdots+2^{k-1}+2^k = 2^k-1+2^k = 2^{k+1}-1$;
(ii) $3+9+27+\cdots+3^k+3^{k+1} = \frac{3}{2}(3^{k+1}-1)+3^{k+1} = \frac{3}{2}(3^{k+2}-1)$;
(iii) $4+12+36+\cdots+4\times3^{k-1}+4\times3^k = 2(3^k-1)+4.3^k = 2(3^{k+1}-1)$;
(iv) $a+ar+\cdots+ar^{k-1}+ar^k = \frac{a(1-r^k)}{1-r}+ar^k = \frac{a(1-r^{k+1})}{1-r}$.

16. (i) $1^2+2^2+\cdots+k^2+(k+1)^2 = \frac{k}{2}(k+1)(2k+1)+(k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$;
(ii) $1^2+4^2+\cdots+(3k-2)^2+(3k+1)^2 = \frac{k}{2}(6k^2-3k-1)+(3k+2)^2 = \frac{(k+1)}{2}[6(k+1)^2-3(k+1)-1]$;
(iii) $2^2+5^2+\cdots+(3k-1)^2+(3k+2)^2 = \frac{k}{2}(6k^2+3k-1)+(3k+2)^2 = \frac{(k+1)}{2}[6(k+1)^2+3(k+1)-1]$;
(iv) $1^3+2^3+\cdots+k^3+(k+1)^3 = \frac{k^2}{4}(k+1)^2+(k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2$;
(v) $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} = \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$. This

can actually be proved more quickly by noticing that $\frac{1}{1\cdot2} = \frac{1}{1} - \frac{1}{2}$, etc. Then all of the dominoes fall in another way.

17. (i) $3^{k+1} = 3\cdot3^k > 3\cdot2^k > 2\cdot2^k = 2^{k+1}$;
(ii) $\left(\frac{1}{4}\right)^{k+1} = \frac{1}{4}\left(\frac{1}{4}\right)^k < \frac{1}{4}\left(\frac{1}{3}\right)^k < \frac{1}{3^{k+1}}$;

The inequalities are reversed if n is negative.

18. (i) $S(k + 1) = (k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1 = S(k) + 3k(k + 1)$. By Example 5, $k(k + 1)$ is divisible by 2. Hence $S(k + 1)$ is divisible by 6.

Alternatively $n^3 - n = (n-1)n(n+1)$. At least one of these must be even and at least one divisible by 3.

(ii) $S(k+1) = 6^{k+1} + 4 = 6^k + 4 + 5 + 6^k = S(k) + 5 \cdot 6^k$. For $k \geq 1$, $5 \cdot 6^k$ is divisible by 10. Hence $S(k+1)$ is divisible by 10. $6^n \equiv 6 \pmod{10}$.

19. $f(k+1) = 3^{2k+2} + 7 = 3^{2k} + 7 + 8t = 8s + 8t$. $\therefore f(k+1)$ is divisible by 8. Or is it easier to notice that $f(n+1) = 3^{2(n+1)} + 7 = 9(3^{2n} + 7) - 56$?

20. More algebra.

21. A subset of $k + 1$ elements either uses the first or it doesn't. If it doesn't, it is a subset of k elements and there are 2^k of these. If it does, the subset minus the first element is a subset of k elements and there are 2^k of these. Altogether there are $2^k + 2^k = 2^{k+1}$ subsets of a $k + 1$ element set.

22. (i) $1 + p_1p_2 \ldots p_{k+1}$ is not divisible by $p_1, p_2, \ldots, p_{k+1}$. It is therefore either a prime (larger than p_{k+1}) or is divisible by a prime larger than p_{k+1}. (See Exercise 7.)

(ii) Use (i).

(iii) Test it out. It should be false. What is the smallest n for which it fails?

23. This follows directly from Exercise 22(i).

24. $(1+x)^{k+1} = (1+x)^k(1+x) \geq (1+kx)(1+x) = 1+(k+1)x+x^2 \geq 1+(k+1)x$.

25. $\cos x \cdot \cos 2x \cdots \cos 2^{k-1}x \cdot \cos 2^k x = \frac{\sin 2^k x}{2^k \sin x} \cdot \cos 2^k x = \frac{1}{2}\frac{\sin 2(2^k x)}{2^k \sin x} = \frac{\sin 2^{k+1} x}{2^{k+1} \sin x}$.

26. The trouble is that if r, s ℕ ℕ, then it is not necessarily true that $r - 1, s - 1$ ℕ ℕ. Even though $\max\{r, s\} = n+1$ implies $\max\{r-1, s-1\} = n$ we cannot conclude that $r - 1 = s - 1$ since the Step 2 assumption only applies to members of ℕ.

27. $7^{2k+2} - 48(k+1) - 1 = 49(7^{2k} - 48k - 1) + 2304k$.

28. $\left[\sqrt{k(k+1)}+1\right]^2 = k(k+1)+\sqrt{k(k+1)}+1 = k^2+k+\sqrt{k(k+1)}+1 \geq k^2+2k+1 = (k+1)^2$.

29. The Fibonacci numbers are u_n, where $u_n = u_{n-1} + u_{n-2}$, with $u_1 = 1 = u_2$. Now $u_{k+1} = u_k + u_{k-1}$, so use a little algebra since u_k and u_{k-1} are obtainable from the question with $n = k$ and $n = k - 1$.

30. The top k discs can be moved to another peg in $2^k - 1$ moves. Then move the largest disc to the third peg. The smaller k discs can be moved to this third peg in $2^k - 1$ moves. Altogether there are $(2^k - 1) + 1 + (2^k - 1) = 2^{k+1} - 1$ moves.

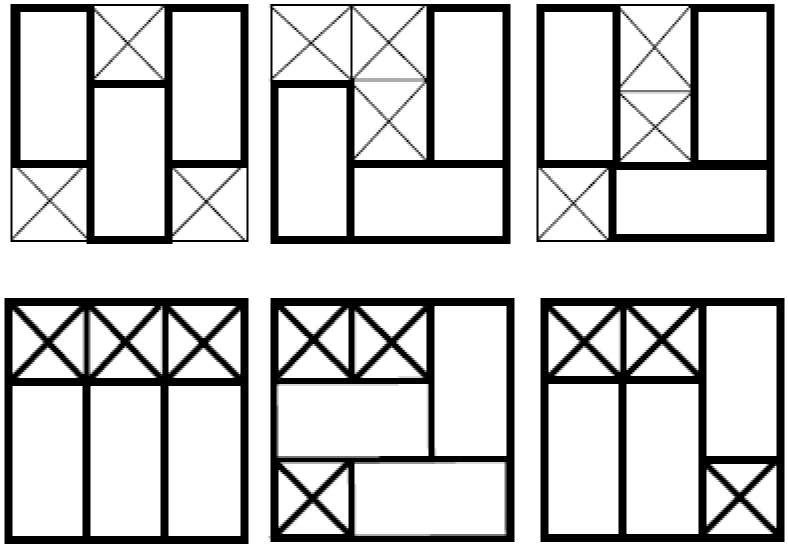
31. (i) True for $n \geq 4$.

(ii) True for $n \geq 4$.

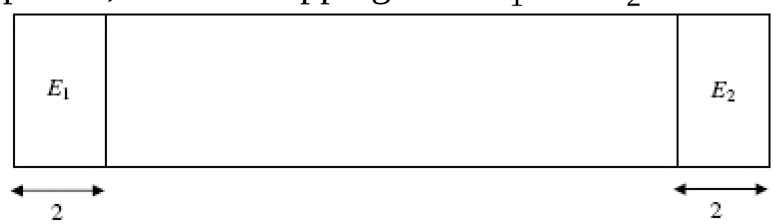
(iii) True for $n \geq 2$.

(iv) 7? 8?

32. We shall refer to such a $(2m + 1)$ x $(2n + 1)$ chessboard with one red square and two black squares removed as a *deleted chessboard*. First, we note that the case $m = n = 1$ is easily handled by exhaustion. Owing to symmetry, there are only six cases that need to be considered, and these are shown below.



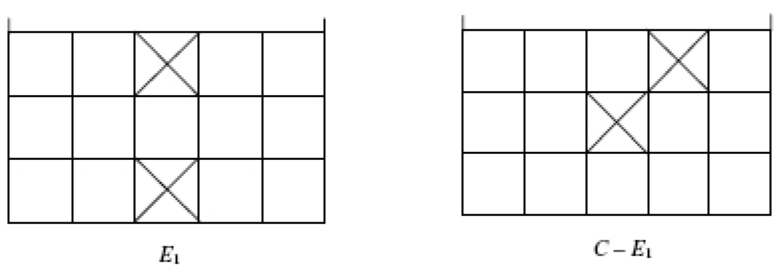
We now proceed by Induction. We are given a $(2m+1) \times (2n+1)$ deleted chessboard C and we may assume that any smaller $(2k+1) \times (2l + 1)$ deleted chessboard which is contained in C may be covered with dominoes. Since at least one of the two dimensions of C is of length at least five, C has two oppositely placed, non-overlapping ends E_1 and E_2 of width two.



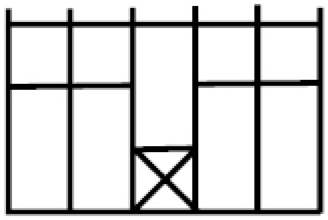
Clearly, we can choose an end containing at most one of the deleted squares of C. Let this end be $E1$ and consider the following two cases.

Case 1. *E_1 contains no deleted square of* C. Then $C–E_1$ contains all three of the deleted squares. By the Induction assumption, $C–E1$ can be covered with dominoes. This covering, together with the obvious one for E_1, yields the desired covering of C.

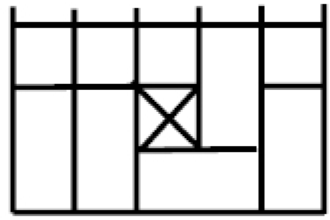
Case 2. *E_1 contains exactly one deleted square of* C. In this case, with the deleted square in E_1 we identify an *associated! square* of the same colour in $C – E_1$ as shown below.



Now delete the associated square in $C – E_1$. By the Induction assumption, there is a domino covering of $C – E_1$ with this deletion. Now C, with its original deletions, may be covered by making use of the covering just found, together with the scheme shown below.



E_1



$C - E_1$

This procedure would fail only in the case where the only choice for the associated square in $C - E_1$ was also deleted. This is impossible in the case of a red square. In the case of a black square, we infer that the one deleted red square is in E_2 and proceed as before.

33. A general law suggested is:

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{(2n-1)2n(2n+1)}{6}, \quad n = 1, 2, 3, \ldots .$$

34. First we observe that $f(1) = 1$ (substitute $m = 2$, $n = 1$ in (4)). Now assume that $f(k) = k$ for $k = 1, 2, \ldots, n-1$. We show that $f(k+1) = k+1$. If $k+1 = 2j$, then $1 \le j < k$ and $f(k+1) = f(2j) = 2j = k+1$.
If $k+1 = 2j+1$, then $1 \le j < k$ and $2j = f(2j+1) < f(2j+2) = f(2(j+1)) = 2f(j+1) = 2(j+1) = 2j+2$. Thus $2j < f(2j+1) < 2j+2$, so $f(2j+1) = 2j+1 = k+1$.

35. The result is valid for $n = 1$. Assume it is valid for $n = k$. Then

$$(1+a_1)(1+a_2)\cdots(1+a_k)(1+a_{k+1}) \ge \frac{2^k}{k+1}(1+a_1+\cdots+a_k)(1+a_{k+1}).$$

We now show that

$$\frac{2^k}{k+1}(1+s)(1+a) \ge \frac{2^{k+1}}{k+2}(1+s+a),$$

where $a = a_{k+1}$, and $s = a_1 + \ldots + a_k$. Multiplying out (and rearranging terms) we obtain

$$2(as-k) + k(a-1)(s-1) \ge 0,$$

and this is valid because $a \ge 1$ and $s \ge k$. There is equality only if $a_j = 1$ for all i. Thus the result is valid for $n = k + 1$ and by Induction for all n.

36. The result is valid for $n = 1$. Assuming its validity for $n = k$, i.e.

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1} = \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{2k-1},$$

we deduce that

$$\begin{aligned} &1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} \\ &= \left(\frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{2k-1}\right) - \frac{1}{2k} + \frac{1}{2k+1} \\ &= \frac{1}{k+1} + \cdots + \frac{1}{2k-1} + \left(\frac{1}{k} - \frac{1}{2k}\right) + \frac{1}{2k+1} \\ &= \frac{1}{k+1} + \cdots + \frac{1}{2k-1} + \frac{1}{2k} + \frac{1}{2k+1}. \end{aligned}$$

37. A generalisation for set A is:

$$\sum_{i=1}^{n} (-1)^{i+1}\,{}^nC_i \frac{1}{i} = \sum_{j=1}^{n} \frac{1}{j}, \quad n = 1, 2, 3, \ldots .$$

This can be proved by Induction. The equality is valid when $n = 1$. Assume that it holds for $n = k$:

$$\sum_{i=1}^{n}(-1)^{i+1}\,{}^{n}C_i\frac{1}{i}=\sum_{j=1}^{n}\frac{1}{j},\quad n=1,2,3,\dots.$$

Then

$$\sum_{i=1}^{k+1}(-1)^{i+1}\,{}^{k+1}C_i\frac{1}{i}$$

$$=\sum_{i=1}^{k+1}(-1)^{i+1}\{{}^{k}C_{i-1}+{}^{k}C_i\}\frac{1}{i}$$

$$=\sum_{i=1}^{k+1}(-1)^{i+1}\,{}^{k}C_{i-1}\frac{1}{i}+\sum_{i=1}^{k}(-1)^{i+1}\,{}^{k}C_i\frac{1}{i}$$

$$=\sum_{j=0}^{k}(-1)^{j}\,{}^{k}C_j\frac{1}{j+1}+\left\{1+\frac{1}{2}+\cdots+\frac{1}{k}\right\}$$

$$=\frac{1}{k+1}\{1-(1-1)^{k+1}\}+\left\{1+\frac{1}{2}+\cdots+\frac{1}{k}\right\}$$

$$=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}+\frac{1}{k+1}.$$

38. The given condition is equivalent to

$$(1+x)^n\equiv 1+x^n \pmod 2. \qquad (*)$$

Now observe that

$$(1+x)^2\equiv 1+x^2 \pmod 2,$$
$$(1+x)^4\equiv(1+x^2)^2\equiv 1+x^4 \pmod 4,$$

and, using Mathematical Induction, we can prove that

$$(1+x)^{2^k}\equiv 1+x^{2^k} \pmod 2,$$

i.e., (*) holds if n is a power of 2. If n is not a power of 2, then $n = 2^{k_1} + 2^{k_2} + \dots$ with at least two distinct k_i's. Then

$$(1+x)^n=(1+x)^{2^{k_1}}(1+x)^{2^{k_2}}\cdots$$
$$\equiv(1+x^{2^{k_1}})(1+x^{2^{k_2}})\cdots \pmod 2,$$

and (*) is not satisfied. Hence ${}^nC_1$, ${}^nC_2$, ${}^nC_3,\dots$, ${}^nC_{n-1}$ are all even integers if and only if n is a power of 2.

39. $9\times\underbrace{109\cdots989}_{n}=9\times\underbrace{109\cdots98900}_{n-1}+9\times 1089.$

40. The result is clearly true for $n = 1$. For $n = 2$, we have to prove $2(1 + x_1x_2) \geq (1 + x_1)(1+ x_2)$. This is equivalent to $(1 – x_1)(1 – x_2) \geq 0$, which is valid with equality if and only if either of x_1 and x_2 equals 1.

Suppose the result holds for all values of n up to $k \geq 2$, with equality occurring under the stated condition. Then, given $0 \leq x_i \leq 1$, i = 1,2,…, k + 1,

$$2^k(1+x_1x_2\cdots x_kx_{k+1})$$
$$=2^{k-1}[2(1+\overline{x_1x_2\cdots x_k}x_{k+1})]$$
$$\geq 2^{k-1}(1+x_1x_2\cdots x_k)(1+x_{k+1})\quad\text{by the } n=2 \text{ case}$$
$$\geq(1+x_1)(1+x_2)\cdots(1+x_k)(1+x_{k+1}),$$

using the result for n = 2 and n = k. If equality occurs, at least k – 1 of the quantities $x_1, x_2,\dots, x_k$ are 1. If only k – 1 of these quantities are 1, then x_{k+1} must equal 1 as well.

41. By letting n = 0,1,2,3, successively, we find that $u_u = 1$, $u_1 = 2$, $u_2 = 2^2$, $u_3 = 2^3$. Consequently, we conjecture that $u_n = 2^n$ for all n; we will establish this result by Induction. We assume that $u_k = 2^k$ for $k = 0,1, 2,\dots, n – 1$. Then from the given relation and the Induction hypothesis,

$$u_k^2 - u_k = \sum_{r=1}^{k} {}^{k+r}C_r u_{k-r} = \sum_{r=1}^{k} {}^{k+r}C_r 2^{k-r}. \qquad (1)$$

If it were known that

$$\sum_{r=1}^{k} {}^{k+r}C_r 2^{k-r} = 2^{2k} - 2^k, \qquad (2)$$

it would then follow from (1) that

$$(u_k - 2^k)(u_k + 2^k - 1) = 0;$$

and since $u_k < 0$, $u_k = 2^k$ which would complete the Induction and establish $u_k = 2^k$ as the unique solution of the problem. Now (2) is easily shown to be equivalent to

$$2^k = \sum_{r=0}^{k} {}^{k+r}C_r 2^{k-r}. \qquad (3)$$

Although (3) is a known binomial identity, we give a proof below. Denote the right side of (3) by a_k. Using the identity

$${}^{k+r}C_r = {}^{k-1+r}C_{r-1} + {}^{k-1+r}C_r,$$

we obtain

$$a_k = \frac{1}{2}\sum_{r=0}^{k} {}^{k-1+r}C_{r-1} 2^{k-r+1} + 2\sum_{r=0}^{k} {}^{k-1+r}C_r 2^{k-1-r}$$
$$= \frac{1}{2}\sum_{s=0}^{k-1} {}^{k+s}C_s 2^{k-s} + 2a_{k-1} + {}^{2k-1}C_k,$$

where we have made the substitution r – 1 = s in the first term. Thus,

$$a_k = \frac{1}{2}(a_k - {}^{2k}C_k) + 2a_{k-1} + {}^{2k-1}C_k = \frac{1}{2}a_k + 2a_{k-1}$$

which reduces to $a_k = 4a_{k-1}$.

Since $a_0 = 1$, we get $a_n = 4^n = 2^{2n}$ by Induction.

42. Three obvious solutions of

$$f(x+y)f(x-y) = (f(x)f(y))^2 \qquad (1)$$

are $f(x) = 0$, 1 or -1.

Setting $y = 0$, we get $(f(x))^2 = (f(x))^2(f(0))^2$, so that, if $f(x) \neq 0$ for some value of x, then $f(0) = 1$ or -1. Since f satisfies (1) if and only if $-f$ does, it suffices to consider the case $f(0) = 1$.

If we put $x = 0$, we get $f(y)f(-y) – (f(y))^2$. If $f(y) \neq 0$, we can divide and get

$$f(-y) = f(y). \qquad (2)$$

Equation (2) still holds if both $f(y)$ and $f(-y)$ are 0, so we have shown that f is an even function. Putting $x = y$, we get

$$f(2x) = (f(x))^4; \quad \text{so} \qquad (3)$$

$$f(x) = 0 \quad \text{implies } f\left(\frac{x}{2}\right) = 0.$$

Thus if f vanishes anywhere, then it vanishes on a set of points approaching 0. Since $f(0) = 1$ and f is continuous, that cannot happen. Consequently $f(x) > 0$ everywhere.

We claim now that for all natural numbers n,

$$f(nx) = (f(x))^{n^2}. \qquad (4)_n$$

For $n = 1$, $(4)_1$ holds trivially; for $n = 2$, $(4)_2$ is equation (3). We prove $(4)_n$ by Induction, setting $y = kx$ in (1):

$$f((k+1)x)f((k-1)x) = (f(x)f(kx))^2 = (f(x))^2(f(kx))^2.$$

We use $(4)k$ and $(4)_{k-1}$ to obtain $(4)_{k+1}$. This will complete the Induction. Setting $x = 1/n$ in $(4)_n$ we get

$$f(1) = \left(f\left(\frac{1}{n}\right)\right)^{n^2}, \quad \text{and thence } f\left(\frac{1}{n}\right) = (f(1))^{1/n^2};$$

and using (4) again, we find $f(m.\ 1/n) = (f\ (1/n))^{m^2}$. So for all positive rational values of x,

$$(5) \qquad f(x) = (f(1))^{x^2}.$$

By continuity (5) holds for positive irrational values of x also. To cover negative values of x we use the fact that both sides of (5) are even. Thus the nonzero solutions of our problems are the functions of the form $\pm a^{x^2}$, $a > 0$.

43. More generally we will show by Induction on n that for any fixed positive integer m there exists a set of n consecutive positive integers each of which is divisible by a number of the form a^m, where a is some integer greater than 1.

For $n = 1$, clearly a^m satisfies the conditions. Assume that for $n = k$, each of the k consecutive numbers $N_1, N_2, \ldots, N_k$ is divisible by an mth power >1. Thus N_i is divisible by $(a_i^m (a_i > 1)$ for $i = 1, 2, \ldots, k$. Let $P = (a_1 a_2 \ldots a_k)^m$. *We now define* $N = N_k +_1 \{((P + 1)^m - 1)^m - 1\}$, *where* $N_k +_1 = N_k + 1$. Then $N + N_1, N + N_2, \ldots, N + N_k +_1$ are $k + 1$ consecutive numbers divisible by af, a_2^m, a_2^m, $(P + 1)^m$, respectively. Hence the desired result is valid by Induction.

44. For $a = 7n$, $7n + 1$, $7n + 2$, $7n + 3$, $7n + 4$, $7n + 5$, $7n + 6$, we have a^2 of the form $7m$, $7m + 1$, $7m + 2$, $7m + 4$ only. The only way for two squares to add to a number which is divisible by 7 is for them to be of the form $7m$. So $x^2 = 7m$ and $y^2 = 7\bar{m}$. But by Exercise 4(c) this gives $x = 7n$ and $y = 7\,7\bar{m}$. Then $(7m)^2 + (7\bar{n})^2 = 49(n^2 + \bar{n}^2)$. Hence $x^2 + y^2$ is divisible by 49.

Is it true that if $x^2 + y^2 + z^2$ is divisible by 7 then $x^2 + y^2 + z^2$ is divisible by 49?

For what t is it true that if $x^2 + y^2$ is divisible by t, then $x^2 + y^2$ is divisible by t^2

45. (a) Let $N = 7a + 2$. For $a = 0, 1, 2, 3, 4, 5, 6, 7, 8$, $7a + 2$ has remainder 2, 0, 7, 5, 3, 1, 8, 6, 4 on dividing by 9. As a increases the same remainders cycle round. So the required n is 58.

(b) I lied! 51 will do the job and so will $51 + 63t$ for any natural number t. (Remember not to trust anyone when it comes to mathematics.)

Chapter 7

Geometry 2

7.1.Cartesian Geometry

Geometry went a long way on the strength of rulers and compasses, polygons and circles, distance and angle. The Greeks established a mountain of knowledge on these objects and Euclid published most of it in his book the "*Elements*". Hence this area of geometry became known as Euclidean geometry. It sought to discover the basic geometrical properties of the world starting from basic assumptions (axioms) about points and lines and the way they behave. Euclidean geometry made a great deal of progress. Some of this can be seen in Chapter 5.

Euclidean geometry worked from axioms via logic to theorems (true statements). It continued to develop long after the Greek era. However, another branch of mathematics had also been developing — this was algebra. In the Seventeenth Century, René Descartes (1596–1650) brought these two branches together when he invented what we know as *cartesian geometry* (and named after Descartes, see MacTutor).

This chapter explores some avenues of cartesian geometry. In particular, lines, the modulus function and the locus of points.

7.2. Lines

In Chapter 5, we looked at triangles and squares but one of the simplest geometrical objects is the line. Most of you have probably done some coordinate geometry. You know about coordinates, axes and so on. You possibly also know about the equation of a line. For instance in Figure 7.1, I've drawn for you $x = 3$, $y = -1$, $y = x$ and $y = -x + 2$.

Now any two points define a line. In other words, there is only one straight line between any two given points. The question is, how do we find the equation of a line?

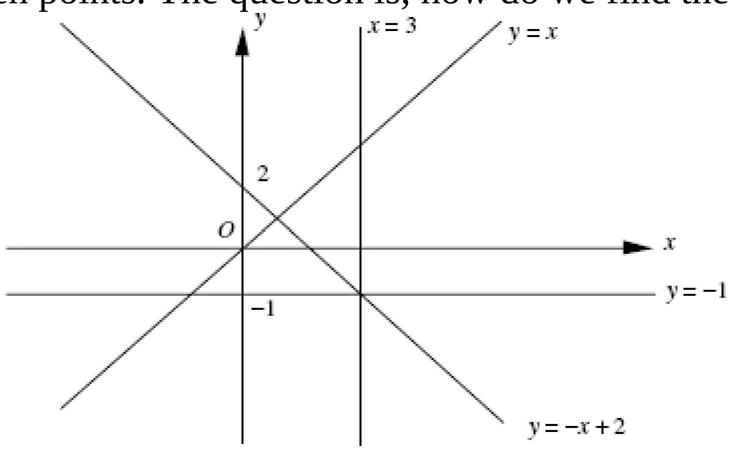


Figure 7.1.

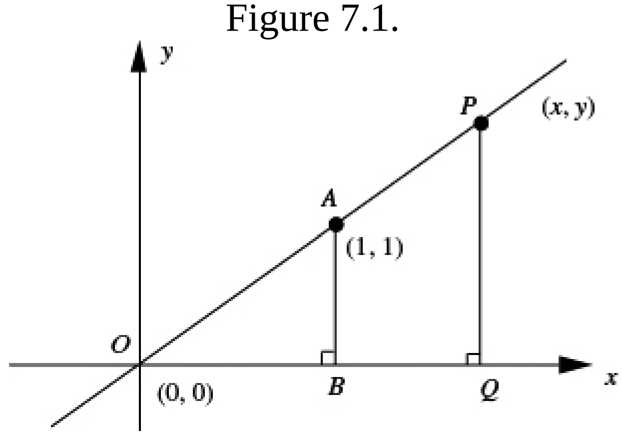


Figure 7.2.

Let's have a look first at lines that go through the origin and some other point.

Example 1. Find the equation of the line which goes through the origin and the point *A* = (1, 1). (See Figure 7.2.)

To do this we need to find a relation between the *x* and *y* value for every point on the line. One way of doing this is to notice that *Δ*'s *OAB*, *OPQ* are similar (see Chapter 5). Hence = $\frac{PQ}{OQ} = \frac{AB}{OB}$. But *AB* = *OB* = 1, *PQ* = *y* and *OQ* = *x*. So we have f = y- This simplifies to *y* = *x*. The equation of the line which goes through (0, 0) and (1,1) is *y* = x.

Exercises

1. Find the equations of the lines that go through the origin and each of the points below.

(i) $(2,2)$; (ii) $(2,1)$; (iii) $(1,2)$; (iv) $(1,-1)$;
(v) $(-1,-1)$; (vi) $(-1,1)$; (vii) $(-2,1)$; (viii) $(0,4)$.

2. Find the equation of the line that goes through the origin and the point (1, m). As *m* changes, what happens to the line?

Finding the equation of a line through two arbitrary points is done in a similar way.

Example 2. Find the equation of the line which passes through *L* = (2,1) and *M* = (3,4). (See Figure 7.3.)

Again we'll try to get an equation linking *x* and y, where (x, y) is a point on the line.

Now *Δ*'s *LMN*, *LPQ* are similar, so $\frac{PQ}{LQ} = \frac{MN}{LN}$ From Figure 7.3, *PQ* = *y -l*, *LQ* = *x* - 2, *MN* = 3 and *LN* = 1. Hence$\frac{y-1}{x-2} = \frac{3}{1}$. So *y* - 1 = 3x - 6. This gives *y* = 3x — 5.

Exercises

3. Find the equation of the lines through the following pairs of points. Where possible express your answer in the form *y* = *mx* + *c*.

(i) $(1,2),(3,4)$; (ii) $(1,3),(4,7)$; (iii) $(2,1),(1,2)$;
(iv) $(-1,2),(-3,6)$; (v) $(-1,-2),(-3,-4)$; (vi) $(1,2),(1,3)$.

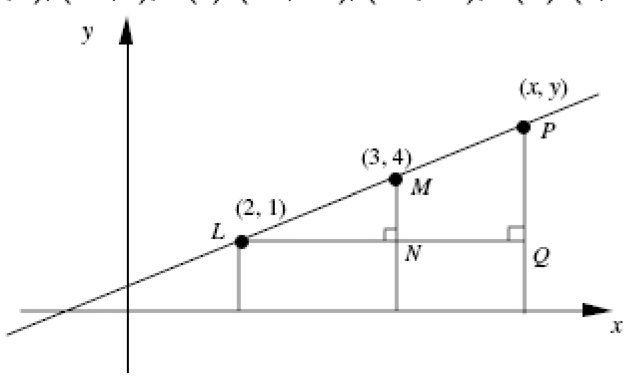


Figure 7.3.

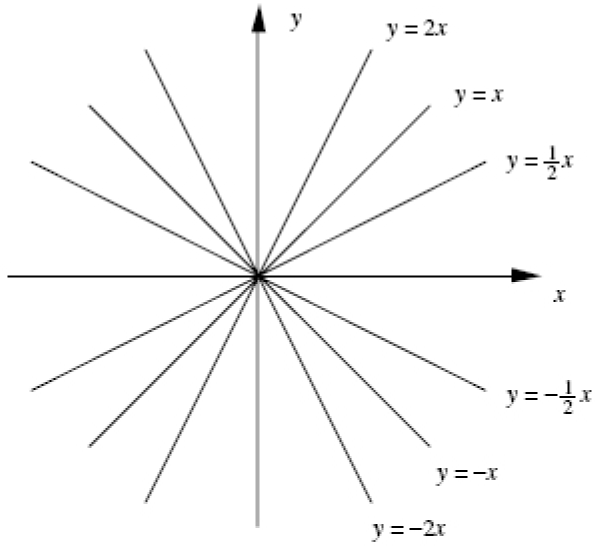


Figure 7.4.

4. Find the equation of the line through the points (xi, y_1) and (x_2, y_2). Are there any problems if $y_1 = y_2$? What if $x_1 = x_2$?

In general a line is parallel to the y-axis and has equation $x = k_1$, or it is parallel to the x-axis and has equation $y = k_2$, or it is of the form $y = mx + c$. The numbers k_1, k_2, m, *c* are all constants.

The significance of the *m* is that it tells how much of a slope the line has. The quantity *m* is called the *gradient* of the line. The effect of a change in *m* is shown in Figure 7.4.

On the other hand, the value *c* is the value of the *y-intercept* of the line. In other words, the line $y = mx + c$ cuts the y-axis at $y = c$.

Exercises

5. Find the gradient and y-intercept of the following lines.
   (i) y = 2x + 4; (ii) y = 4x - 2; (iii) 2y = x - 1; (iv) 0 = 4x + 8y + 7.
   Sketch these lines on a set of cartesian axes.
6. Find the gradient of the line through the points (x_1, y_1) and (x_2, y_2).
7. Sketch the following pairs of lines and determine the angles at which they meet.
   (i) *y* = x, *y* = —x; (ii) *y* = x +1, *y* = —x + 4.
8. (a) Find the equation of the line through the origin which is perpendicular to y = 2x.
   (b) Find all possible lines which are perpendicular to *y* = 2x.
9. Repeat Exercise 8 with the line *y* = —3x.
10. Let *m* be any non-zero real number.
   (a) Find the equation of any line which is perpendicular to the line *y* = *mx*.
   (b) Find all possible lines which are perpendicular to the line *y* = *mx*.

So we see that lines in the plane are sets of points like {(x, y): *y* = x}. But what happens if we change the equality to an inequality? What is {(x, y):*y > x}?*

We have to find all those points (x, y) for which the y-value is greater than the x-value. Now (1, 2) is one such point. You can see that it lies *above* the line *y* = *x* in Figure 7.5.

Other such points are (–2,1) and (–2, –1). They both lie above $y = x$. But any point (x, y) with $y > x$, lies above the line y = x. So {(x, y): $y > x$} is the whole region above the line $y = x$.

In practice we only need to test one point to find out the region that we're looking for. If the point that we test satisfies the inequality, then so do *all* of the points in this region. If the point doesn't satisfy the inequality then the region that we want is on the *other* side of the line.

We represent this region in the plane by shading in the part above the line y = x. Since the line $y = x$ is *not* part of the region {(x, y): $y > x$}, we draw y = x as a dotted line. This is all shown in Figure 7.6.

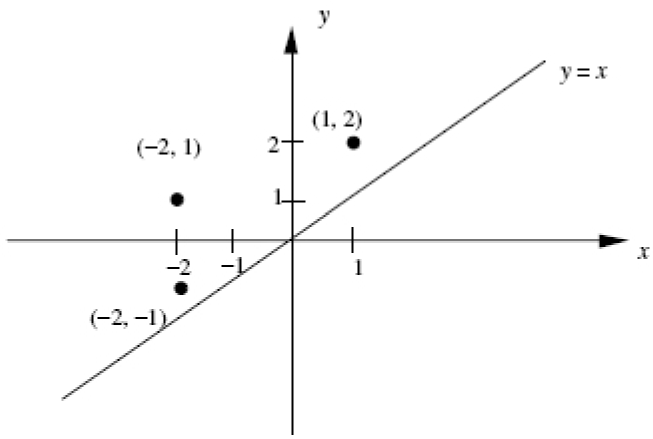


Figure 7.5.

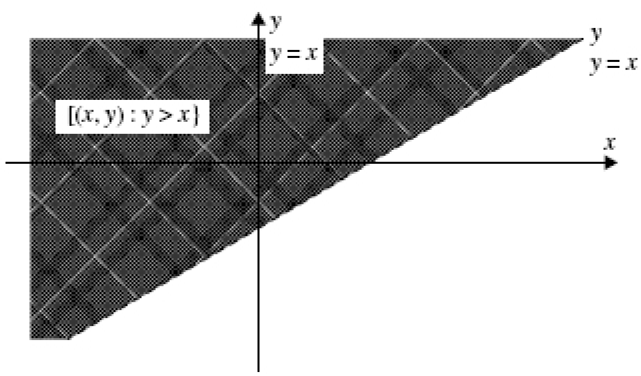


Figure 7.6.

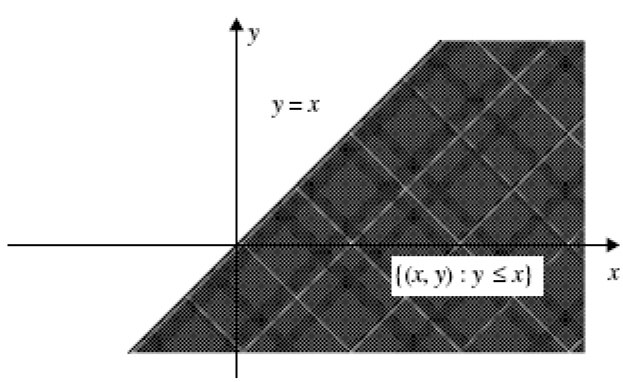


Figure 7.7.

If the boundary line is actually part of the region under consideration, then we draw it as a solid line. We show {(x, y): $y < x$} in Figure 7.7.

Exercises

11. Sketch the following regions.

(i) $\{(x,y): y < 2x\}$;
(ii) $\{(x,y): y \geq 4x\}$;
(iii) $\{(x,y): y > 3x + 2\}$;
(iv) $\{(x,y): y \leq 2x + 3\}$;
(v) $\{(x,y): y \leq 1 - x\}$;
(vi) $\{(x,y): y > -2x - 4\}$;

(vii) $\{(x,y): y > x\} \cap \{(x,y): y > -x\}$;
(viii) $\{(x,y): y < x\} \cap \{(x,y): y \geq -x + 1\}$;
(ix) $\{(x,y): x + y + 4 > 0\}$.

7.3. Modulus

At this stage we bring in the complication of the modulus sign. For reasons which I hope you'll learn to appreciate (if not actually love) we define

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

We read $|x|$ as the "*modulus* (or *magnitude* or *absolute value)* of x".

So |5| = 5, |74 ∠ 3| =74 . 3, | —1| = 1 and |—37 ∠ 89| = 37 ∠ 89.

The whole point about $|x|$ is that it tells us how big x is. If you like, it tells us its magnitude.

Exercises

12. Write down the numerical value of the following.

(i) |17|; (ii) |—21 |; (iii) |—99|;
(iv) |0|; (v)|7| + |6|; (vi) |7| + |—6|;
(vii) | —8| + |—5|.

13. Which of the following is true for all real numbers a and b

$$|a+b| \leq |a| + |b| \quad \text{or} \quad |a+b| \geq |a| + |b|?$$

Can $|a + b| = |a| + |b|$?

14. Which of the following equalities hold for all values of *a?*

(i) 3|a| = |3a|; (ii) —3|a| = |—3a|;
(iii) |*a* — 5| = |5 — a|; (iv) |a| + | —5| = |a| — 5.

Generalise where possible.

Do these things lead to any interesting graphs? What does the graph of $y = |x|$ look like?

Now $y = |x|$ is the same as $y = x$ for $x > 0$. On the other hand, for $x < 0$ it's the same as $y = -x$. So the graph of $y = |x|$ looks like the V shape in Figure 7.8.

The graph of $y = |x - 1|$ can be found by breaking things up into two parts. Now for $x – 1 \geq 0$, $|x – 1| = x – 1 < 0$, $|x - 1| = -(x - 1) = 1 - x$. So

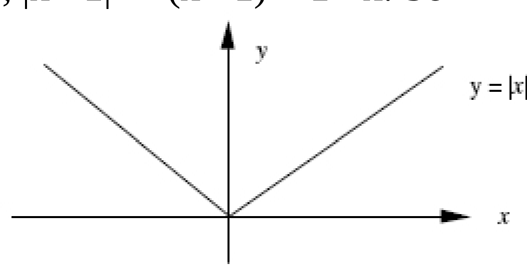


Figure 7.8

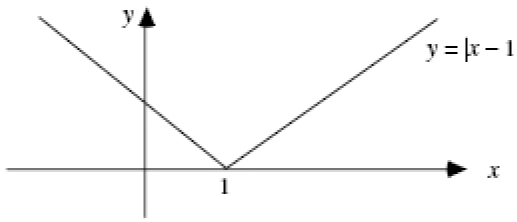


Figure 7.9

$|x-1| = -(x-1) = 1-x$. So

$$y = |x-1| = \begin{cases} x-1, & \text{for } x \geq 1, \\ 1-x, & \text{for } x < 1. \end{cases}$$

The graph is shown in Figure 7.9. It's actually a translation of $y = |x|$ by one unit to the right.

Exercises

15. Sketch the following graphs.
  (i) y = |x — 2|; (ii) y = |x — 3|; (iii) y = |x + 2|;
  (iv) y = |2x|; (v) y = |3x — 3| ; (vi) y = |1 — 2x|.
16. Which of the following pairs of graphs are the same?
  (i) y = 2|x|, y = |2x|; (ii) y = —2|x|, y = |—2x|;
  (iii) y = |*x* — 4|, y = |4 — x|; (iv) y = |*x*| + |— 4|, y = |*x*|— 4.
17. By considering the four regions where $x \geq 0$, $y \geq 0$ and x > 0, *y* 0 and x < 0, $y \geq 0$ and x < 0, *y* 0, sketch the graph of |*y*| = |*x*|.
18. (a) Sketch the following graphs
  (i) |*y*| = |*x* — 1|; (ii) |*y*| = |2x|;
  (iii) |*y* — 1| = |*x*|; (iv) |*y*| = |3x|.
  (b) Find the equation in modulus form of the two perpendicular lines which pass through (5, 3), given that one line has gradient 1.

One final example.

Example 3. Sketch {(x, y): |*y* — *x*| + |*y*| = 2}.

To be able to sketch this we first notice that we have two modulus signs. Both of these have two things happening to them depending upon whether the expression inside them is positive or negative. So we first have to find the four regions (4 = 2 × 2) into which these conditions divide the plane. We then have to look at the values of the modulus signs in these regions to see what sort of graph we've got. It's not hard, just a bit tedious. However the surprising result at the end is worth the effort. We work it out this way.

The regions we want are, *y* — *x* > 0, *y* > 0; *y* — *x* > 0, *y* 0; *y* — *x* 0, *y* > 0; and *y* — *x* 0, *y* 0. We show these regions in Figure 7.10.

If *y* — *x* > 0, *y* > 0, the equation |*y* — *x*| + |*y* | = 2 becomes *y* — *x* + *y* = 2 or 2*y* — *x* = 2. This simplifies to *y* = ½*x* + 1. So the part of the whole graph in region I (and region I only) is the line *y* = ½*x* + 1.

If *y* — *x* > 0, *y* > 0, the equation |*y* — *x*| + |*y* | = 2 becomes *y* — *x* — *y* = 2 or *x* = —2. This goes in region II. So we will need to draw *x* = —2 in region II only.

If $y — x$ 0, y > 0, the equation $|y — x| + |y\ | = 2$ becomes $x — y + y = 2$ or $x = 2$. We use that part of the line $x = 2$ which lies in region III.

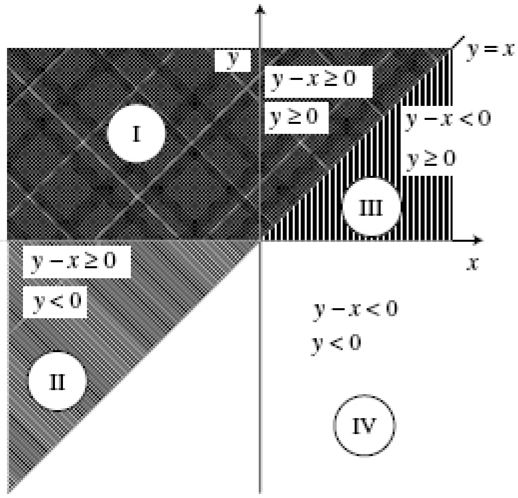


Figure 7.10

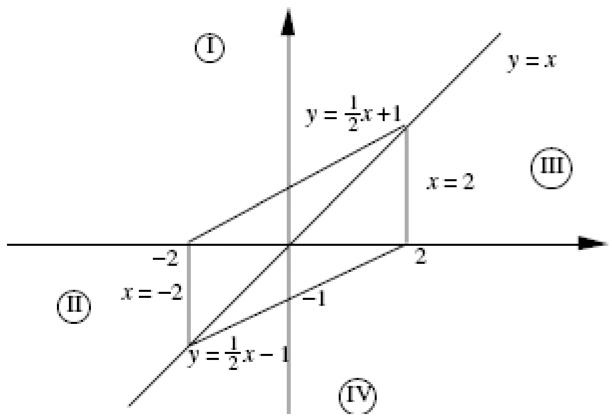


Figure 7.11

Finally, if $y– x$ 0, y > 0 the equation $|y— x| + |y| = 2$ becomes $x– y— y = 2$. We use the part of this line which is in region IV.

We put all this information together in Figure 7.11. Surprisingly we come up with what looks like a parallelogram.

Now we've got to the end, you might like to go through this again just to make sure you've mastered all of the steps. You should note though, that $|y — x| + |y| = 2$ is the equation of a parallelogram. Why? Isn't that neat!

Exercises

19. Sketch the following sets and identify the shapes in these graphs.
   (i) {(x, y): $|x| + |y| =4$};
   (ii) {(x, y): $|x — y| + |x + y| =4$};
   (iii) {(x, y): $|x — 2y| + |2x + y| =4$};
   (iv) {(x, y): $|x — y| + |x + y — 2| =4$}.
20. Sketch the following squares *OABC* and describe them using modulus signs.
   (i) $A = (2,0)$, $B = (2,2)$, $C = (0,2)$;
   (ii) $A = (— 1,0)$, $B = (— 1,1)$, $C = (1,0)$.
21. Sketch the following squares *ABCD* and describe them using modulus signs.
   (i) $A = (1,0)$, $B = (1,2)$, $C = (— 1,2)$, $D = (— 1,0)$;
   (ii) $A = (1, —1)$, $B = (1,1)$, $C = (— 1,1)$, $D = (—1, —1)$;
   (iii) $A = (1,0)$, $B = (0,1)$, $C = (— 1,0)$, $D = (0, —1)$;

(iv) $A = (1, 0)$, $B = (2, 1)$, $C = (1, 2)$, $D = (0, 1)$.

22. Given any square anywhere in the plane, how would you find its equation using modulus signs?

23. Sketch the following sets and identify their shape.

(i) {x, y):|y—x|+|x|= 2};

(ii) {x, y):\y — 2x \ + \y\ = 4};

(iii) {x, y):\y — x \ + \yf 2x \ =6};

(iv) {x, y):|y—x|+|y— 2x| =6}

(v) {x, y):\3y — x \ + \x+ 3y \ == 6};(vi) {x, y):2\3y — x \ +x +3y \= 6}

24. Sketch the following quadrilaterals *ABCD* and describe them using modulus signs.

(i) $A = (1, 0)$, $B = (2, 1)$, $C = (1, 1)$, $D = (0, 0)$;

(ii) $A = (1, —1)$, $B = (2,1)$, C = (— 1,1), $D = (—2, —1)$;

(iii) $A = (1, —1)$, $B = (3, —1)$, C = (4,1), $D = (2,1)$.

25. (a) Show how to find the equation of any parallelogram using modulus signs.

(b) Is it possible to express any quadrilateral as an equation using modulus signs?

(c) What polygons have equations that can be expressed in terms of modulus signs? This is an open ended investigation. Start with triangles. You've done four-sided polygons so after triangles try pentagons, hexagons and so on.

26. Are there any values of *b* for which the equations

$$y = |x| \text{ and } y = 6 - |b - x|$$

have an infinite number of points in common? If so, find them; if not, say why not.

7.4. Loci: One Fixed Point

Most objects that move do so within certain constraints. Cars usually stick to roads or they invariably come to grief. Planes are not very good under water. What goes up must come down.

In this section we look at points that move under certain constraints in the plane. The result is called the *locus* of the point. (The plural of locus is *loci* not locuses.)

The simplest way to start is to look at a point *P* which moves so that it is a fixed distance from a fixed point. Clearly *P* moves in a circle. The fixed distance is the radius of that circle.

Exercises

27. On a set of cartesian axes, using whatever instruments you think might be appropriate, draw the loci of the following points that are the given distance from the given point.

(i) 5 from $O = (0, 0)$; (ii) 10 from *O*;

(iii) 4 from *O*; (iv) 4 from $C = (1, 1)$;

(v) 5 from $C = (1, 2)$; (vi) 4 from $C = (— 1,1)$;

(vii) 8 from $C = (—2, —3)$. (Use appropriate units to suit your graph paper.)

28. The point *P* moves so that it is always a distance 4 from the fixed point *C*. If the locus of *P* passes through (– 1, 0) and (7, 0) find the coordinates of *C*.

29. The point *P* moves so that it is always a distance 5 from the fixed point *C* = (a, 4). If the locus of *P* passes through (–2, 0) and (4, 0), find a.

30. The point P moves so that it is always a distance 2 from the fixed point C = (2, b). Find b if the locus of P passes through (1, 0) and (3, 0).
31. A point P moves so that its distance from the fixed point C is 5. The locus of P passes through the points (0, 0) and (6, 0). Find all possible positions of C.
32. A point P moves so that its distance from the fixed point C is 13. The locus of P passes through the points (0, —1) and (0,9). Find the coordinates of C.
33. A point P moves so that its distance from a fixed point C is 13. The locus of P passes through the points (2, –3) and (2,7). Find the coordinates of C.
34. A point P moves so that its distance from a fixed point C is 13. The locus of P passes through the points (1,2) and (11,26). Find the coordinates of C.
35. A point P moves so that its distance from a fixed point C is 25. The locus of P passes through the points (7, –1) and (32, 24). Find the coordinates of C.

I think we're just about ready now to find the equation of a circle, given its radius (the fixed distance of the original locus problem) and its centre (the fixed point of the locus problem).

Example 4. Find the equation of the circle centre O and radius 2.

Look at Figure 7.12. Let P = (x, y) be any point on the circle. Then OP = 2 since the circle is of radius 2. By Pythagoras, $x^2 + y^2 = 2^2$. So the equation of the circle is simply $x^2 + y^2 = 4$.

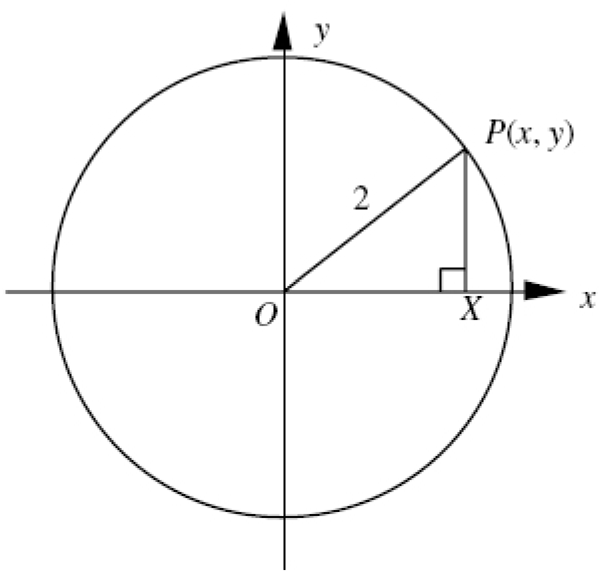


Figure 7.12

The locus of the point P which moves so that it is a fixed distance 2 from the fixed point O is a circle of radius 2. Any point (x, y) on this locus has the x and y linked by the equation $x^2 + y^2 = 4$.

Exercises

36. Find the equations of the loci where P moves so that it is a fixed distance r from the point C, where

(i) $r = 1$, $C = (0, 0)$; (ii) $r = 5$, $C = (0, 0)$;
(iii) $r = \sqrt{2}$, $C = (0, 0)$; (iv) $r = a$, $C = (0, 0)$;
(v) $r = 1$, $C = (1, 0)$; (vi) $r = 2$, $C = (1, 2)$;
(vii) $r = 3$, $C = (-1, 1)$; (viii) $r = \sqrt{2}$, $C = (-1, -3)$;
(ix) $r = a$, $C = (2, 1)$; (x) $r = a$, $C = (s, t)$.

37. What is the locus of a point that moves so that it is equidistant from the points (1, 0), (—1, 0)?

38. What is the locus of a point that moves so that it is equidistant from the two points (0, 1), (0, 3)?

39. What is the locus of a point that moves so that it is equidistant from the two points (1, 0), (0, 1)? *P*(*x*, y)

Summarising, for the record, using Pythagoras we see that the equation of the locus of the point which moves so that it is a fixed distance *r* from
the fixed point (x_1, y_1) is

$$(x - x_1)^2 + (y - y_1)^2 = r^2.$$

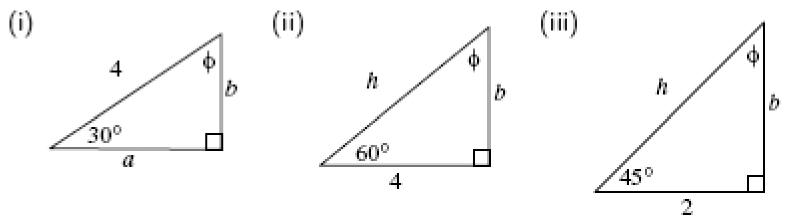
Equivalently, this is the equation of the circle, radius r, centre (x_1, y_1).

7.5. The Cosine Rule

We start this section with an Exercise.

Exercise

40. Find all the unknown angles and sides.



It turns out that cos *θ* is useful in determining unknown sides or angles in triangles which are not right angled. This is because of the Cosine Rule. For the sides and angles in Figure 7.13, it turns out that we have

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This is known as the *Cosine Rule.*

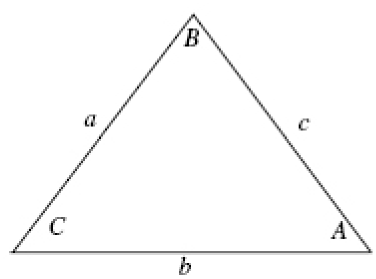
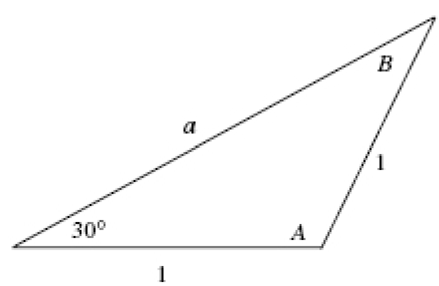


Figure 7.13.

Example 5. Find the unknown sides and angles in the triangle below.



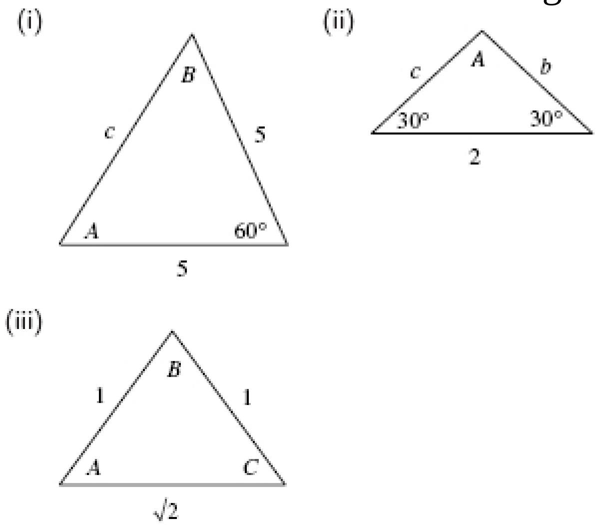
Since the triangle is isosceles *B* = 30°. Then *A* =180° - 60° = 120°.

Now $a^2 = 1^2 + 1^2$ - 2 x 1 cos 120° = 2 - 2 x (-0.5) = 3.

Hence $a = \sqrt{3}$.

Exercises

41. Find all the unknown sides and angles in the triangles below.



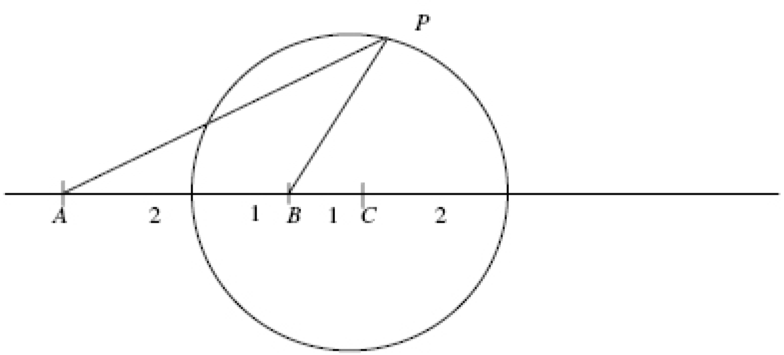
42. Pythagoras' Theorem is usually quoted as follows: In a right angled triangle the square on the hypotenuse equals the sum of the squares on the other two sides.
Show that if the square of one side of a triangle is equal to the sum of the squares on the other two sides, then the triangle is a right angled triangle.

43. Prove that the Cosine Rule is true.

44. Discover, then prove, a Sine Rule for triangles.

45. Let $y = m_1x +$ ci and $y = m_2x + c_2$ be any two lines where neither m_1 nor m_2 is zero. Show that the two lines are perpendicular if and only if $m_1 m_2 = -1$.

46. In the situation in the diagram below, use the Cosine Rule to prove that the ratio of *AP* to *BP* is 2 no matter where *P* is on the circle. (C is the centre of the circle.) (Do you need to know that $A = (0, 0)$ and $B = (3, 0)$?)



7.6. Loci: Two Points

So let's move on to *two* fixed points. What is the locus of a point that moves so that it is equidistant from *two* fixed points?

In Exercises 37, 38 and 39 we saw that, in each case, the locus was a straight line. Is this always the case?

Exercises

47. What is the locus of a point that moves so that it is equidistant from the pairs of points below?
(i) $A = (1,0)$, $B = (3,0)$; (ii) $C = (1,0)$, $D = (1,4)$;
(iii) $E = (2, 0)$, $F = (0, 2)$; (iv) $G = (2, 0)$, $H = (0,4)$.

48. In all cases so far we have found that the locus of a point that moves so that it is equidistant from two points, is a straight line. Can this be proved? Is it true?
If it is true, use Euclidean arguments to prove it. If it's false find a counterexample.

We'll now use coordinate geometry to show that the locus of a point which moves so that it is equidistant from two fixed points is a straight line.

Let the fixed points be (a, b), (c, d). Let P = (x, y) be a point equidistant from (a, b) and (c, d). Then ?s/$(x - a)^2 + (y - b)^2$ = y/$(x-c)^2 + (y -d)^2$.

$$\sqrt{(x-a)^2+(y-b)^2}=\sqrt{(x-c)^2+(y-d)^2}.$$

Hence

$$x^2-2ax+a^2+y^2-2by+b^2=x^2-2cx+c^2+y^2-2dy+d^2. \qquad (1)$$

If $b = d$, then $a \neq c$ or we only have one fixed point. So $2x(a - c) = a^2 — c^2$. This simplifies to x = ½(a + c), the equation of a line perpendicular to the x-axis (and the line between the two fixed points).

If $b = d$, then $d - b = 0$ so we may divide both sides of (1) by 2(d - b) to give
Again this is the equation of a straight line.

Exercises

$$y=x\left(\frac{a-c}{d-b}\right)+\frac{1}{2}\left(\frac{c^2+d^2-a^2-b^2}{d-b}\right).$$

49. Show that all the loci of Exercise 47 are lines that are the perpendicular bisectors of the line segments joining the two fixed points.
Is this always the case?
50. Given two fixed points the simplest locus is that of a point moving so that it is equidistant from the two fixed points.
Exercise 46 suggests that if P is such that AP: PB is 2, then P lies on a circle.
(a) For the fixed points A, B of Exercise 47, find the equation of the locus of P such that AP : PB = 2.
(b) What is the locus of the point P which moves so that PB : AP = 2?
(c) Repeat (a) and (b) for the other pairs of fixed points of Exercise 47.
51. Given any fixed points A and B, what is the locus of P such that AP: PB = 2?
52. Repeat Exercise 51 with “3” replacing “2”.
53. Guess, sorry conjecture, what will happen if “2” in Exercise 51 is replaced by any real number “k”.
Prove your conjecture.
(Have you taken into account all possible values of k? Does it make any sense for k to be negative?)

The next kind of locus we can try that is associated with fixed points is the “length of string” locus. Take a piece of string, two drawing pins, a piece of hardboard, some paper and a pencil. Put the paper on the board and then put a drawing pin in either end of the string to fix it to the board. Don't stretch the string between the pins; keep it loose. Now put your pencil against the string and pull the string tight. Move the pencil round the paper so that the string is always kept as tight as possible.

What is the shape of the locus that is traced out by the pencil? What is therefore the locus of the point of the pencil?

We illustrate the situation in Figure 7.14. You should find that the pencil makes a complete closed curve. It actually looks a bit like a circle that has been sat on. Is it? Has it?

Exercises

54. Suppose the drawing pins are at the points *A* and *B*, and the length of the string is *k*. Construct the "length of string" locus for the following points and values of *k*. (Choose your own units.)

(i) *A* =-1,0),*B*= (1,0),*k*=6;
(ii) *A* =-1,0),*B*= 1, 0),*k*=9;
(iii) *A* =-2,0),*B*= (2, 0),*k*=9;
(iv) *A* =-2,0),*B*= (2, 0),*k*=12;
(v) *A* =-2,0),*B*= (2, 0),*k*=15.

For each locus find the coordinates of the points where it crosses the x- and y-axes. (You will need to use Pythagoras' Theorem again to calculate the y-intercepts.)

55. Continue your own investigation of the "length of string" locus.

Keep *A* and *B* fixed. What is the effect of changing *k*? Is there a smallest value of *k*? Is there a largest value of *k*?

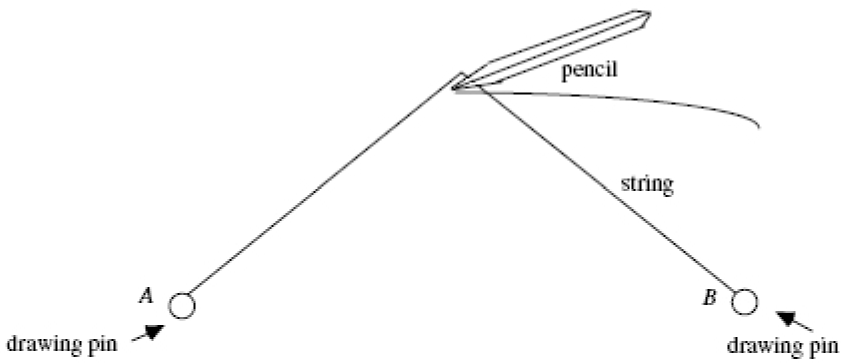


Figure 7.14

What shape are you producing? Are you convinced it's a closed curve? Is it a circle? Could it ever be a circle?

Let's find the cartesian equation of the "length of string" locus.

Example 6. Find the equation of the locus of Exercise 54(i).

Here our drawing pins are at *A* = (-1,0) and *B* = (1,0). The length of string is 6. Let *P* = (x, y) be any point on the locus. Then we know that√*AP* + *PB* = *k* = 6.

Now $AP = (x + 1)^2 + y^2$ and $PB = \sqrt{(x - 1)^2 + y^2}$, so we have

$$\sqrt{(x+1)^2+y^2}+\sqrt{(x-1)^2+y^2}=6. \qquad (2)$$

Equation (2) is a mess. Have you ever come across anything like this with *two* square roots? Even if we squared it we'd still have one square root left. To get rid of that we'd have to square again!

I'm sorry. There's nothing for it but to do it. Here goes. Squaring both sides of (2) gives

$$(x+1)^2+y^2+2\sqrt{(x+1)^2+y^2}\sqrt{(x-1)^2+y^2}+(x-1)^2+y^2=36.$$

If we keep the square roots on the left, and square again, we'll finally get rid of all the square roots. So let's tidy up and square again.

$$\begin{aligned} &2\sqrt{(x+1)^2+y^2}\sqrt{(x-1)^2+y^2} \\ &\quad = 36-(x+1)^2-y^2-(x-1)^2-y^2 \\ &\quad = 36-[x^2+2x+1+y^2+x^2-2x+1+y^2] \\ &\quad = 34-2x^2-2y^2. \end{aligned}$$

So

$$\sqrt{(x+1)^2+y^2}\sqrt{(x-1)^2+y^2} = 17-x^2-y^2.$$

Let's square again like we did last summer.

$$[(x+1)^2+y^2][(x-1)^2+y^2] = (17-x^2-y^2)^2.$$

The algebra we've got left to do here, boggles the mind. Take a deep breath, or a short break, or use a CAS program, and then…

$$\begin{aligned} &(x+1)^2(x-1)^2+(x+1)^2y^2+y^2(x-1)^2+y^4 \\ &\quad = 17^2-34(x^2+y^2)+(x^2+y^2)^2 \end{aligned}$$

$$\begin{aligned} &\therefore (x^2-1)^2+y^2[x^2+2x+1+x^2-2x+1]+y^4 \\ &\quad = 17^2-34x^2-34y^2+x^4+2x^2y^2+y^4 \\ &\therefore x^4-2x^2+1+2x^2y^2+2y^2+y^4 \\ &\quad = 17^2-34x^2-34y^2+x^4+2x^2y^2+y^4. \end{aligned}$$

Ah! Blessed relief. At least we can do some cancelling. We lose all the x^4's, y^4's and x^2y^2's. Then we've only got

$$-2x^2+1+2y^2 = 17^2-34x^2-34y^2.$$

This simplifies to

$$8x^2+9y^2 = 72.$$

so

$$\begin{aligned} 8(3)^2+9(0)^2 &= 8\times 9 = 72, \\ 8(-3)^2+9(0)^2 &= 8\times 9 = 72. \end{aligned}$$

That's considerably simpler than we had any right to expect. But is it right? Perhaps we should check a few points. We know from Exercise 58(i) that the curve crosses the x-axis at (3,0) and (-3,0). Do these points satisfy equation (3)?

$$\begin{aligned} 8(3)^2+9(0)^2 &= 8\times 9 = 72, \\ 8(-3)^2+9(0)^2 &= 8\times 9 = 72. \end{aligned}$$

Yes. These points do satisfy the equation.

We also know, that $(0, 2\sqrt{2})$ and $(0, -2\sqrt{2})$ also lie on the locus. Do these satisfy the equation (3) too?

Exercises

56. Find the equations of the loci of Exercise 54. All your equations should end up being as simple as equation (3).

Check your equations by determining whether or not they are satisfied by the x-and y-intercepts that you found in Exercise 54.

57. So far all the "length of string" loci that we have considered have had equations of the form $\alpha x^2 + \beta y^2 = \gamma$.

(a) Suppose our "length of string" locus has equation $4x^2 + 9y^2 = 36$ and the drawing pins wereon the *x*-axis and symmetrically placed about the *y*-axis. Find the position of the drawing pins and the length of the string.

(b) Repeat (a) with the following equations

(i) $x^2 + 4y^2 = 4$; (ii) $x^2 + 9y^2 = 9$;

(iii) $5x^2 + 9y^2 = 20$; (iv) $7x^2 + 8y^2 = 56$.

58. Suppose the drawing pins are placed at $A = (-c, 0)$, $B = (c, 0)$ and the string is of length *k*. Show that the equation of the "length of string" locus has the form $\alpha x^2 + \beta y^2 = \gamma$.

59. Show that any "length of string" locus lies between two circles. How are these circles related to the *x*- and *y*-intercepts of earlier exercises? Show that a point on the "length of string" locus has coordinates that partly come from the smaller of these two circles and partly from the larger.

Exercise 59 and equations of the form $\alpha x^2 + \beta y^2 = \gamma$, where $\alpha \neq \beta$, should convince you that the "length of string" loci are not circles. They are in fact *ellipses*. This is a shape that is like one of the cross-sections of a rugby ball. It is also pretty close to the shape of the orbit of the planets around the sun.

The technical jargon for the position of the drawing pins is the *foci* of the ellipse. So in Exercise 58, *A* is the position of one focus and *B* is the position of the other. In the planetary situation, the sun sits at one of the foci of each planet's orbits.

Exercises

60. In all our work so far on ellipses, we have taken our foci to lie on the *x*-axis. Investigate what equations are obtained when the foci are on the *y*-axis and symmetrically placed about the *x*-axis.
In such situations do the ellipses still lie between a pair of circles? Given a pair of circles with centres at the origin, show the two ellipses that lie between them. (Can more than two ellipses lie between these circles?)

61. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is commonly used for an ellipse which is symmetrically placed about the origin.

(a) If $a > b$, find the position of the foci and the "length of the string" which formed the ellipse.

(b) Repeat (a) if $b > a$.

(c) What happens if $a = b$?

62. Which reminds me, the "length of string" loci arise because the sum of the distances from *P* to two given points is a constant. What happens if the *difference* between the distances from *P* to two points is a constant? To help you answer this suppose the fixed points are *A*, *B* and the difference is *k*. Find the equation of the locus of *P* such that $|AP - PB| = k$, where

(i) $A = (-1,0)$, $B = (1,0)$, $k = 1$;

(ii) $A = (-2,0)$, $B = (2,0)$, $k = 3$;

(iii) $A = (-3,0)$, $B = (3,0)$, $k = 4$.

Do we still have a closed curve?

63. What is the locus of a point that moves so that it is equidistant from two fixed *lines?*

64. What is the locus of a point that moves so that it is equidistant from *three* fixed *points?*

65. What is the locus of a point that moves so that the sum of its distances from three fixed points is a constant?

66. What is the locus of a point that moves so that its distance from a fixed line is equal to its distance from a fixed point?

67. Is the area of the equilateral triangle on the hypotenuse of a right angled triangle, equal to the sum of the areas of the equilateral triangles on the other two sides? Generalise.

68. Find the equations of the loci of the points that move so that they are equidistant from the fixed points *A* and the fixed lines L.
(i) *A* = (0, 0) and *L* is y = 6; (ii) *A* = (0, 0) and *L* is x = -4.

69. Find the equations of the loci of the points that move so that they are equidistant from the fixed lines *L* and *M*.
(i) *L* is y = 4 and *M* is y = 6;
(ii) *L* is y = 4 and *M* is x = -4;
(iii) *L* is *y* = 2x and *M* is *y* = 4x;
(iv) *L* is y = 2x + 3 and *M* is y = -3x + 2.

7.7. Conics

Look at the double cone in Figure 7.15. It has a circular horizontal cross- section.

If you take a horizontal cut through it, the exposed interior face will be a circle.

A cut at an angle, like that of *A*, will expose a face with an elliptical boundary — the section here is an *ellipse*. Cuts like *B*, which are parallel to the “side” of the double cone produces parabolic sections — the boundary of the exposed face is in the shape of a *parabola*. Vertical cuts like *C* will produce *hyperbolic* cross-sections.

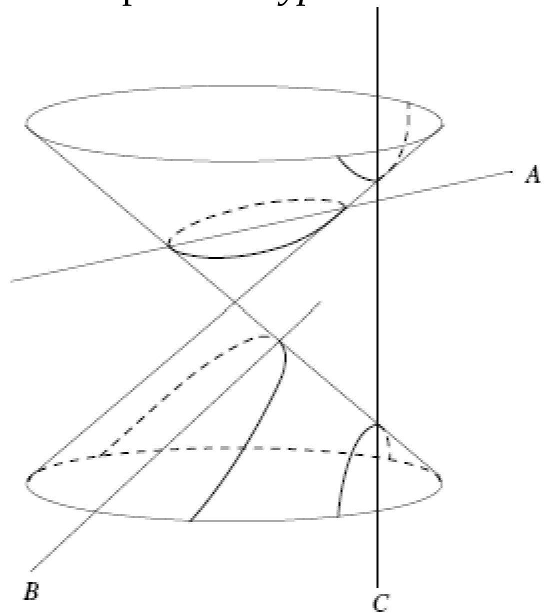


Figure 7.15.

Double cones are difficult to model in wood but single cones are not hard. You may well have one in your school that is even cut in the ways I've considered in Figure 7.15.

Because of their links to the cone via cross-sections, the circle, the ellipse, the parabola and the hyperbola are called *conic sections*. These shapes all appear somewhere in the last section.

You can find out more about conics by looking on the web or by looking at a geometry book in a library.

7.8. Solutions

1.

(i) $y = x$; (ii) $y = \frac{1}{2}x$; (iii) $y = 2x$; (iv) $y = -x$;
(v) $y = x$; (vi) $y = -x$; (vii) $y = -\frac{1}{2}x$; (viii) $x = 0$.

2. $y = mx$; see Figure 7.4 to find out what happens as m changes.

3.

(i) $y = x + 1$; (ii) $y = \frac{4}{3}x + \frac{5}{3}$; (iii) $y = -x + 3$;
(iv) $y = -2x$; (v) $y = x - 1$; (vi) $x = 1$.

4. Using the approach of Example 2 you should see that $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$ or equivalently $\frac{y-y_2}{y_1-y_2} = \frac{x-x_2}{x_1-x_2}$. If $x_1 \neq x_2$ and $y_1 \neq y_2$ this can be written in the form

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \left(\frac{y_1x_2 - y_2x_1}{x_2 - x_1}\right).$$

(If you didn't get either of the equations of the first sentence, it is still possible that you are not wrong. Does your equation give the equation of the second sentence above?)

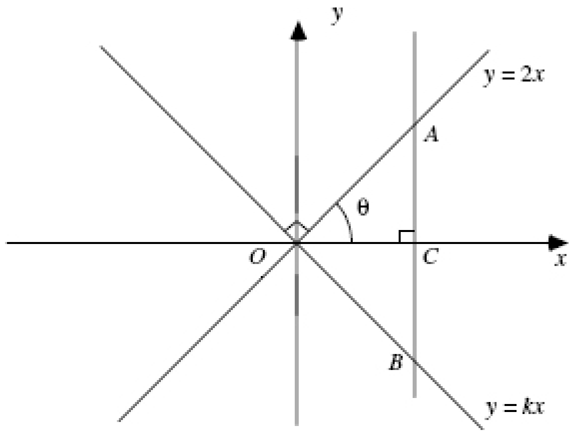
In the case $y_1 = y_2$, we get a horizontal line. Every point on this line has y_1 as its y-value. Hence the equation of such a line is $y = y_1$.

If $x_1 = x_2$, we have a vertical line. On such a line, every point has the same x-value. The line's equation is therefore $x = x_1$.

5. (i) gradient: 2; y-intercept: 4; (ii) 4; –2;
(iii) $\frac{1}{2}$; $-\frac{1}{2}$ (the equation must be written in the form $y = \frac{1}{2}(x - 1)$);
(iv) $-\frac{1}{2}$; $-\frac{7}{8}$.

7. (i) 90°; (ii) 90°; (iii) 90°; (iv) 90°.

8. (a)

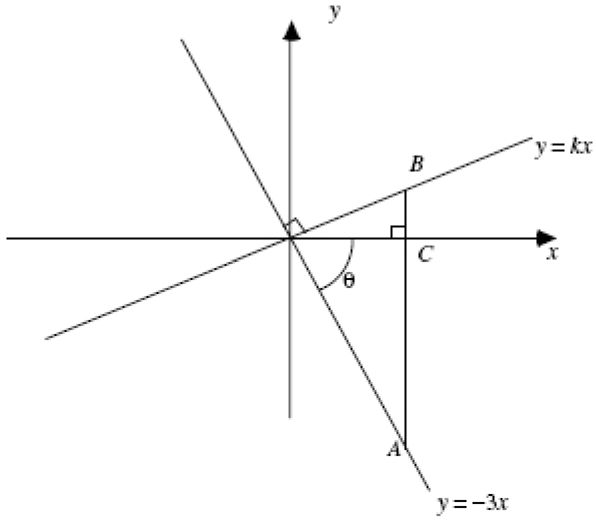


In the diagram, $y = kx$ is the required line. Let A be any point on the line $y = 2x$. Let C be the point where the line through A perpendicular to the x-axis meets the x-axis and let B be the point where this perpendicular meets the line $y = kx$.

Since angle AOB = 90°, then $\triangle$'s AOC, OBC are similar. (You should be able to show this for yourself. You only need to prove that the triangles have the same angles.) Hence $\frac{BC}{OC} = \frac{AC}{OC}$. But $\frac{AC}{OC}$ gives the gradient of the line $y = 2x$, so $\frac{AC}{OC} = 2$. On the other hand, $\frac{BC}{OC}$ is the magnitude of the gradient of the line $y = kx$. Since this gradient is negative, then its value is $-\frac{AC}{OC} = -\frac{1}{2}$. The line $y = kx$ is therefore $y = -\frac{1}{2}x$.

(b) Every line perpendicular to $y = 2x$ is parallel to $y = -\frac{1}{2}x$. Hence all the lines perpendicular to $y = 2x$ have gradient $-\frac{1}{2}$. Hence their equations are all of the form $y = -\frac{1}{2}x + c$.

9. (a)



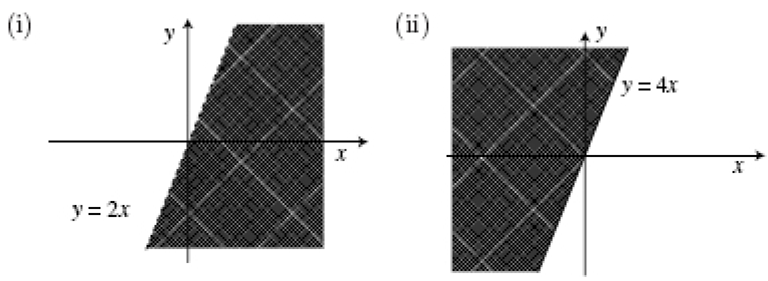
Using a similar argument to that of Exercise 8(a) we see $\frac{BC}{OC} = \frac{1}{3}$. Hence the gradient of the line $y = kx$ is $\frac{1}{3}$.

(b) All possible lines perpendicular to $y = -3x$ have equation $y = \frac{1}{3}x + c$.

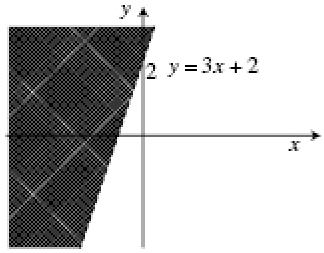
10. (a) Exactly the same arguments used in Exercise 8(a) will give $y = -\frac{1}{m}x$ provided $m \neq 0$.

$y = -\frac{1}{m}x + c$.

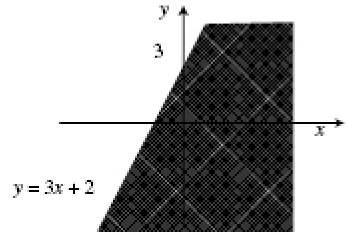
11.



(iii)



(iv)



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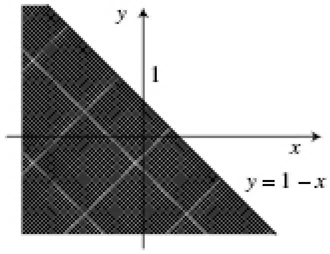

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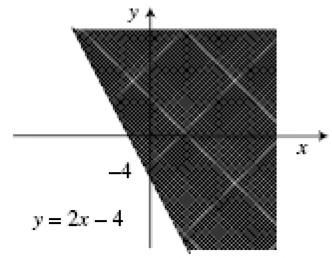
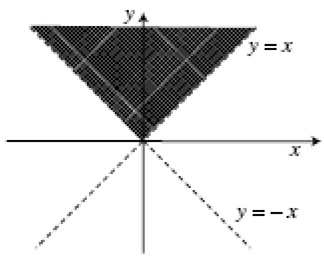
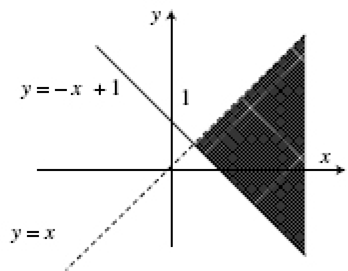


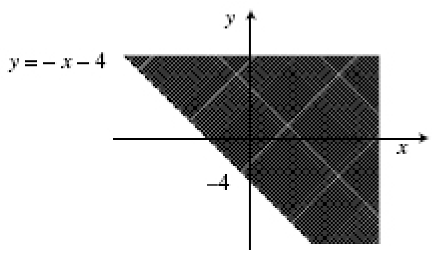
(vii)



(viii)



(ix)



12. (i) 17; (ii) 21; (iii) 99; (iv) 0; (v) 13; (vi) 13; (vii) 13.

13. Clearly $|8+(-5)| \ngeq |8|+|-5|$. Hence $|a + b| > |a| + |b|$ cannot be true for all real numbers.

On the other hand $|a + b| \leq |a| + |b|$ is true for all real numbers. Test the four cases $a > 0$, $b > 0$; $a > 0$, $b < 0$; $a < 0$, $b > 0$; $a < 0$, $b < 0$.

From your work in these four cases you will see that $|a + b|$ can equal $|a| + |b|$ if a and b are both positive or both negative or both zero.

($|a + b| < |a| + |b|$ is sometimes referred to as the *Triangle Inequality.* Why? See Chapter 5.)

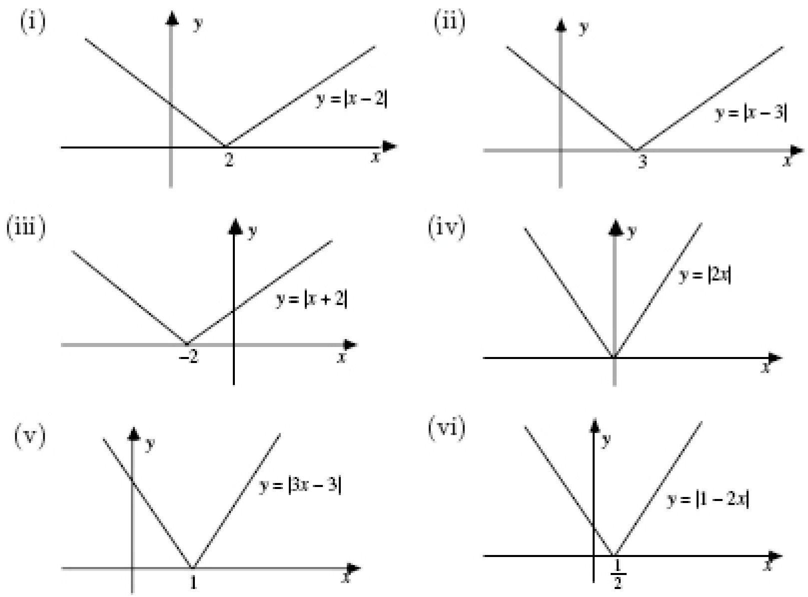
14. (i) For $a > 0$, $|a| = a$ and $|3a| = 3a$, so $3|a| = |3a|$. For $a < 0$, $|a| = -a$ and $|3a| = -3a$, so $3|a| = |3a|$. This argument can be used to show that $k|a| = |ka|$ for all non-negative values of k.

(ii) Since $|-3a|$ is always positive and $-3|a|$ is always negative, $-3|a| \neq |-3a|$. However, for all negative values of k we can show that $k|a| = -|ka|$.

(iii) $|5 - a| = |-(a - 5)| = |-5|$. It can be shown that $|ka| = |k||a|$, which generalises the generalisation of (i).

(iv) This is clearly false if a is zero, for instance.

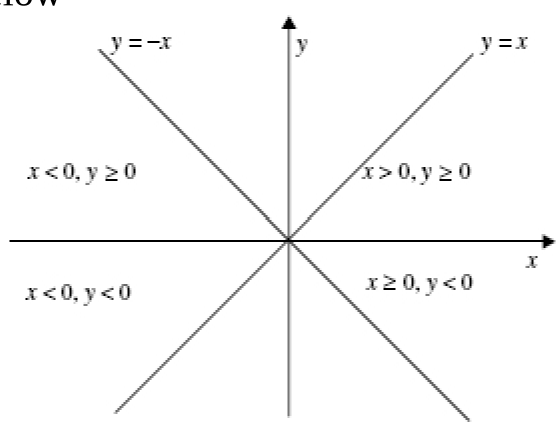
15.



16. (i) and (iii) are the same (see Exercise 14).
(ii) and (iv) are not the same (again see Exercise 14).

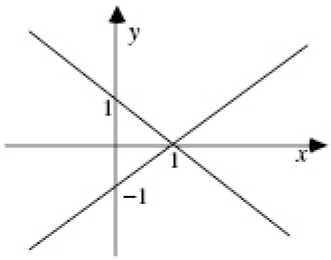
17.
$x > 0$, $y > 0$ gives $|y| = y$, $|x| = x$ so $y = x$;
$x > 0$, $y < 0$ gives $-y = x$ or $y = -x$;
$x < 0$, $y > 0$ gives $y = -x$;
$x < 0$, $y < 0$ gives $-y = -x$ or $y = x$.

We sketch the graph below

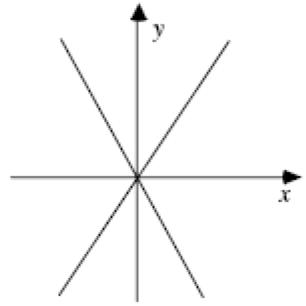


The four regions $x > 0$, $y < 0$, etc. are the four quadrants into which the x- and y-axes divide the plane. We insert the appropriate part of $y = x$ or $y = -x$ in the appropriate quadrant to get the graph above. This shows two intersecting perpendicular lines.

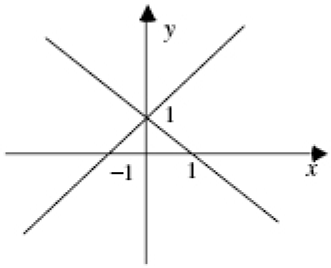
18. (a) (i)



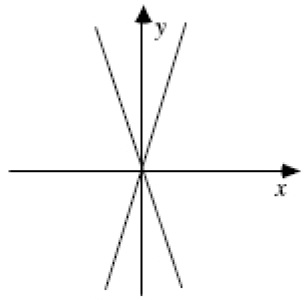
(ii)



(iii)

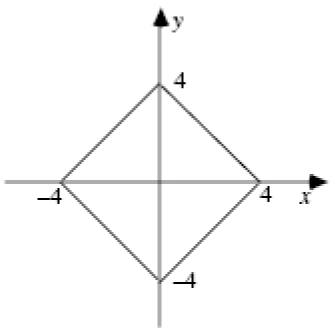


(iv)



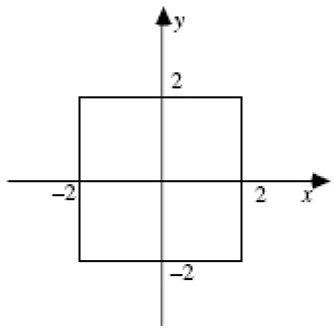
(b) $|y-3| = |x-5|$.

19. (i)



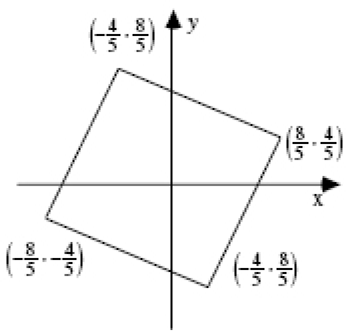
a square

(ii)



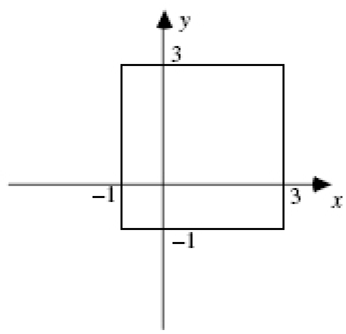
a square

(iii)



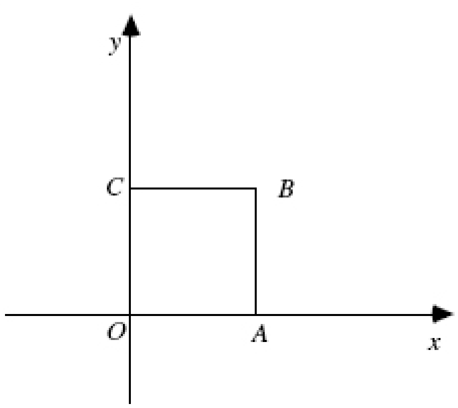
a square

(iv)



a square

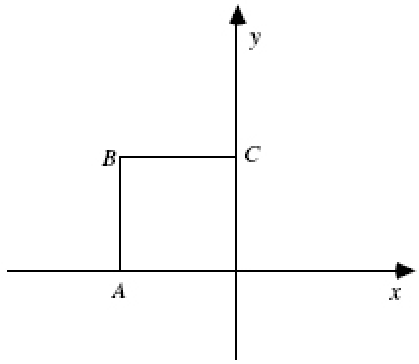
20. (i)



$|x-y| + |x+y-2| = 2.$

At the moment this answer can be achieved via guessing from Exercise 19.

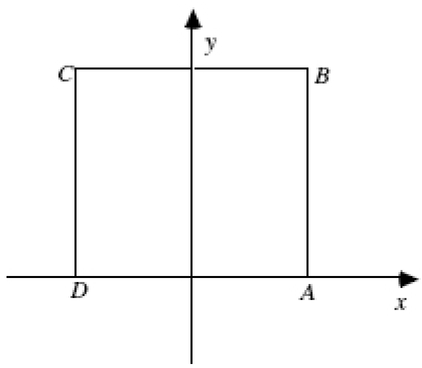
(ii)



$|x+y|+|x-y+1|=1.$

(After this amount of guessing you might be able to find the equation of *any* square.)

21. (i)



$|y-x-1|+|y+x-1|=2.$

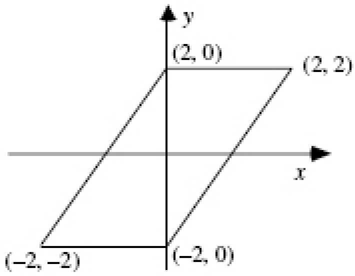
(What have diagonals got to do with all this?)

(ii) $|y-x|+|y+x|=2$; (iii) $|x|+|y|=1$; (iv) $|x-1|+|y-1|=1$.

22. First find the equations of the diagonals of the square. If these are $y = m_1x + c_1$ and $y = m_2x + c_2$ then the equation of the square is $a|y - m_1x - c_1| + b|y - m_2x - c_2| = 1$. The values of *a* and *b* will depend on the size of the square in question.

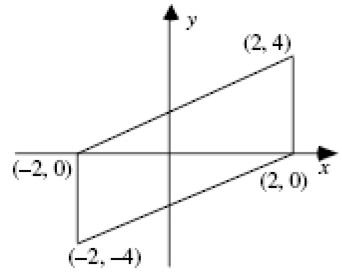
((1) What is the relation between m_1 and m_2? (2) Actually for fixed m_1, m_2, c_1, c_2, as *a* and *b* are varied in a fixed ratio you get an infinite set of “concentric” squares. What happens if you change *a* and *b* independently? (3) How do the diagonals of the square come into the picture? (4) What is the relation between the gradient of the diagonals? (5) How do you determine a given square from an infinite set of “concentric” squares?)

23. (i)



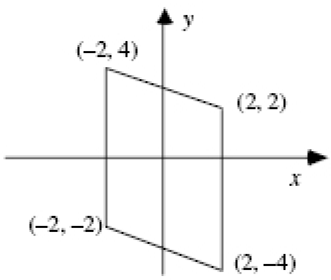
a parallelogram

(ii)



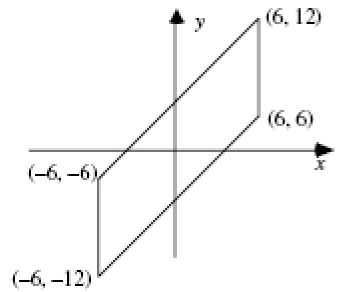
a parallelogram

(iii)



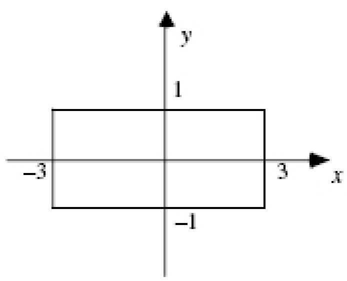
a parallelogram

(iv)



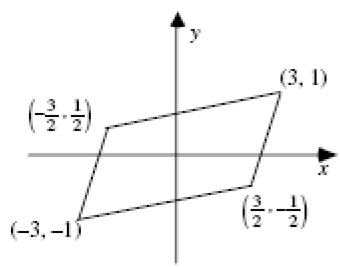
a parallelogram

(v)



a rectangle

(vi)



a parallelogram

(It would be interesting to see what happened if you sketched $a|3y - x| + |x + 3y| = 6$ for various values of a.)

(Again the diagonals seem to play an important role here.)

24. (i) $|x-1|+|2y-x|=1$; (ii) $|y+x|+|2y-x|=3$;
(iii) $|3y-2x+5|+|y+2x-5|=4$; (iv) $2|3y-x|+\frac{1}{2}|x+3y|=6$
or $4|3y-x|+|x+3y|=12$ or $\frac{1}{3}|3y-x|+\frac{1}{12}|x+3y|=1$ etc.

(If you did the extra work in Exercise 23(vi) you should have found this easily.)

25. (a) $a|y - m_1x - c_1| + b|y - m_2x - c_2| = 1$ where $y = m_1x + c_1$, $y = m_2x + c_2$ are the diagonals of the parallelogram and a and b have to be found for each quadrilateral.

(Can you actually prove this?)

(b) I don't think so. I conjecture that you need the quadrilaterals to be parallelograms. What do you think of that conjecture?

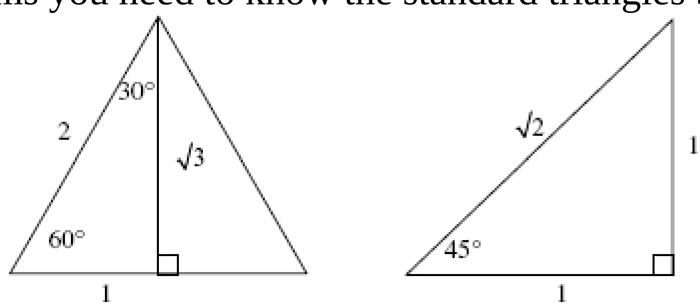
(c) What did you find? How did things go for things odd?

26. Sketch the two graphs with $b = 0$, say. Then move the second graph until it overlaps with the first. This is the same as translating the graph with $b = 0$. The values 6 and -6 look interesting.

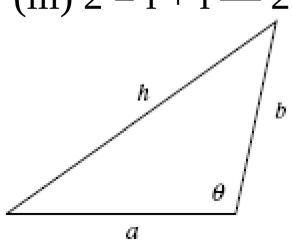
27. These are all circles. The centres of the circles are the given points and the radii are the fixed distances.

28. The line segment between (-1, 0) and (7, 0) is 8 units long. Hence this is a diameter of the circle involved. Thus $C = (3, 0)$.

29. From Chapter 5 we know that the perpendicular bisector of a chord of a circle passes through its centre. Hence $C = (1,4)$ and $a = 1$.
30. The points $C = (2, b)$, $(2, 0)$ and $(1,0)$ (or $(3, 0)$) form a right angled triangle with hypotenuse 2. Hence $b = \pm\sqrt{3}$. (There are two possible points C. Why?)
31. $C = (3, 4)$ or $(3, -4)$. (See the last Exercise.)
32. $C = (12, 4)$ or $(-12, 4)$.
33. $C = (14, 2)$ or $(-10, 2)$.
34. Since $(1, 2)$ and $(11, 26)$ are a distance 26 apart, they form a diameter of the circle. Hence $C = (6, 14)$.
35. Since the distance between the two given points is $25\sqrt{2}$ and the radius of the circle is 25, C, $(7, -1)$ and $(32, 24)$ are on the vertices of a right angled triangle with sides 25, 25, $25\sqrt{2}$. Hence $C = (32, -1)$ or $(7, 24)$.
36. (i) $x^2 + y^2 = 1$; (ii) $x^2 + y^2 = 25$;
(iii) $x^2 + y^2 = 2$; (iv) $x^2 + y^2 = a^2$;
(v) $(x-1)^2 + y^2 = 1$; (vi) $(x-1)^2 + (y-2)^2 = 4$;
(vii) $(x+1)^2 + (y-1)^2 = 9$; (viii) $(x+1)^2 + (y+3)^2 = 2$;
(ix) $(x-2)^2 + (y-1)^2 = a^2$; (x) $(x-s)^2 + (y-t)^2 = a^2$.
37. We're obviously now moving in another direction. Let the point be $P = (x, y)$. Then we know that $(x-1)^2 + y^2 = (x+1)^2 + y^2$. If we simplify this we get $x = 0$. The locus is therefore the y-axis. (Or was this obvious from the start?)
38. Here we have $x^2 + (y-1)^2 = x^2 + (y-3)^2$. This simplifies to $y = 2$.
39. $(x-1)^2 + y^2 = x^2 + (y-1)^2$ simplifies to $y = x$.
40. To do these problems you need to know the standard triangles below.



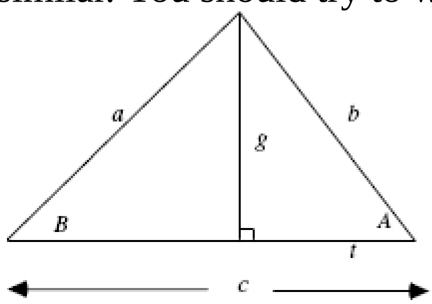
41. (i) $c^2 = 5^2 + 5^2 - 2 \cdot 5 \cdot 5\cos 60° = 50 - 50\cos 60° = 25$. Hence $c = 5$ and therefore $A = B = 60°$. (Actually it's easier to see from the start that $A = B = 60°$, so c *has* to be 5.)
(ii) $A = 120°$ and $b = c$. Hence $4 = 2b^2 - 2b^2 \cos 30°$. Therefore $b \approx 3.86$;
(iii) $2 = 1 + 1 - 2\cos B$. Therefore $B = 90°$ and $A = C = 45°$.



This result is known as the *converse* of Pythagoras' Theorem.
In the triangle shown, assume that $a^2 + b^2 = h^2$.
No, apply the cosine rule: $h^2 = a^2 + b^2 - 2ab\cos\theta$. But since $a^2 + b^2 = h^2$, $\cos\theta = 0$. Hence $\theta = 90°$.
(Beware! We may be on sand here. Does a proof of the Cosine Rule depend on what we are trying to prove? If it does we have gone in a circle and have proved nothing. What

do *you* think?)

43. We prove the Cosine Rule for an acute angle θ. The proof for an acute angle is similar. You should try to work that out for yourself.



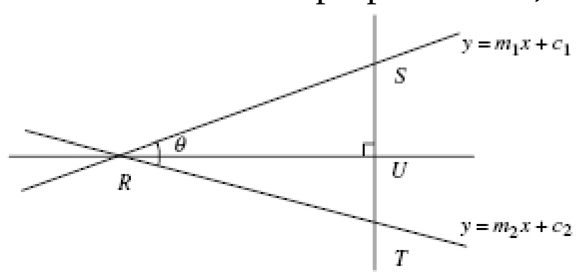
Now by Pythagoras $g^2 = a^2 — (c — t)^2$ and $g^2 = b^2 — t^2$. Hence $2tc = b^2 + c^2 — a^2$. Further $\cos A = \frac{t}{b} = \frac{2tc}{2bc} = \frac{b^2+c^2-a^2}{2bc}$. Simplifying we get the Cosine Rule.

(**Note.** We proved this using Pythagoras' Theorem. Hence the converse of Pythagoras that we proved in the last exercise has been proved. There is no flaw in our argument.)

44. The sine rule says that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Using the triangle in Exercise 43 we see that $g = b \sin A = a \sin B$. Hence $\frac{a}{\sin A} = \frac{b}{\sin B}$. The rest of the rule follows by dropping the perpendicular from angle *A* and using a similar argument on the two triangles so created.

45. If the two lines are perpendicular, then Exercise 10 shows us that $m_1 m_2 = -1$.



Now we'll assume that $m_1 m_2 = —1$ and prove that the two lines are perpendicular. Suppose the lines meet at *R* in an angle θ. Then $ST^2 = RS^2 + RT^2 — 2RS$? $RT \cos \theta$.

If we are cunning and choose *U* so that $RU = 1$, then $SU = m_1$ and $UT = |m_2|$. (I'm assuming for simplicity that $m_1 > 0$.) In $\triangle RSU$ we then have

$$RS^2 = 1 + m_1^2 \quad \text{and} \quad RT^2 = 1 + m_2^2.$$

Now

$$ST^2 = (m_1 + |m_2|)^2 = m_1^2 + 2m_1|m_2| + |m_2|^2$$
$$= m_1^2 + 2m_1\frac{1}{m_1} + \frac{1}{m_1^2} \quad (\text{since } m_1m_2 = -1).$$

Hence

$$ST^2 = m_1^2 + 2 + \frac{1}{m_1^2} = \left(m_1 + \frac{1}{m_1}\right)^2.$$

On the other hand

$$RS^2 + RT^2 = (1 + m_1^2) + (1 + m_2^2)$$
$$= (1 + m_1^2) + \left(1 + \frac{1}{m_1^2}\right) = \left(m_1 + \frac{1}{m_2}\right)^2.$$

So back to the Cosine Rule…

$$\left(m_1 + \frac{1}{m_1}\right)^2 = \left(m_1 + \frac{1}{m_1}\right)^2 - 2RS \cdot RT \cos \theta.$$

Clearly $\cos\theta = 0$, so $\theta = 90°$. Hence if $m_1 m_2 = -1$, then the two lines are perpendicular.

46. Use the Cosine Rule on Δ's *APC* and *BPC*. Let $\angle BCP = 0$. Then

$$\begin{aligned} BP^2 &= PC^2 + BC^2 - 2PC \cdot BC \cos\theta \\ &= 4 + 1 - 4\cos\theta \ (PC \text{ is a radius}) \\ &= 5 - 4\cos\theta, \end{aligned}$$

and

$$\begin{aligned} AP^2 &= PC^2 + AC^2 - 2PC \cdot AC \cos\theta \\ &= 4 + 16 - 16\cos\theta \\ &= 20 - 16\cos\theta = 4(5 - 4\cos\theta) = 4BP^2. \end{aligned}$$

Hence the result follows.

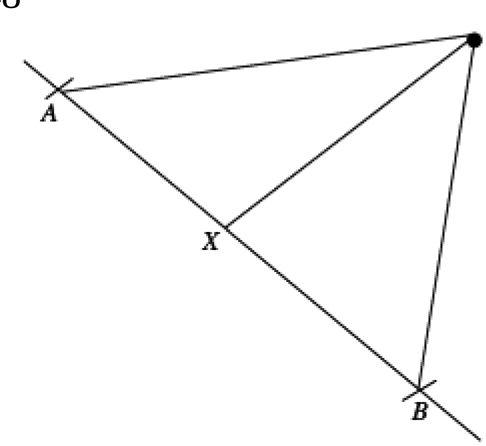
47. (i) the straight line $x = 2$;

(ii) the line $y = 2$;

(iii) $(x - 2)^2 + y^2 = x^2 + (y - 2)^2$ gives the line $y = x$;

(iv) $(x - 2)^2 + y^2 = x^2 + (y - 4)^2$ gives the line $y = \frac{1}{2}x + \frac{3}{2}$.

48



Let *P* be any point such that *AP* = *PB*, where *A* and *B* are arbitrary fixed points. We will show that *P* lies on the perpendicular bisector of *AB*.

Consider Δ's APX, *BPX*, where *X* is the midpoint of the line segment *AB*. Now *AP* = *BP*, and *AX* = *XB* given. Clearly *PX* = *PX*. Hence Δ's APX, *BPX* are congruent (SSS). Hence $\angle AXP = \angle BXP$. But $\angle AXP + \angle BXP = 180°$, so $\angle AXP = \angle BXP = 90°$.

We have proved that *PX* is perpendicular to AB, so *P* lies on the perpendicular bisector of *AB*. Hence all points which are equidistant from *A* and *B* lie on a line (the perpendicular bisector of *AB*). The required locus is therefore a straight line.

49. I'm too lazy to do all of these. So here goes with (iv). (The rest are the same but easier.)

Now the midpoint of *GH* is (1, 2). This lies on the line $y = \frac{1}{2}x + \frac{3}{2}$. The gradient of *GH* is – 2 and of the line is ½. Since ½ (– 2) = – 1, the line segment *GH* and the line y = 1 2x + 3 2 are perpendicular. Hence $y = \frac{1}{2}x + \frac{3}{2}$ is the perpendicular bisector of *GH*.

Exercise 48 shows that this is always the case.

50. (a) Let *P* = *(x, y)*. If *AP* : *PB* = 2 we have $AP^2 = 4PB^2$. Hence

$$x^2 + y^2 = 4[(x - 3)^2 + y^2].$$

Simplifying gives

$$3x^2 - 24x + 3y^2 + 36 = 0. \qquad (4)$$

If you know the algebraic trick of completing the square then you'll see that

$$3x^2 - 24x = 3(x^2 - 8x) = 3[(x - 4)^2 - 16] = 3(x - 4)^2 - 48.$$

If you don't know the trick I think you should be able to check that what I've done is correct.

So going back to equation (4) we get

$$3(x-4)^2 - 48 + 3y^2 + 36 = 0.$$
$$\therefore\ 3(x-4)^2 + 3y^2 = 12.$$
$$\therefore\ (x-4)^2 + y^2 = 4.$$

This is clearly a circle, centre (4, 0) and radius 2.

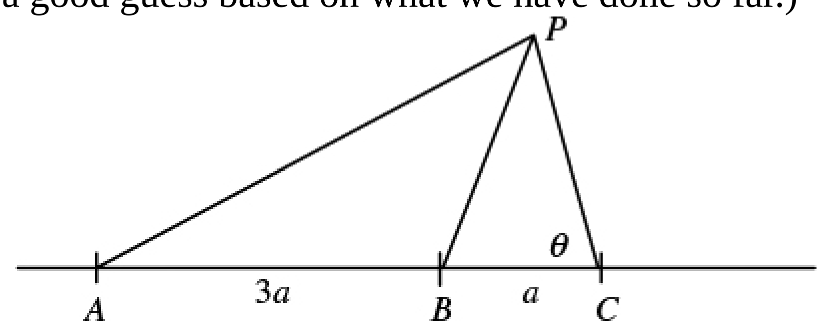
(b) $4AP = PB$ gives $4[x^2 + y^2] = (x-3)^2 + y^2$ which simplifies to $(x+1)^2 + y^2 = 2^2$ — another circle.

(c) (i) $AP : PB = 2$ gives $\left(x - \frac{11}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2$;
$PB : AP = 2$ gives $\left(x - \frac{1}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2$.

(ii) $CP : DP = 2$. $(x-1)^2 + y^2 = 4[(x-1)^2 + (y-4)^2]$. This simplifies to $(x-1)^2 + \left(y - \frac{16}{3}\right)^2 = \left(\frac{8}{3}\right)^2$;
$DP : CP = 2$. $4[(x-1)^2 + y^2] = (x-1)^2 + (y-4)^2$. This simplifies to $(x-1)^2 + \left(y - \frac{4}{3}\right)^2 = \frac{32}{9}$.

These are both circles. Similar results hold for (iii) and (iv).

51. Let *A* and *B* be any two distinct points. Suppose they are a distance 3a apart. Let *BC* = *a*, where *C* is on the line *AB* extended past B. (Here I'm cheating a little by letting *BC* = *a* but it is a good guess based on what we have done so far.)



We will show that *PC* is a constant for all *P* such that *AP* : *PB* = 2. Using the Cosine Rule on triangles *APC, BPC* gives

$$AP^2 = PC^2 + AC^2 - 2PC \cdot AC\cos\theta$$
$$= PC^2 + 16a^2 - 8a \cdot PC\cos\theta$$
$$BP^2 = PC^2 + BC^2 - 2PC \cdot BC\cos\theta$$
$$= PC^2 + a^2 - 2a \cdot PC \cdot \cos\theta$$

Hence

$$PC^2 + 16a^2 - 8a \cdot PC\cos\theta = 4[PC^2 + a^2 - 2a \cdot PC\cos\theta].$$

Simplifying gives *PC* = 2a. Since this is a constant, *P* moves so that it is always equidistant from the fixed point *C*. Hence the locus of *P* is a circle.

(**Note.** (1) You can use the cartesian method of proof but the algebra gets a bit messy.
(2) *BP* : *AP* = 2 must also give a circle. Just interchange the roles of *A* and *B* in the proof above.
(3) *A* and *B* are not necessarily on a horizontal line.)

52. Surely a circle again. Try the argument of the last exercise with *AB* = 8a and *BC* = *a*. You should find that *PC* = 3a, so the locus of *P* is a circle of radius 3*a*, centre *C*.

53. **Conjecture.** *For k positive, the locus of P is such that AP : BP = k is a circle. The centre C of this circle is on the line AB extended a distance* $\frac{AB}{(k^2-1)}$ *past B.*

Comment. This conjecture looks good for k = 2, 3. Does it have any obvious drawbacks though? As with all conjectures you now either prove it or come up with a counter-example. I'll go away and come back later when you've had a chance to think.

54. You should find answers that are close to the following ones.

(i) x-intercepts: $(\pm 3, 0)$, y-intercepts: $(0, \pm\sqrt{8})$;
(ii) $(\pm\frac{9}{2}, 0)$, $(0, \pm\sqrt{77})$; (iii) $(\pm\frac{9}{2}, 0), (0, \pm\sqrt{65})$;
(iv) $(\pm 6, 0), (0, \pm\sqrt{32})$; (v) $(\pm\frac{15}{2}, 0), (0, \pm\frac{1}{2}\sqrt{209})$.

55. With A and B fixed and k increasing, the magnitude of the y-intercepts and that of the x-intercepts get closer in value. Consequently the closed curve becomes more circular. However, no matter how large k becomes, the curve is never actually a circle.

From the physical constraints of the problem it is clear that k must be at least as big as the distance between A and B. If k equals this distance what locus do we get? What is the relation between the x-intercept and k?

56.

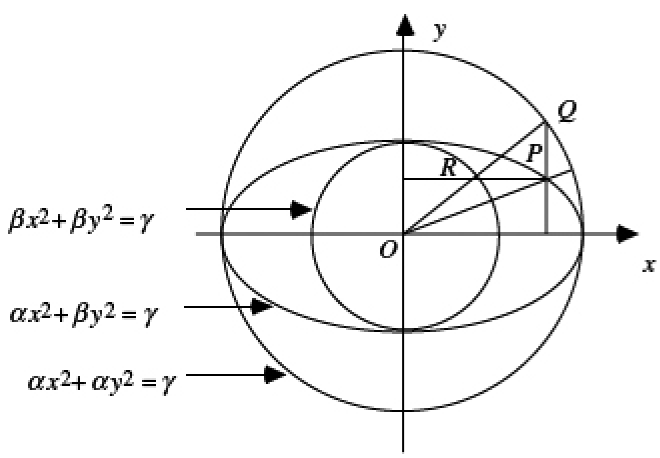
(ii) $\frac{4x^2}{81} + \frac{4y^2}{77} = 1$; (iii) $\frac{4x^2}{81} + \frac{4y^2}{65} = 1$;
(iv) $\frac{x^2}{36} + \frac{y^2}{32} = 1$; (v) $\frac{4x^2}{225} + \frac{4y^2}{209} = 1$.

57. (a) Find the x- and y-intercepts. The length of the string is 6. (This is twice the x-intercept. Why?) If the drawing pins are at (±c, 0) then $c = \sqrt{5}$ (using y-intercept information). So the drawing pins are at $(\pm\sqrt{5}, 0)$.

(b) (i) Stringlength: 2, drawing pins: $(\pm\sqrt{3}, 0)$; (ii) 6, $(\pm\sqrt{8}, 0)$;
(ii) 4, $(\pm\frac{1}{3}\sqrt{124}, 0)$; (iv) $2\sqrt{8}$, $(\pm 1, 0)$.

58. You should get $\frac{4x^2}{k^2} + \frac{4y^2}{k^2 - 4a^2} = 1$, so $\alpha = \frac{4}{k^2}$ and $\beta = \frac{4}{k^2 - 4a^2}$.

59



(Here we are assuming that $\alpha < \beta$.)

Draw the line ORQ. Then the vertical through Q and the horizontal through R, intersect at P, a point on the ellipse. So P has the x-coordinate of Q and the y-coordinate of R.

52. **(Revisited)** I knew you'd find this eventually. By now you've had a chance to think over my conjecture. For a start, you should have found that k = 1 causes difficulties. We already know that if $AP = PB$, then the locus of P is a line. The fact that we were looking at a quantity $\frac{AB}{(k^2-1)}$ should have alerted you to this.

This means we need k > 1. However, at this stage I think the conjecture can be proved using the Cosine Rule as we did in Exercise 55.

But what does $0 < k < 1$ mean? If $AP: PB = k$ then $PB : AP = \frac{1}{k}$ and $\frac{1}{k} > 1$. So in this case we just interchange the roles of A and B and our circle reappears but with its centre on the "*A* side" of *AB*.

You might like to think about what happens for $k > 1$ as k approaches 1. The centre C moves further and further away from B. I suppose in the limit you might think of C as reaching infinity so that the straight line we get when $k = 1$ is somehow the arc of a circle with infinite radius.

As k passes through 1, does C reappear at infinity (or at least a very long way off) but on the "*A* side" of *AB*?

(Use some technology to animate this situation and see what it looks like.)

60. Just rotate the situation for foci on the x-axis, through 90°.

For the "two ellipses" situation, take the x-coordinates of R (see Exercise 63's solution) and the y-coordinate of Q to get a point on the new ellipse.

You can actually get an infinite number of ellipses by rotating the diagram slowly.

61. (a) First note that the x-intercepts are $(\pm a,0)$ and the y-intercepts are $(0, \pm b)$. Hence the length of string is $2a$ and the foci are at $(\pm\sqrt{a^2 - b^2}, 0)$.

It's at this point you see why a needs to be bigger than b. If it were the other way round we would be trying to find the square root of a negative number in order to find the position of the foci.

(b) You should sketch this situation. Now the long axis of the ellipse is vertical. Consequently the string length is 2b and the foci are at $(0, \pm\sqrt{b^2 - a^2})$.

(c) If a = b, then we have a circle, centre the origin and radius a. (Here the two foci coincide to become the one centre.)

62.

(i) $12x^2 - 4y^2 = 3$; (ii) $144x^2 - 343y^2 = 324$;

(iii) $5x^2 - 4y^2 = 20$.

None of these are closed curves. They actually consist of two branches.

Such curves are popularly known as *hyperbolas*.

63. If the lines are parallel, then the locus is another line parallel to these two and mid-way between them.

For skew lines the locus is the two lines that bisect the two adjacent angles formed by the skew lines. (What is the relation between the angle between the original pair of lines and the angle between the lines of the locus?)

64. Suppose P moves so that it is equidistant from A, B and C. From Exercise 48, P lies on the perpendicular bisectors of *AB* and *BC*. If A, B, C are not collinear then this gives us a unique point. If A, B, C are collinear it gives us no point.

65. I don't know but I'd like to find out. I conjecture that, depending on the constant, it is a closed curve. However, rushing into algebra containing three square roots is extremely off-putting. Has anyone got any better ideas? (Try a CAS program.)

66. This is a *parabola*. It isn't a closed curve. It looks a bit like a part of a hyperbola.

67. Surely it is.

68. (i) $y = 3 - \frac{x^2}{12}$ — a a parabola;

(ii) $y^2 = -8x + 16$ — also a parabola but with a horizontal axis of symmetry.

69. (i) $y = 5$; (ii) $y = -x$; (iii) $y = 3x$; (iv) $y = 5$.

Chapter 8
Some IMO Problems

8.1. Introduction

This chapter is slightly different from the others in that it is the only one that looks specifically at four problems and makes no effort to introduce any new mathematics. The problems too, are ones that have been used or proposed at International Mathematical Olympiads (IMO). The aim here is to give you a chance to have a go and compare your ability with the best students in the world. Because the questions are hard, I provide some hints and suggestions in case you get stuck. Complete solutions are provided eventually.

The main reason for choosing the problems that I have is that they are all questions in which progress can be made by trial and error and looking at special cases. This is not always the case with IMO problems. Usually you'll need to know some geometry, some number theory or whatever, before you get started. But the problems I've chosen here can be done after a little experimentation. Hopefully the hints will help you see how experimenting with maths problems can sometimes lead you to a solution.

8.2. What is the IMO?

There are mathematics competitions held in a large number of countries throughout the world. In some countries there are regional competitions, and in some there are national competitions, while in some others there are both. The supreme maths competition available to secondary school students, however, is the International Mathematical Olympiad. This is open to any country that can assemble six or fewer students (20 years and under) to travel to the country hosting the IMO in that particular year.

Each participating country may send questions it has devised to the host nation. From these questions, approximately 30 are selected for consideration by the Jury — the collection of team leaders, who gather in the host country a few days before the arrival of their teams. From these short-listed questions, six are chosen and on each of two consecutive mornings, three questions are attempted by the students in a 4½ hour marathon exam.

After they have completed the two mornings of competition the students are entertained by their hosts, while the team leaders and their deputies mark their team's attempts. These marks have to be justified before a panel of people from the host country.

Approximately the top 50% of students gain a medal of some description. The lower half of the medal winners get bronze medals, the top one- sixth get gold and the rest get silver. The IMO has been going since 1959. It started as a competition between eastern bloc countries and by the turn of the 21st century some 90 countries from all over the world competed.

One of the singular features of the IMO is that once a team arrives at a predetermined point in the host country, accommodation and meals are both provided free of charge. It maybe this that is the reason for the very friendly atmosphere in which the IMO take place. It may, of course, simply be that people are people wherever they come from and the majority of us put on our best behaviour when we are someone's guests.

Finally let me say that the IMO is not an end in itself. I think most of the team leaders would not be involved just for the sake of the six who they take to IMOs. Generally the effort is all about encouraging students to think about mathematics. The IMO serves as a pinnacle to attract the best students of all countries but in the process of finding these students, hopefully a large number of students of all levels of ability are introduced to more mathematics than they would meet in school.

8.3. PHIL 1

The following problem was proposed by the Philippines at the 30th IMO.

Problem 1. *Prove that the set* {1,2,…,1989} *can be expressed as the disjoint union of* 17 *subsets* $A_1, A_2, \ldots, A_{17}$ *such that*

(i) *each* A_i *contains the same number of elements, and*

(ii) *the sum of all elements of each* A_1 *is the same for* i =1,2,…, 17.

The IMO organisers thought that the following two alternative forms should be considered by the Jury.

Problem 2. *Prove that the set* {1,2,…,1989} *can be expressed as the disjoint union of* $A_1, A_2, \ldots, A_{117}$ *such that*

(i) *each* A_i *contains the same number of elements, and*

(ii) *the sum of all elements of each* A_i *is the same for* i = 1,2,…, 117.

Problem 3. *Let* $M = \{1,2,\ldots, n\}$. *Prove necessary and sufficient condition(s) for the number m, so that M can be expressed as the disjoint union of m subsets* A_i, $i = 1,2,\ldots,$ *m, such that*

(i) *each* A_i *contains the same number of elements, and*

(ii) *the sum of all elements of* A_i *is the same for* $i = 1, 2, \ldots, m$.

Problem 2 was used in the 30th IMO as Question 1. I refer to it in the section heading as PHIL 1 because that was its name in the early Jury sessions.

It occurs to me that some of you may never have heard of "necessary and sufficient". Actually, it's the same as "if and only if".

You all know Pythagoras' Theorem. It can be stated as

> "*A triangle is a right angled triangle if and only if the square of the hypotenuse (h) is equal to the sum of the squares of the other two sides (a, b).*"

This is because *if* the triangle is right angled, *then* $h^2 = a^2 + b^2$ and *if* $h^2 = a^2 + b^2$, *then* the triangle is right angled. (I talked about this in the last chapter.)

But we can also state Pythagoras' Theorem in terms of necessary and sufficient.

> "*A* necessary and sufficient *condition for a triangle to be right angled is that* $h^2 = a^2 + b^2$."

Think of it this way. If the triangle is right angled, then $h^2 = a^2 + b^2$. There is no other choice for h, a, b. They have to be linked by $h^2 = a^2 + b^2$. It is *necessary* that $h^2 = a^2 + b^2$.

On the other hand, if $h^2 = a^2 + b^2$, then the triangle is right angled. In other words, to get a right angled triangle all we have to know is that $h^2 = a^2 + b^2$. It is enough, that is, it is *sufficient* for our purposes — the getting of right angle-ness — that $h^2 = a^2 + b^2$.

So that's what necessary and sufficient is all about. In Problem 3, then, you have to find some condition "blah" (or conditions blah, blah and blah), such that blah implies that M can be broken into the A_i's as required (blah is sufficient). You also have to show that if M can be broken up into A_i's as described, then blah follows (blah is necessary).

You should now sit down with pencil and paper for a day or so and see how far you can get. If you think you can solve Problem 2 see what you can do with Problem 3. If Problem 2 escapes you then go to p. 254 for some hints.

8.4. MON 1

The following problem was submitted for the 30th IMO by Mongolia. It didn't make the final six but was considered by the Jury. Below I have reformulated the question in terms of graphs. This formulation doesn't make the question any easier or more difficult but it does make it nicer to state.

For background on graph theory, see Chapter 3. Some of the basic concepts and ideas discussed there may help you to solve this problem but I have given a few more ideas on graphs here. Some of these might help you understand the solution to the problem but you might still be able to solve the problem without them.

Recall that a *graph* is simply a collection of vertices joined by edges. I've shown four in Figure 8.1.

If all vertices are joined to all other vertices, then we say that the graph is *complete*. We denote the complete graph on n vertices by K_n. Hence C is K_5 but B is not K_4 (there is an edge missing).

We can put two graphs G and H together to make their *union*, $G \cup H$, just by drawing them next to each other. So $D = K_3 \cup K_4$.

A *spanning subgraph* H of G is one with the same vertices as G but only a subset of the edges. Hence B can be thought of as a spanning subgraph of K_4 and A as a spanning subgraph of K_5.

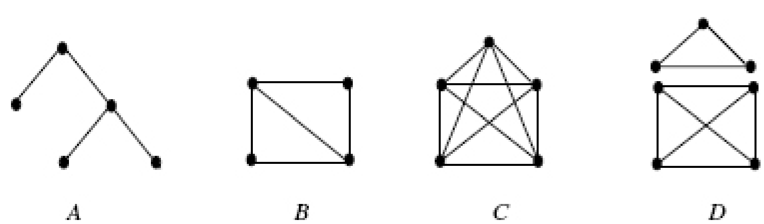


Figure 8.1.

Actually A is a *tree*. That is, it doesn't have any cycles — you can't go from any chosen vertex in A to any other along edges of A and then get back by another, different such route. However, you can get from any vertex to any other vertex along edges of A. This "getting between" and "no cycles" are the two things that make a graph a tree. Clearly B and C are not trees because they have cycles. D fails both because it has cycles and because you can't get from a vertex of the K_3 to a vertex of the K_4 using edges of D.

All this talk leads to the fact that A is a *spanning tree* of K_5.

Now onto Problem 4.

Problem 4. *A graph on seven vertices has the property that, given any three vertices, at least two are joined by an edge.*

What is the smallest number of edges in such a graph?

Find all such graphs.

Now work through this problem for yourself. If you haven't solved it after a day or so, then look at the Hints on p. 256.

To make things easy for ourselves, I will refer to the idea that among any 3 vertices there is at least one edge, as the *triple property*.

8.5. MON 6

The problem on which we base this section and the discussion of Section 9 was also submitted by Mongolia to the 30th IMO. It was not included in the final six problems because the idea had been used in another competition. However, this doesn't mean that it isn't an IMO standard problem.

Problem 5. *A positive integer is assigned to every square of an m × m chessboard. These numbers can be changed by adding an integer to two adjacent squares, provided such additions produce non-negative numbers*.

Find necessary and sufficient conditions on the original positive integers, so that after a finite number of such additions, all numbers on the board are zero.

(Two squares are adjacent if they share a common edge.)

Now work on. Previous questions should have given you a clue as to how to proceed. Hints can be gathered from p. 258.

8.6. UNK 2

The following problem was posed at the 29th IMO by the United Kingdom.

Problem 6. *A function f is defined on the positive integers, n, by*

$$f(1) = 1, \quad f(3) = 3, \quad f(2n) = f(n)$$
$$f(4n+1) = 2f(2n+1) - f(n)$$
$$f(4n+3) = 3f(2n+1) - 2f(n).$$

Determine the number of positive integers n, less than or equal to 1988, *for which f(n) = n*.

This is a nice problem for several reasons. The idea behind the function is interesting and even when you've found *f* there's still a little bit of work to do. In addition, even if you don't immediately see what *f* is, a little perspiration should lead you to finding it.

The hints start on p. 259 but by now you should be able to work out for yourself the first few things to try.

8.7. Hints — PHIL 1

In a question like this you can hunt around for inspiration by trying small cases and then looking for a pattern. This is one of the most basic of problem solving techniques and is possibly one of the most useful.

Problem 2′. *Find two disjoint subsets* A_1, A_2 of $M = \{1,2,3,4\}$ *whose union is M where* $|A_1| = |A_2|$ *and where the sum of the elements of* A_1 *equals the sum of the elements of* A_2.

Incidentally $|A_1|$ means the number of elements in the set A_1 and disjoint means that A_1 and A_2 have no elements in common.

Surely you'll quickly see that $A_1 = \{1,4\}$ and $A_2 = \{2, 3\}$ will do. (In fact I'm pretty sure that they're the only ones that will do.)

The aim of the next few exercises is to give you a feel for how you might play with the ideas of Problem 2, until you have assembled enough ammunition to be able to solve it.

Exercises

1. Find disjoint sets A_1, A_2, A_3 whose union is $M = \{1, 2, 3, 4, 5, 6\}$ such that $|A_1| = |A_2| = |A_3|$ and such that the sum of the elements in each of A_1, A_2, A_3 is the same.
   Is there only one solution?
2. Find disjoint sets A_1, A_2, A_3, A_4 whose union is $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|A_i|$ is the same for $i = 1, 2, 3, 4$ and the sum of all elements of A_i is the same for $i = 1, 2, 3, 4$.
   Is there only one solution?
3. Now answer Problem 3 if $n = 2m$.
   Is your solution unique?

Just to save ourselves a lot of writing I'm going to bring in a definition at this point. I'll say that $\{A_1, A_2,\ldots, A_m\}$ is a *partition* of the set M if $\cup^m_{i=1} A_i = M$ and if $A_i \cap A_j = \emptyset$ unless $i = j$. Here $\cup^m_{i=1} A_i = A_1 \cup A_2 \cup\ldots\cup A_m$.

So a partition is a collection of subsets of a set whose union is the whole of the set and such that no two subsets have any elements in common (they are all disjoint). For instance, in Problem 2', $\cup^2_{i=1} A_1 \cup A_2 = M$ and $A_1 \cap A_2 = \emptyset$. Hence $\{A_1, A_2\}$ is a partition of $\{1, 2, 3, 4\}$.

Again the aim behind the next set of Exercises is to see how to play with a problem until you get on top of it. Try leading up to a general solution.

Exercises

4. Show that your solutions of Exercises 1, 2, 3 are partitions of M in each case.
5. Rephrase Problem 3 using the word "partition".
6. Partition the set $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ into three sets of equal size so that the sum of the elements in each set is the same. {Before you start, what is going to be the size of the sum of the elements in each set of the partition?}
   Is your solution unique?
7. Repeat Exercise 6 with a partition of four equal subsets.
   Is your solution unique?
8. Show that $M = \{1,2,\ldots, 9\}$ can be partitioned into 3 equal size sets so that the sum of the elements in each set is the same.
   Can you show that this can only be done in two ways?
9. Partition $M = \{1,2,\ldots, 15\}$ into five equal subsets the sum of whose elements are equal.
   Is your solution unique?
10. Can you see how to partition $M = \{1,2,\ldots, 3m\}$ into m subsets of the same size so that the sum of the elements in each set is the same?
    If so, do it. If not, try some more specific values of m before returning to the general case.
    Is the solution unique?

11. Partition $M = \{1,2,\ldots, 5m\}$ into m subsets of equal size, the sum of the elements in each set being the same.
12. Repeat Exercise 11 with $M = \{1,2,\ldots, 7m\}$.
13. Solve Problems 1 and 2.
14. Based on your work above, what do you conjecture is (are) the required necessary and sufficient condition(s) for Problem 3.
Prove your conjecture.
When the required partitions exist are they ever unique?

But almost any problem can be extended. We start off with an extension that isn't really.

Exercises

15. Let $M = \{2^i : i = 1,2,\ldots n\}$. For what n can the set M be partitioned into sets A_i of equal size such that the *product* of the elements in each set A_i is the same?
16. Let $M = \{1,2,\ldots, 14\}$. If $A_i \subseteq M$, let π_i be the product of all of the elements of A_i.
(a) Partition M into sets A_i of size 2 so that $\sum_{i=1}^{7} \pi_i$ is as small as possible.
(b) Partition M into sets A_i of size 2 so that $\sum_{i=1}^{7} \pi_i$ is as large as possible.
(c) Repeat (a) and (b) with subsets of size 7.
Generalise.

8.8. Hints — MON 1

If you can't immediately see how to tackle a problem like this (and most of us can't), then try a smaller one and work your way up. Smaller cases are usually more manageable and at the same time they give you a feel for the problem and may suggest a line of attack.

Exercises

17. Try the problem first with 3, 4, 5 and 6 points. (There's really no point starting with 2 points because Problem 4 is about triples of points.)
The answer for 3 should be clear.
For 4 is 1 edge enough? Do we need 3 edges? What do the smallest graphs look like?
18. Conjecture what a smallest graph on 7 vertices must look like. Prove that the minimum graph must have at most that many edges.
Can you prove that the minimum graph has precisely that many edges?

Naturally though we can't stop there. It is clear that this problem can be generalised. So push on to the generalisation.

Exercise

19. What is (are) the smallest graph(s) (in terms of edges) on n vertices which obey(s) the triple property?

A graph on n vertices is said to be *minimal* with respect to the triple property if it has the triple property but none of its subgraphs on n vertices does.

For instance, $J = K_3 \cup K_5$ is minimal with respect to the triple property for all graphs on 8 vertices. Clearly it satisfies the triple property. However, if any edge is removed from J the triple property is lost.

The concept of minimality is a common one in mathematics. It tells us something about the smallest member of a string of objects. The string of objects with J as its minimal element is the set of graphs obtained from *J* by adding edges which don't exist in *J*.

Exercises

20. Find all the graphs on 7, 8, and 9 vertices, which are minimal with respect to the triple property.
21. What are the graphs on *n* vertices which are minimal with respect to the triple property?
22. A graph is said to satisfy the *quadruple property* if given any *four* vertices at least *two* are joined by an edge.
    Describe the minimal graphs with the quadruple property. What is the smallest number of edges among all such graphs?
23. A graph is said to satisfy the *m-ple property* if given any *m* vertices at least *two* are joined by an edge.
    Describe the graphs on *n* vertices which are minimal with respect to the *m*-ple property.
24. A graph is said to satisfy the *triangle property* if there is a triangle joining three of any four given vertices.
    Describe the graphs on *n* vertices which are minimal with respect to the triangle property.
25. A graph is said to satisfy the *t-clique property* if the subgraph induced by any $t + 1$ vertices contains a complete graph on *t* vertices.
    Describe the graphs on *n* vertices which are minimal with respect to the *t-clique property*.

It is clear that we could go on forever with this line of generalisations. Insert your own properties of graphs and see if you come up with some interesting results or interesting graphs.

8.9. Hints — MON 6

Once again I suggest that you proceed by stages with this problem. There's no point in looking at a 1 × 1 chessboard, so start with a 2 × 2. Also drop the non-negative condition. It's a little unnecessarily restrictive to start with.

Exercises

26. In the following 2 × 2 boards, a positive integer is placed in each square. By adding integers to pairs of neighbouring squares, which boards can be reduced to ones which contain all zeros? For the others, keep a record of how far you can get towards all zeros.

(i)

1	2
3	4

(ii)

2	5
3	6

(iii)

5	13
3	10

(iv)

1	2
4	3

(v)

11	4
15	8

(vi)

11	5
15	8

27. In the 2 × 2 board below, *a, b, c, d* are positive integers. By adding various integers to pairs of adjacent squares, find a relation between *a, b, c, d,* so that the numbers in each square eventually become zero.

a	b
c	d

28. Repeat Exercise 26 but this time only allow non-negative numbers to appear in each square at any stage.
29. Repeat Exercise 27 but this time only allow non-negative numbers to appear in each square at any stage.
30. Can the following one row board be reduced to all zeros by successively adding integers to neighbouring squares, without any square ever containing a negative integer?

1	6	3	2	5	1

31. Solve Problem 5 for the case $m = 1$ and arbitrary n. (Beware the problems of Exercise 30.)
32. Solve Problem 5 for the case $m = 2$, $n = 3$.
(If you find this too hard at first, go through the preliminary stages of putting in specific values for the original numbers. Then do the general case as in Exercises 27 and 31. This should give you a conjecture at least.)
33. Solve Problem 5 for the case $m = n = 3$.
34. Solve Problem 5.
35. Once again arbitrarily assign positive integers to the squares of an $m \times n$ chessboard. Change these numbers by progressively adding the same integer to a pair of neighbouring squares. Never allow any square to contain a negative number.
How close can you get to reducing all the numbers to zero?
Can you predict this at the start?
36. Solve Problem 5 for the case when positive real numbers are initially assigned to each square and we may add any real number to an adjacent pair of squares so that no square ever contains a negative number.

8.10. Hints — UNK 2

If you have no idea what f is at least you know that you should be trying to find it. The best way to start is to draw up a table of values.

Throughout the exercises, f refers to the function of Problem 5.

Exercises

37. Determine $f(n)$ for all $n \le 30$.
38. In the range of Exercise 37, for what n is $f(n) = n$? For what n is $f(n) \ne n$?
Can you see any patterns in any of these?
39. Without calculating $f(31)$ do you think it is 31 or something else? Why?
Now calculate $f(31)$.

At this stage (or perhaps much earlier) you will realise that if you had a computer at your disposal you could get it to solve the problem for you. All you need to do is to program it to calculate $f(n)$ for all $n \le 1988$ and check which of these values is the same as n. Indeed if you are desperate you could do the calculations yourself by hand.

In the meantime let's press on to find an analytical solution.

Exercise

40. Determine $f(2^m)$ for all m.
Determine $f(2^m \pm 1)$ for all m.

Have we now covered all the values in Exercises 37 and 39?

In my youth I knew a doughnut shop which sold rather delicious doughnuts. On the wall was, roughly speaking the following poem:

"*As you wander on through life my friend May this always be your goal. Keep your eye upon the doughnut And not upon the hole*."

In mathematics sometimes it pays to look at the hole. Even the holes tell you something about the doughnuts they're attached to.

Exercises

41. Note that there are some pairs n_1, n_2 that get interchanged by f. In other words there are n_1, n_2 such that $f(n_1) = n_2$ and $f(n_2) = n_1$. Try to identify as many of these as you can.

    Is there any pattern here? Are these any help in trying to find f?

42. Since $f(2n) = f(n)$ you've probably stricken even numbers off of your Christmas card list. How does $f(2n) = f(n)$ fit in with the interchanging of Exercise 41?

By now if you haven't worked out what f is and if you haven't stolen a look at the answer, you're probably extremely perplexed. Give it all another 24 hours to switch itself around in your head.

Exercises

43. Just for something different, and to give you a totally new perspective on life, express 11 and 13 in bases 2, 3 and 4. Look for similarities.

    Repeat with 19, 23, 25, 29.

44. (a) So what do you think flipping f is doing?

    (b) Whatever you think it is, show that f satisfies the defining relations in Problem 6.

    (c) Can any other function satisfy those relations?

45. Now you know what f is, for what n is $f(n) = n$?

    Count all such n that are less than or equal to 1988.

46. How many numbers less than or equal to 1988 exist such that $f(f(n)) = n$?

47. Produce a similar problem to Problem 6 which is based on the number 3.

48. The function g is defined on the natural numbers and satisfies the following rules:

    (i) $g(2) = 1$;

    (ii) $g(2n) = g(n)$ and $g(2n + 1) = g(2n)$ for all natural numbers n.

    Let n be a natural number such that $1 \le n \le 1989$. Calculate M, the maximum value of $g(n)$. Also calculate how many values of n satisfy $g(n) = M$.

(Irish Mathematical Olympiad 1989)

Of course, you need to have a certain basic mathematical knowledge before you tackle these problems — that's always going to be true. You can't do much in life on zero knowledge. But most of you know enough mathematics to be able to solve the problems posed here, by yourself. What you may not have had was an idea of *how* to tackle the problems. Now you know how to worry problems to death before they do the same to you.

8.11. Solutions

1. $A_1 = \{1,6\}$, $A_2 = \{2,5\}$, $A_3 = \{3,4\}$.

Since the sum of the elements of M is 21 then each set A_i must have a sum of 7. Sums of 7 can only be obtained from M in three ways. Hence, to within labelling of the A_i there is only one solution to this problem.

2. $A_i = \{i, 9 - i\}$ for $i = 1, 2, 3, 4$.

   The argument of Exercise 1 shows that this solution is unique.

3. $A_i = \{i, 2m + 1 - i\}$ for $i = 1, 2, \ldots, m$.

   Clearly the solution is unique.

4. In each case $M = \cup^{m}_{i=1} A_i$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

5. Let $M = \{1, 2, \ldots, n\}$. Prove necessary and sufficient condition(s) for the number m so that M can be partitioned into m equal size subsets the sum of whose elements are the same.

6. The sum of the elements of M is 78. Since we are to have three subsets in the partition and they are to have the same sum, the sum of each A_i is 26.

   One solution is $A_i = \{i, 7 - i, i + 6, 13 - i\}$ for $i = 1, 2, 3, 4$. However, you should be able to see that there are at least 15 different partitions that will do the job. This is because we have essentially taken the partition of M into 6 equal subsets with the same sum of 13, and combined them.

   There are more solutions yet. For instance, $A_1 = \{1,6,8,11\}$, $A_2 = \{2,5,7,12\}$, $A_3 = \{3,4,9,10\}$, does not come by combining partitions whose sum is 13. There is no way that both A_1 and A_2 can be subdivided into 2-element subsets any of whose sums is 13.

   So how many partitions exist for this problem?

7. For this partition, if it exists, we need a sum of $78 \div 4$. Clearly, no such partition exists.

8. Suppose $1 \in A_1$. Let $A_1 = \{1, a, b\}$. A quick count shows that $1 + a + b = 15$ so $a + b = 14$. The only pairs in M which give 14 are 5, 9 and 6, 8.

   Case 1. $A_1 = \{1,5,9\}$. Suppose $2 \in A_2$ and $A_2 = \{2, c, d\}$. Then $c + d = 13$. This forces c and d to be 6 and 7. (We can't use 9 and 4 because 9 is already in A_1. Similarly we can't use 5 and 8.) So $A_2 = \{2, 6, 7\}$ and we are left with $A_3 = \{3, 4, 8\}$.

   Case 2. $A_1 = \{1, 6, 8\}$. So let $A_2 = \{2, c, d\}$. Because $c + d = 13$ and 6 and 8 have been used in A_1, c and d are 4 and 9. So $A_2 = \{2, 4, 9\}$. This forces $A_3 = \{3,5,7\}$.

   So M can be divided in precisely two ways.

9. $A_1 = \{1,8,15\}$, $A_2 = \{2,9,13\}$, $A_3 = \{3,10,11\}$, $A_4 = \{4,6,14\}$, $A_5 = \{5,7,12\}$.

   There are at least two other solutions. Try to find them. One of them doesn't have all of 1, 2, 3, 4, 5, in different sets of the partition.

10. First we notice that the sum of elements of M is ½$3m(3m + 1)$. Hence the sum for each set of the partition is $3/2(3m+1)$. This is only an integer if m is odd. So we need m to be odd. In other words it is *necessary* that m be odd.

    So suppose m is odd. Can we do the partitioning? Try to generalise Exercise 9. Take $A_1 = \{1, a, b\}$. Now in Exercise 9, $b = 3m$. Since we must have $1 + a + b = 3/2(3m + 1)$ we see that $a = ½(3m + 1)$. Because m is odd, a is an integer.

So, in generalising the partition given in Exercise 9, let $A_i = \{i, a_i, b_i\}$ with $b_i = 3m - 2(i - 1)$ for $1 \le i \le ½(m + 1)$ and $b_i = 3m - 2(i - 1) + m$ for $½(m + 3) \le i \le m$ and with $i + a_i + b_i = 3/2(3m + 1)$. Hence $a_i = ½(3m - 1) + i$ for $1 \le i \le ½(m + 1)$ and $a_i = ½(m - 1) + i$ for $½(m + 3) \le i \le m$.

We now need to check that no number is repeated twice. Since $1 \le i \le m$ then there are no repeats in the *i*th term.

Now $½(3m - 1) + l \le a_i \le ½(3m - 1) + ½(m + 1)$ for $1 \le i \le ½(m + 1)$ and $½(m - 1) + ½(m + 3) \le a_i \le ½(m - 1) + m$ for $½(m + 3) \le i \le m$. Hence $½(3m + 1) \le a_i \le 2m$ for $1 \le i \le ½(m + 1)$ and $(m + 1) \le a_i \le ½(3m - 1)$ for $½(3m + 1) \le i \le m$. So the a_i's take on all values from $m+1$ to $2m$ and so no two a_i's are the same and no i_1 and a_{i2} can be equal.

Finally $2m + 1 < b_i < 3m$ for $1 < i$ $½(m + 1)$ and these are all odd, and $2m + 2 \le b_i \le 3m - 1$ for $½(m + 3) \le i \le m$ and these are all even. So the b_i's take all values from $2m + 1$ to $3m$ and so i_1, a_{i2} and b_{i3} can never be equal.

So the sets A_i as defined, do indeed form a partition with each set A_i having the required sum. In this proof it was enough to know that m was odd. So to obtain a partition of the required type it is *sufficient* to assume that m is odd.

Hence a necessary and sufficient condition for $M = \{1,2,\ldots, 3m\}$ to be partitioned into sets of equal size, so that the sum of the elements of each set is the same, is that m be odd.

On the uniqueness side it is easy to take one of the other partitions of Exercise 9 and generalise it. Unlike Exercise 3, the solution to the "3*m*" problem is not unique.

11. I assume that before you tackle this problem you will have tried to achieve a partition for the cases where $m = 3, 5, 7, \ldots$. When you've discovered a pattern you should form a conjecture and try to prove it. In other words, you should repeat (unless you've suddenly got insight into this problem and that may well happen) the steps leading to the proof of Exercise 10.

Claim. *A necessary and sufficient condition for a partioning of* $M = \{1,2,\ldots, 5m\}$ *of the type required is that m be odd.*

Proof. *m odd is necessary. The sum of the elements of each set has to be* $\frac{5m(5m+1)}{2m} = \frac{5}{2}(5m+1)$*. This number has to be an integer so we need m to be odd.*

m odd is sufficient. Assume m is odd. Then let $\{A_i\}$ be a partition of $3m$ into m sets as described in the proof of Exercise 10. Let $B_i = \{3m + i, 5m - (i - 1)\}$ for $i = 1,2,–, m$. Finally let $C_i = A_i \cup B_i$.

Now the sum of each of the elements in A_i is $3/2(3m+ 1)$ and the sum of the two elements of B_i is $8m + 1$. Hence the sum of all the elements of C_i is $3/2(3m + 1) + (8m + 1) = 5/2(5m + 1)$ as required.

Because the elements of A_i are the integers from 1 to $3m$ and the elements of B_i are the integers from $3m + 1$ to $5m$ (no two of which are equal) then $\cup^m_{i=1} C_i = M$.

Further $C_i \cap C_j = (A_i \cup B_i) \cap (A_j \cup B_j) = (A_i \cap A_j) \cup (B_i \cap B_j)$. Now we know from Exercise 10 that for $i \neq j$, $A_i \cap A_j = \emptyset$. It is clear that $B_i \cap B_j = \emptyset$ for $i \neq j$ since $3m + i$ covers the integers from $3m + 1$ to $4m$ while $5m - (i - 1)$ covers $4m + 1$ to $5m$.

Hence, given m odd, the sets C_i partition M in the required way.

(Clearly this partition is not unique.)

12. By dividing the sets of size 7 into one triple and two pairs, we again see m odd is a necessary and sufficient condition for the right type of partition to exist.

13. **Problem 1.** First note that $1989 = 117 \times 17$. One possible solution is to take A_i from Exercise 10 to cover the first $3m$ integers where $m = 17$. Each such A_i has sum 78.

Now note that $B_i = \{52 + (j - 1) + 57i, 1989 - (j - 1) - 57i: j = 1,2,\ldots, 57\}$ has sum $2041 \times 57 = 116337$.

Let $C_i = A_i \cup B_i$. Then the sum of the elements in C_i is $116337 + 78 = 116415$ (as required by $\frac{1989 \times 1990}{2 \times 17}$). Further $C_i \cap C_i = \emptyset$ and $\cup^{17}_{i=1} C_i = M$.

Problem 2. The same sort of argument works again. Take the A_i from Exercise 10 to cover the first $3m = 381$ integers. Then take $B_i = \{382 + (j - 1) + 7i, 1989 - (j - 1) - 7i: j = 1,2,\ldots, 57\}$.

Let $C_i = A_i \cup B_i$. Check that the C_i partition M as required.

14. Clearly n must be divisible by m or we cannot partition M into sets of equal size. Let $n = mt$.

Since the sum of the elements in each set of the partition is equal, $\frac{n(n+1)}{2m}$ must be an integer. Now $n = mt$ so if n is odd, $n + 1$ is even and $2m$ is a factor of $n(n + 1)$. On the other hand, if n is even, $n + 1$ is odd and $n + 1$ is not divisible by m. We thus require $2m$ to be a factor of n. This will happen unless t is odd.

The required necessary and sufficient condition is that either n is odd or n is even and $\frac{n}{m}$ is even. Alternatively the condition is either m and t are both odd or t is even. (Note that the problems of Exercises 10, 11 and 12 are special cases of m and t being odd.)

The proof for $t = 1$ is obvious. For t even we use the proof of Exercise 3. For all t odd, $t > 1$, $t = 2s + 3$ and we can apply the proof technique of Exercises 10, 11 and 12.

The partitions are only unique when $t = 1$ or 2.

15. Since $2^i 2^j = 2^{i+j}$, in other words, we add the indices when we multiply, this question is exactly the same as Problem 3.

16. (a) If $A_i = \{i, 15 - i\}$, $i = 1,2,\text{–}, 7$ then we minimize $\sum_1^7 \pi_i$. This is seen by noting that $ij + i'j' - ij' - i'j = (i' - i)(j' - j)$. Assume that $i < i'$. Then $ij' + i'j$ is greater than $ij + i'j'$ if $j' < j$. Therefore if $i < i'$ and $j' < j$ we need to put $ij + i'j'$ into $\sum \pi_i$ rather than $ij' + i'j$.

(b) The argument above shows that we need to put the high numbers together. Hence we require the partition $A_i = \{2i, 2i - 1\}$ for $i = 1,2,\ldots,7$.

(c) The argument of (a) implies that the partition $A_1 = (1, 3, 5, 8,10,12,14\}$, $A_2 = \{2, 4, 6, 7, 9,11,13\}$ minimises the sum of the products and $B_1 = \{1, 2, 3,4, 5, 6, 7\}$,

B_2 = {8, 9,10,11,12,13,14} maximises the sum of the products.

17. |VG| = 3. Clearly we only need one edge. There is a unique smallest edge graph here — the graph on three vertices with one edge.

|VG| = 4. Let VG = $\{a_1, a_2, a_3, a_4\}$. Suppose |EG| = 1 and $a_1a_2 \in$ EG. Then a_1, a_3, a_4 do not contain an edge between them. Hence we need |EG| > 1.

Let EG = $\{a_1a_2, a_3a_4\}$. Checking out the various possibilities we see that this graph is a required smallest graph. Any other two-edge graph on 4 vertices is of the form EG = $\{a_1a_2, a_2a_3\}$. Then a_1, a_3, a_4 do not satisfy the triple property. Hence there is a unique smallest graph here too.

|VG| = 5. Let's try to build up from what we know. Assume VG = $\{a_1, a_2, a_3, a_4, a_5\}$. Now in G_{a5} (G with vertex a_5 removed), by the 4-vertex case we must have at least two edges. So suppose a_1a_2, $a_3a_4 \in$ EG. Consider G_{a1}. This causes us to add a_2a_5 if we adopt the strategy of adding the fewest number of edges at a time. Now consider G_{a2}. This forces $a_1a_5 \in$ EG.

At this stage we have $G = K_3 \cup K_2$. Checking all sets of 3 vertices in G we see that they contain at least one edge. But can we find a graph with the triple property which has only 3 edges?

Let VH = $\{a_1, a_2, a_3, a_4, a_5\}$ and |EH| = 3. Now H does not have a spanning tree and is therefore not connected. If H has two isolated vertices, then these vertices plus any other vertex, disobey the triple property. Otherwise H has two components, one of which is not a complete graph, so two vertices in this component are not joined by an edge. These two vertices and a vertex in the other component do not satisfy the triple property.

So there are four edges in the smallest graph and that graph is $K_3 \cup K_2$ and is unique.

|VG| = 6. Because $K_3 \cup K_3$ has fewer edges than $K_4 \cup K_2$ then guess that the minimal graph here has 6 edges. We will assume that the minimal graph H has 5 edges and hope for a contradiction.

If H is connected it is a tree. If H has more than 2 endvertices (vertices of degree 1), then any 3 of these vertices do not satisfy the triple property. Hence H is P_6, a tree with no vertex of degree bigger than 2, and the 2 end vertices plus one vertex not adjacent to an endvertex again disobey the triple property.

If H is not connected then it is easy to find 3 vertices which do not satisfy the triple property.

We now look at the problem.

MON 1.

Conjecture. *The unique smallest graph is* $K_4 \cup K_3$ *which has* 9 *edges.*

Comment.

(1) Let G be a minimal graph. If G has two components they must be complete. (Why?)

(2) $K_4 \cup K_3$ has fewer edges than $K_5 \cup K_2$.

(3) The existence of $K_4 \cup K_3$ shows that if G is the smallest graph on 7 vertices with the triple property, then |VG| ≤ 9.

Claim 1. If *G is smallest, then* |EG| > 8.

Proof. We suppose that |EG| ≤ 8 and obtain a contradiction. Now the sum of the degrees of the vertices of G = 2|EG|. Hence the sum of degrees is less than or equal to 16. Hence there is at least one vertex with degree less than 3. Suppose this vertex is a_1. Then deg $a_1 \leq 2$ and so there are four vertices a_2, a_3, a_4, a_5 in G which are not adjacent to a_1. Now if any two of a_2, a_3, a_4, a_5 are not adjacent, these two vertices along with a_1 disobey the triple property.

Hence a_1, a_6, a_7 share at most 2 edges, so 2 of a_1, a_6, a_7 are not adjacent. These two with one of a_2, a_3, a_4, a_5 must then disobey the triple property.

□

Claim 2. *If G is smallest, then* |EG| = 9.

Proof. This follows from Claim 1 and the fact that $K_4 \cup K_3$ satisfies the triple property.

□

Claim 3. *$G = K_4 \cup K_3$ is the unique smallest graph*.

Proof. Now |EG| = 9 so suppose there exists H on 7 vertices with the triple property. Since $\sum_{v \in VH}$ deg v = 2|EH| = 18, then there exists a vertex in H with degree less than 3. Let this vertex be a_1 and let A_1 be the set of vertices joined to a_1 and let A_2 be the remaining vertices.

All vertices in A_2 must be adjacent, otherwise a_1 along with two non-adjacent vertices of A_2 disobey the triple property. If $|A_2| \geq 5$, then |EH| ≥ 10. Hence $|A_2| \leq 4$.

Since $|A_1| \leq 2$ and $1 + |A_1| + |A_2| = 7$ we must have $|A_1| = 2$ and $|A_2| = 4$.

By the triple property, if the two vertices of A_1 are not joined, then every vertex of A_4 is joined to at least one vertex of A_1. But this gives |EH| > 9.

Hence the two vertices of A are adjacent and so $H = G$.

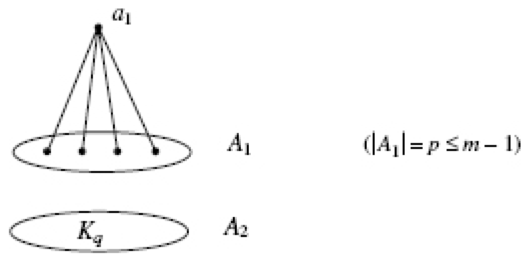
19. **Case 1.** Suppose n is even. Let n = 2m. Then $G = K_m \cup K_m$ satisfies the triple property.

We now show that G is the unique smallest graph with the triple property.

Assume H satisfies the triple property and |EH| ≤ |EG| = $m(m - 1)$. Since $\sum_{v \in VH}$ deg v = 2|EH| ≤ $2m(2m - 1)$. H contains a vertex of degree ≤ $m - 1$. Let a_1 be the vertex of minimum degree p say, in H. Let A_1 be the vertices adjacent to a_1 and let A_2 = VH – ($\{a_1\} \cup A_1$), with $|A_1| = 2m - p - 1$.

By the triple property (see the argument in Claim 3 of Exercise 18) all vertices of A_2 are joined.

Hence the graph on A_2 is K_q. The situation for H so far is shown in the diagram below.



Now

$$2|\mathrm{EH}| = \sum_{v \in \mathrm{VH}} \deg v = p + \sum_{a \in A_1} \deg a + \sum_{b \in A_2} \deg b$$
$$= p + \sum_{a \in A_1} \deg a + q(q-1) + \sum_{b \in A_2} [(\deg b) - (q-1)].$$

This last line follows since every vertex in A_2 has degree at least $q - 1$ and K_q has ½ $q(q - 1)$ edges. Given that a_1 is the vertex of minimum degree in H, then deg $a \geq p$ for all $a \in A_1$ Hence $\sum_{a \in A_1} \deg a \geq p^2$.

Hence $2|\mathrm{EH}| \geq p + p^2 + q(q-1) + \sum_{b \in A_2} [(\deg b) - (q-1)]$.

We now note that for 2|EH| to be minimum $\sum_{a \in A_1} \deg a = p^2$ and deg $b = q - 1$ for all $b \in A_2$. Hence the smallest value of |EH| is $\frac{1}{2}[p + p^2 + q(q-1)]$ In this case deg $a = p$ for all $a \in A_1$, so $A_1 \cup \{a_1\}$ induces a complete graph K_{p+1}.

Hence $H = K_{p+1} \cup K_q$, where $p + 1 + q = 2m$.

Claim. *Among all graphs $K_{p+1} \cup K_q$, where $p + 1 + q = 2m$ with the fewest edges is $K_m \cup K_m$.*

Proof. We use a trick here. Since $p \leq m - 1$, $p + 1 \leq m$. . Let $p + 1 = m - a$ for $a \geq 0$.. Then $q = m + a$.

$$|\mathrm{EH}| - |\mathrm{EG}| = \tfrac{1}{2}[(m-a)(m-a-1) + (m+a)(m+a-1) -$$

Now $2m(m-1)] = a^2$.

Since $|\mathrm{EH}| \leq |\mathrm{EG}|$ we must have $a^2 + |\mathrm{EG}| \leq |\mathrm{EG}|$, , in which case $a = 0$ and $H = G = K_m \cup K_m$.

□

Case 2. Suppose n is odd. Let $n = 2m + 1$. Then $G = K_m \cup K_{m+1}$ satisfies the triple property.

The proof is almost exactly the same as for Case 1.1 think you should be able to do it for yourself without my help.

20. Using previous arguments we get,

on 7 vertices : $K_1 \cup K_6$, $K_2 \cup K_5$, $K_3 \cup K_4$;

on 8 vertices : $K_1 \cup K_7$, $K_2 \cup K_6$, $K_3 \cup K_5$, $K_4 \cup K_4$;

on 9 vertices : $K_1 \cup K_8$, $K_2 \cup K_7$, $K_3 \cup K_6$, $K_4 \cup K_5$.

21. On n vertices we have $K_5 \cup K_{n-s}$, for $s = 1, 2, \ldots, t$, where t is the integral part of ½n. This has already been proved in Exercise 19.

22. The graphs $K_s \cup K_t \cup K_{n-s-t}$, where $1 \leq s \leq t \leq u$, where u is the integral part of $\frac{1}{3}n$, certainly satisfy the quadruple property and are minimal.

Now show that there are no other minimal graphs. To do this follow the pattern of Exercise 18. In the quadruple property case, A_2 will be $K_a \cup K_b$ plus perhaps some extra edges. The number of edges is least if we have just three complete graphs.

The smallest number of edges arises when s, t and $n - s - t$ are as equal as possible given n.

23. Here we have $\left(\bigcup_{i=1}^{m} K_{a_i}\right) \cup K_b$, where $n = b + \sum_{i=1}^{m} a_i$. The usual arguments apply.

The smallest number of edges is achieved when the a_i and $\sum_{i=1}^{t} a_i$ are as equal as possible.

24. If you can't see how to do this straightaway, then try to solve the problem for 7, 8, 9 and 10 vertices. This should lead you to a conjecture.

Claim. *The unique minimal graph is* $K_1 \cup K_{n-1}$.

Proof. Suppose G is a minimal graph which is not connected. If a_1, a_2 and b_1, b_2 are in distinct components of G, then there is no triangle containing any three of these vertices.

If G is disconnected, then it has two components, one of which is a single vertex, a say. If b_1, b_2, b_3 are in the other component and b_1 is not joined to b_2, then the triangle property is violated. Hence the component containing b_1, b_2, b_3 is complete and $G = K_1 \cup K_{n-1}$.

Suppose then that G is connected. There do not exist distinct vertices a_1, a_2, b_1, b_2 such that a_1 is not joined to a_2 and b_1 is not joined to b_2. This is because the triangle property is not satisfied by $\{a_1, a_2, b_1, b_2\}$ in this case. Hence all edges of $EK_n - EG$ are adjacent to a single vertex. Thus G contains $K_1 \cup K_{n-1}$ as a subgraph.

□

25. See Exercise 24.

26. Those which can be reduced to zeros are (i), (ii), (v).

$$\text{(iii) gives } \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array};$$

$$\text{(iv) gives } \begin{array}{|c|c|}\hline 0 & 2 \\ \hline 0 & 0 \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 2 & 0 \\ \hline \end{array};$$

$$\text{(vi) gives } \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array} \text{ or } \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}.$$

27. $a + d = b + c$.

28. You obtain the same answer.

29. The answer you get is still $a + d = b + c$.

30. $$\begin{array}{c} 1\,6\,3\,2\,5\,1 \\ 0\,0\,0\,4\,5\,1 \end{array} \rightarrow \begin{array}{c} 0\,5\,3\,2\,5\,1 \\ 0\,0\,0\,0\,1\,1 \end{array} \rightarrow \begin{array}{c} 0\,5\,5\,4\,5\,1 \\ 0\,0\,0\,0\,0\,0. \end{array} \rightarrow$$

So the answer is yes. The trick is to know when (and where) to add rather than subtract.

31. From now on, I will assume that the top left-hand square of the various chessboards is black. Let S_b be the sum of the numbers on the black squares and let S_w be the sum of the numbers on the white squares.

Claim. *In order for us to be able to reduce all the integers to zero without introducing negative numbers, it is necessary and sufficient that* $S_b = S_w$.

Proof. *The condition* $S_b = S_w$ *is necessary*. Since at each operation we add the same integer to a black and white square, the sums on the black and white squares are always changed by the same amount. To end with a total of zero on both the black and the white square we need to start with $S_b = S_w$.

The condition $S_b = S_w$ *is sufficient.* We assume that $S_b = S_w$ and then show how to reduce all entries to zero. We proceed by induction on n.

Step 1. If $n = 1$, then there are no white squares so $S_w = 0$. But $S_b = S_w = 0$, so the board is already reduced to zeros.

Step 2. Assume that for every $1 \times k$ board, if $S_b = S_w$ then the reduction can take place.

Step 3. Let $n = k + 1$. So we assume that $S_b = S_w$ for a $1 \times (k + 1)$ board. Suppose the board starts

b_1	w_1	b_2	w_2	...

If $b_1 \leq w_1$, then add $-b_1$ to the b_1 and w_1 squares. This gives a board

0	$w_1 - b_1$	b_2	w_2	...

Forgetting about the zero, we have a $1 \times k$ board with $S'_b = S_b - b_1 = S_w - b_1 = S'_w$. By Step 2, this smaller board can be reduced to zeros. Hence we can reduce the $1 \times (k + 1)$ board to zeros. If $b_1 > w_1$, then add $b_1 - w_1$, to the w_1 and b_2 squares to give

b_1	b_1	$b_2 + (b_1 - w_1)$	w_2	...

Now we have the situation of b_1 being less than or equal to the new "w_1" so we can use the argument above to reduce this case to zeros.

The fact that $S_b = S_w$, is therefore sufficient to prove the required result.

□

32. The necessary and sufficient condition is still $S_b = S_w$. The necessity follows from Exercise 31. The sufficiency follows by induction on n.
33. See Exercise 32.
34. The necessary and sufficient condition is $S_b = S_w$. The necessity follows from Exercise 31. To prove the sufficiency use induction on mn using at Step 2 the more powerful version of induction that assumes things can be done for all $mn \leq k$. Then show that the first row (or column) can be reduced to zeros.
35. You should be able to show that if $|S_b - S_w| = S \neq 0,$ then every square except one can be reduced to zero. The only non-zero square will contain S. Further, S will be on a black square if $S_b > S_w$ and on a white square if $S_w > S_b$.
36. Really the problem has got nothing to do with integers. We get the same answer for this question as we do for Exercise 34. Even the proof is the same.

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	1	1	3	1	5	3	7	1	9	5

n	11	12	13	14	15	16	17	18	19	20
$f(n)$	13	3	11	7	15	1	17	9	25	5

37.

n	21	22	23	24	25	26	27	28	29	30
$f(n)$	21	13	29	3	19	11	27	7	23	15

38.

n	1	3	5	7	9	15	17	21	27
$f(n)$	1	3	5	7	9	15	17	21	27

39. $f(31) = 31$.

40. $f(2^m) = 1$. This is easily proved by induction.

$$f(2^m + 1) = f(4 \cdot 2^{m-2} + 1) = 2f(2^{m-1} + 1) - f(2^{m-2}).$$

Induction should now convince you that $f(2^m + 1) = 2^m + 1$. Similarly $f(2^m - 1) = 2^m - 1$.

Unfortunately this doesn't count for the fact that $f(21) = 21$. But we've made some progress.

41. Various pairs that interchange are (11, 13), (19, 25), (23, 29). In fact for n odd it looks as if either $f(n) = n$ or $(n, f(n))$ are an interchangeable pair. So how do we tell the two apart?

42. The strange thing about $2n$ is that it's *twice* n.

43.

Number	Base 2	Base 3	Base 4
11	1011	102	23
13	1101	111	31
19	10011	201	103
23	10111	212	113
25	11001	221	121
29	11101	1002	131

That base 2 column looks flipping interesting.

44. (a) Only you can answer this question. (b) and (c) are yours too.

45. Well up to now maybe you don't, so I had better tell you. The function f converts n to its binary (base 2) form, then reverses the order of that binary number and then converts the reordered number to its decimal form.

If you haven't seen this up till now, then go back and check it all out. So look at f (1). Now $f(1)$. Now $1 = (1)_2 \to (1)_2$ after reversing, which is 1 in base 10. So $f(1) = 1$.

$$3 = (11)_2 \to (11)_2 = 3. \text{So } f(3) = 3.$$

$$2n = 2 \times n = (10)_2 \times (?)_2 = (?0)_2 \to (0\cup)_2 = (\cup)_2 = f(n).$$

So $f(2n) = (n)$.

$$\begin{aligned} 4n + 1 = 4 \times n + 1 &= (100)_2(?)_2 + (1)_2 = (?00)_2 + (1)_2 \\ &= (?01)_2 \to (10\cup)_2 = (1 \cup 0)_2 - (\cup)_2 = (10)_2(1\cup)_2 - (\cup)_2 \\ &= 2f(2n+1) - f(n). \end{aligned}$$

So $f(4n + 1) = 2f(2n + 1) - f(n)$.

$$\begin{aligned} 4n + 3 &= (100)_2(?)_2 + (11)_2 = (?11)_2 \to (11\cup)_2 \\ &= (1 \cup 0)_2 + (1\cup)_2 - (\cup 0)_2 \\ &= (11)_2(1\cup)_2 - (\cup 0)_2 = 3f(2n+1) - 2f(n). \end{aligned}$$

Hence $f(4n + 3) = 3f(2n + 1) - 2f(n)$.

So our “base 2 and reverse” function does satisfy the defining equations of Problem 6. But is it the only function to do this?

The answer is perhaps not. Whatever some other function g might be which satisfies the defining equations, it would be such that $g(1) = 1 = g(2)$ and $g(3) = 3$. Since all other function values are defined recursively in terms of the images of 1, 2, 3 then any other function g would have to agree with f everywhere. Hence the function defined is indeed the “base 2 and reverse” one.

After all that, we see that $f(n) = n$ if and only if n is a palindrome in base 2. Since $1988 = (11111000100)_2$, we only have to count the base 2 palindromes up to this stage. So we've got 1, 11, 101, 111, 1001, 1111, 10001, 10101, 11011,11111,…

There are a total of 92 of these palindromes. You can either list them or find a simple way to count them all.

46. Isn't it true that every odd number either has $f(n) = n$ or has $f(f(n)) = n$?

Of course there's no need to worry about even numbers.

47. What did you find?

48. After Problem 6 you should leap onto this problem and destroy it.

The current function is still about base 2. The value of $f(n)$ is the number of ones in the binary form of n. (Prove this.)

Now $2^{11} = 2048 > 1989 > 1024 = 2^{10}$. Hence $M = 10$. There are five solutions of $f(n) = M$. They are $n = 1023, 1535, 1791, 1919$ and 1983.

Index

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